Homework 3

Due Feb 5, 2025

Solutions and Rubric

1.6 : 1, 2, 3, 6, 11, 26

2.1 : 1, 3, 9, 12, 13, 15, 16, 17

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 40 points
- So final score is $\left(\frac{\text{score}}{40}\right) \cdot 8$

§1.6, 1.	Graded	15	Graded
2.	Completion	16	Completion
3.	Not graded	17	Completion
6.	Not graded		
11.	Graded		
26.	Graded		
§2.1, 1.	Graded		
3.	Not graded		
9.	Not graded		
12.	Not graded		
13.	Not graded		

Question 1 (FIS, 1.6, nr. 1)

Graded

a.) False, by convention $\text{Span}(\phi) = \{0\}$ so generates, and since ϕ is LI it is a basis for $\{0\}$

b.) True, because if V is finitely generated, we can extend a singleton to a basis.

c.) False, every finitely generated vector space has a finite basis.
Consider $F[x] = \{$ all polys with coeff in $F\}$

d.) False, $\{ [\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}] \}$ and $\{ [\begin{smallmatrix}-1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\\frac{1}{2}\end{smallmatrix}] \}$ are two different bases of $\mathbb{R}^2$.

e.) True, by cor 1, page 47.

f.) False, $\{1, x, \dots, x^n\}$ is a basis for $P_n(F)$ so dimension $n+1$.

g.) False, $M_{m,n}(F)$ has a $m \cdot n$ dimension.

h.) True.

i) False, we need linear independence too

j) True, since a maximal linearly indep set is upper bounded by $\dim(V)$

k) True, $\{0\}$ and V, $n > 0$.

l) True, since if S was not LI,
then span(S) has dim $< n$,
contradicts spanning V

Question 2 (FIS, 1.6, nr. 2)

Completion

Determine if the following sets in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$:

- Two ways to fail bases:
  - Not Linearly independent
  - Not spanning.
- OR show we have a maximal L.I. set.

- Observe that this question is equivalent to asking if the L.T. given by the matrix is injective & surjective.
- I.e is the L.T. bijective?
- I.e. is the matrix of the L.T. invertable?

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right\}$

Observe that the question of LI is to determine if $\exists$ non-trivial $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ s.t

Note

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = 0.$$

That is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0.$$

2.e Does the Linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{bmatrix}$$ have trivial kernel? (equivalently injective?)

- The question about spanning is can given any $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$ can we find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ s.t

$$(x_1)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (x_2)\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + (x_3)\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} ?$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

That is, is $A : \mathbb{R}^3 \to \mathbb{R}^3$ surjective?

• Fix $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & -4 \\ -1 & 1 & 3 \end{bmatrix}$

a) One checks if A is invertible or not

- Yes A is invertable.

b.)

c.) Continue

d.)

e.)

Question 3 (FIS, 1.6, nr. 3)

- Similar idea as above, but with polynomials.

Determine if the following are bases for $P_2(\mathbb{R})$:

Failure occurs when

① Not linearly independent in $P_2(\mathbb{R})$

② Not spanning set of $P_2(\mathbb{R})$.

a.) $\{-1-x+2x^2,\ 2+x-2x^2,\ 1-2x+4x^2\}$

① Can we find non-trivial $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

s.t

$$\alpha_1(-1-x+2x^2)+\alpha_2(2+x-2x^2)+\alpha_3(1-2x+4x^2)=0$$

$$\Rightarrow (-\alpha_1+2\alpha_2+\alpha_3)\underline{(1)}+(-\alpha_1+\alpha_2-2\alpha_3)\underline{x}$$

Std basis

$$+(2\alpha_1-2\alpha_2+4\alpha_3)\underline{x^2} = 0$$

0 iff all coeff is zero

$$\Rightarrow \begin{bmatrix} -\alpha_1 + 2\alpha_2 + \alpha_3 \\ -\alpha_1 + \alpha_2 - 2\alpha_3 \\ 2\alpha_1 - 2\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & -2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So again the question of linearly independence is equivalent to asking if the $A : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$ has trivial Kernel?

(Question about injective)

4 Check A is invertable $\Rightarrow$ $\mathrm{Ker}(A) = \{0\}$

$\Rightarrow$ the set of vectors is L.I.

To determine spanning, means given

$p = \sum^{2} \beta_i x^i \in P_2(\mathbb{R})$, can we find $\alