Homework 2

Due Jan 29, 2025

Solutions and Rubric

1.4 : 1, 2, 3, 5, 6, 12

1.5 : 1, 3, 4, 5, 7, 10, 16, 18, 20

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 40 points
- So final score is $\left(\frac{\text{score}}{40}\right) \cdot 8$

§1.4	1.	Graded	16. Not graded
	2.	Completion	18. Not graded
	3.	Completion	20. Graded
	5.	Completion	
	6.	Not graded	
	12.	Graded	
§1.5.	1.	Graded	
	3.	Not graded	
	4.	Not graded	
	5.	Not graded	
	7.	Not graded	
	10.	Graded	

Question 1 (FIS, 1.4, nr. 1)

Graded

a) True, since $0 = \sum_{i=1}^{n} 0 v_i$, $\forall\, v_1, \dots, v_n \in S$ where $S \neq \emptyset$.

b) False, $\text{span}(\emptyset) = \{0\}$ by convention (page 30)

c.) True, let $W := \bigcap \{U \mid U \text{ subspace of } V \text{ containing } S\}$.
Since $\text{span}(S)$ is a subspace containing S, $W \subseteq \text{span}(S)$. But since $\text{Span}(S)$ smallest subspace containing S (thm 1.5), and U is a subspace containing S, $\text{span}(S) \subseteq W$.

d.) False, any non-zero constant.

e) True

f) False, $2x_1 + x_2 = 3$, $5x_1 + x_2 = 3$ has no solution

Question 2 (FIS, 1.4, nr. 2)

Completion

a.)

$$
\begin{aligned}
2x_1 - 2x_2 - 3x_3 &= -2 \\
3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\
x_1 - x_2 - 2x_3 - x_4 &= -3
\end{aligned}
$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 3 & 0 \\ 3 & -3 & -2 & 5 \\ 1 & -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix}$$

$\Rightarrow$ Solve $Ax = b$, with row reduction

$$x = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

b.) Complete

c.) Complete

d.) Complete

e.) Complete.

Question 3 (§ 1.4, nr 3)

Completion

Given vectors in $\mathbb{R}^3$, determine if the first vector is in the span of the other 2.

a.) Yes,

$$V_1 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

That is solve $a, b \in \mathbb{R}$ in

$$\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} a + 2b \\ 3a + 4b \\ -b \end{bmatrix}.$$

By R3, $b = -3$.

By R_1, $a + 2(-3) = -2 \Rightarrow a = 4$

Thus $V_1 = 4V_2 - 3V_3$.

(Unique solution)

b.) Yes, $V_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Solve $a, b \in \mathbb{R}$ (if possible)

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = a \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3a + 2b \\ 2a - b \\ a - b \end{bmatrix}$$

$$\begin{aligned} -3a + 2b &= 1 \\ 2a - b &= 2 \\ a - b &= -3 \end{aligned} \xrightarrow[R_3 + \frac{1}{3}R_1]{R_2 + \frac{2}{3}R_1} \begin{aligned} -3a + 2b &= 1 \\ 0a + \tfrac{1}{3}b &= \tfrac{8}{3} \\ 0a - \tfrac{1}{3}b &= -\tfrac{8}{3} \end{aligned}$$

$\Rightarrow b = 8$

$\Rightarrow -3a + 16 = 1$

$\Rightarrow -3a = -15$

$\Rightarrow a = 5$

Thus $V_1 = 5V_2 + 8V_3$

c.) Complete

d.) $V_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

Solve $a, b \in \mathbb{R}$ (if possible)

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + b\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} a+b \\ 2a-3b \\ -3a+2b \end{bmatrix}$$

$$\begin{aligned} R_1 \quad a + b &= 2 \\ R_2 \quad 2a - 3b &= -1 \\ R_3 \quad -3a + 2b &= 0 \end{aligned} \xrightarrow[R_3 + 3R_1]{R_2 - 2R_1} \begin{aligned} a + b &= 2 \\ -5b &= -5 \\ 5b &= 6 \end{aligned}$$

$$\xrightarrow[\frac{1}{5}R_2]{-\frac{1}{5}R_2} \begin{aligned} a + b &= 2 \\ b &= 1 \\ b &= 6/5 \end{aligned}$$

Thus no solution exists, and V_1 is not in $\text{Span}(V_1, V_2)$.

e) Complete

f.) Complete .

Question 4 (§ 1.4 , nr 5)

Not graded

Determine wheter the given vector is in the span of S:

a.) $V = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

That is can we solve for $a, b \in \mathbb{R}$ in

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = (a) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (b) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad ?$$

$$\Rightarrow \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \underset{3 \times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} a \\ b \end{bmatrix}}$$

By inspection $a = 1, b = -1$ solves the system.

b.) Complete

c.) Complete

d.) Complete

e.) $p = -x^3 + 2x^2 + 3x + 3$

$S = \{ x^3 + x^2 + x + 1,\ x^2 + x + 1,\ x + 1 \}$

Observe :

$$x^3 + x^2 + x + 1 - (x^2 + x + 1) = x^3 \in \text{span}(S)$$
$$x^2 + x + 1 - (x + 1) = x^2 \in \text{span}(S)$$

So $\text{span}(S) = \{ x^3, x^2, x+1 \}$

Solve $a, b, c \in \mathbb{R}$

$$-x^3 + 2x^2 + 3x + 3$$
$$= ax^3 + bx^2 + c(x+1)$$

Thus $a = -3,\ b = 2,\ c = 3$

shows $p \in \text{span}(S)$.

f.) $p := 2x^3 - x^2 + x + 3$

$S := \{ x^3 + x^2 + x + 1 , x^2 + x + 1 , x + 1 \}$

Observe that for $a, b, c \in \mathbb{R}$

$$\begin{aligned} &2x^3 - x^2 + x + 3 \\ &= a(x^3 + x^2 + x + 1) + b(x^2 + x + 1) + c(x + 1) \\ &= ax^3 + (a+b)x^2 + (a+b+c)x + (a+b+c)(1) \end{aligned}$$

Then $a + b = -1$, $a + b + c = 1$, $a + b + c = 3$.

But $-1 + c = 1 \Rightarrow c = 2$,

and $-1 + c = 3 \Rightarrow c = 4$, a contradiction.

Thus no such $a, b, c \in \mathbb{R}$ exists,

and $p \notin \operatorname{span}(S)$.

g.) $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right.$

To get 4 in the bottom right we need

$4 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then we need $-2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

to get 2 in the top right, and

we have. And we need $3 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$.

Then

$$3 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & 4-2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix},$$

as required to show A is in the span of S.

Question 5 (§ 1.4, nr 6)

Completion

Observe that $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = F^3$

since $\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix} = 2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix} = -2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Showing $\operatorname{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

$= \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$= F^3.$

Question 6 (§ 1.4 , nr 12)

Graded

Let V be a vector space, and $W \subseteq V$.
Then W is a subspace of V iff $\mathrm{span}(W) = W$.

<u>Proof</u>: Since $\mathrm{Span}(W)$ is a subspace of V, we know $\mathrm{Span}(W) = W$ implies W is a vector space of V.

For the other direction, suppose W is a subspace of V.
First let $w \in W$. Then $1 \cdot w = w$, showing w is a linear combination of elements in W.
Thus $w \in \mathrm{Span}(W)$ and $W \subseteq \mathrm{Span}(W)$.

To show $\mathrm{Span}(W) \subseteq W$, we apply Thm 1.5 page 31, to know that $\mathrm{span}(W)$ is contained in every subspace that contains W. Since W is a subspace, containing itself, it follows $\mathrm{span}(W) \subseteq W$.

Hence $\mathrm{Span}(W) = W$, as required $\boxtimes$

Rubric:

- 1 point : $\text{span}(W) = W \Rightarrow W$ is a subspace of V

If W subspace, then

- 3 point : $\text{span}(W) \subseteq W$
- 1 point : $W \subseteq \text{span}(W)$

} use of double set inclusion.

(*) At least two ways to show $\text{span}(W) \subseteq W$.

1. $\text{Span}(W)$ is the smallest subspace of V containing W.
2. Induction on # terms in a linear combination.

Section 1.5: Linear Dependence and Independence

Question 7 (§ 1.5, nr 1.)

Graded

a.) False, $S = \left\{ \begin{array}{c} v_1 \\ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \right\}$ is linearly dependent since $0 = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$, but

v_3 not a linear combination of v_1 and v_2.

b.) True, because $1 \cdot 0 = 0$ is a non-trivial lin. comb.

c.) False, there is no vectors to chose from in $\emptyset$.

d.) False, $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \subseteq S$, when $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is L.I.

e.) True, converse of 1.6.

f.) True.

Question 8 (§ 1.5, nr 3.)

Not graded

Show that the following subset of $M_{2\times 3}(F)$ is linearly dependent:

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Observe that

$$(1)\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Question 9 (§ 1.5, nr 4.) Not graded

Show that $\{e_1, e_2, \ldots, e_n\} \subseteq F^n$ is linear indp

where $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, only 1 in j^{th} comp.

Pf: Suppose

$$\sum_{i=1}^{n} \alpha_i e_j = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow \alpha_i = 0 \quad \forall\, i = 1, \ldots, n.$

Hence $\{e_i\}_{i=1}^{n}$ linearly independent in F^n.

Question 10 (§ 1.5, nr 5.)

Completion

Recall $P_n(F)$ denote the vector space of polynomial of degree at most n.

Show that $\{1, x, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

We say $\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} b_i x^i$ in $P_2(F)$ iff

$a_i = b_i$ for all $i = 0, 1, \ldots, n$.

So suppose $\alpha_0(1) + \alpha_1 x + \ldots + \alpha_n x^n = 0$.
By definition of equality to the zero polynomial that means $\alpha_i = 0$ for all $i = 0, 1, \ldots, n$.

Hence $\{1, x, \ldots, x^n\}$ is linearly independent.

Question 11. (§ 1.5, nr 7.)

Not Graded

Find a linearly independent set that generates the subspace of diagonal matrices in $M_{2\times 2}(F)$.

Consider $S := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\mathrm{Span}(S) = \mathrm{Diag}_{2\times 2}(F)$ since

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also S is linearly independent since

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

iff

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

iff

$\alpha_1 = 0$ and $\alpha_2 = 0$.

Question 12. (§ 1.5, nr 10.)

Graded

Find three linecry dependent vectors in $\mathbb{R}^3$ s.t non of the three are scalor multiples of each other:

Consider

$$(2)\begin{bmatrix}1\\0\\-1\end{bmatrix}+(3)\begin{bmatrix}0\\-2\\5\end{bmatrix}=\begin{bmatrix}2\\-6\\13\end{bmatrix}$$

$$\left\{\underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{v_1},\underbrace{\begin{bmatrix}0\\-2\\5\end{bmatrix}}_{v_2},\begin{bmatrix}2\\-6\\13\end{bmatrix}\right\}$$

2 pts for not being scalor multiple of each other

Non is a scalor multiple of the other since they don't have zero in same component.

3 pts for linear dependence.

$$(2)\begin{bmatrix}1\\0\\-1\end{bmatrix}+(3)\begin{bmatrix}0\\-2\\5\end{bmatrix}+(-1)\begin{bmatrix}2\\-6\\13\end{bmatrix}=0$$

Showing dependence

Question 13. (§ 1.5, nr 16.) Not Graded

Let V be a vector space, $S \subseteq V$. Show that S is linearly independent iff every finite subset of S is linearly independent.

Proof: Every subset of a linearly independent set is linearly independent follows from the corollary on page 39.

For the converse, suppose that every finite subset of S is linearly independent.

(we need to show that the only way to express the zero vector as a L.C. of elements of S is the trivial one)

Take $s_1, s_2, \ldots, s_n \in S$, $\alpha_1, \ldots, \alpha_n \in F$ and suppose $\sum \alpha_i s_i = 0$.

To show S is linearly independent we need to show $\alpha_i = 0$ for all $i = 1, \ldots, n$.

Well $\{s_1, \ldots, s_n\} \subseteq S$ a finite subset of S. By our assumption, $\{s_1, \ldots, s_n\}$ is linearly independent. Thus $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. But this just showed S is linearly indpt. $\square$

Question 14 (§ 1.5, nr 16.)

Not Graded

Given a $S \subseteq P(F)$ of non-zero polynomials s.t no two polynomials in S have the same degree. Prove that S is linearly independent.

<u>Proof</u>: To derive a contradiction, suppose S is linearly dependent. That means there exists distinct $p_1, p_2, \dots, p_k \in S$, and $\alpha_1, \dots, \alpha_k \in F$, at least one $\alpha_i \neq 0$ s.t

$$\sum_{i=1}^{k} \alpha_i p_i = 0 .$$

By relabeling we can assume $\alpha_1, \alpha_2, \dots, \alpha_{\tilde{k}} \neq 0$, and $\deg(p_1) < \deg(p_2) < \dots < \deg(p_{\tilde{k}})$, where strict inequality comes from our assumption on S. Let $M := \deg(p_{\tilde{k}})$.

Since $\deg(p_i) < M$ for all $i = 1, \dots, \tilde{k}-1$ it follows that $\alpha_{\tilde{k}} x^M$ is the only term in $\sum_{i=1}^{k} \alpha_i p_i$ raised to the power M.

Thus $\alpha_{\tilde{k}} x^M = 0 \implies \alpha_{\tilde{k}} = 0$, a contradict.

Hence S is linearly independent. $\boxed{S}$

Question 15 (§ 1.5, nr 20.) Graded

Show that $f(t) = e^{rt}$, $g(t) = e^{st}$ where $s \neq r$ are lineary independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Pf: to derive a contradiction, suppose e^{rt}, e^{st} are lineary dependent. That is $\exists\, a, b \in \mathbb{R}$ s.t at least one non zero say $a \neq 0$;

$$a e^{st} + b e^{rt} = 0 \quad , \quad \forall t \in \mathbb{R}.$$

The $a e^{s(0)} + b e^{s(0)} = 0 \implies a = -b$.

But then $a e^{st} = -b e^{rt} = a e^{rt}$

$$\implies e^{st} = e^{rt} \quad , \forall t \in \mathbb{R}$$

Since the exponential is injective this means

$st = rt$, $\forall t \in \mathbb{R}$. In particular $t = 1$ s.t $s = r$, contradiction.

Hence $\{e^{st}, e^{rt}\}$ lineary independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ whenever $s \neq t$. $\boxtimes$

Rubric

- 2pts setting up $ae^{st} + be^{rt} = 0$
- 2pts for justifying why $a = b = 0$, or proceed via contradiction.
- 1 pt for using injection of exp.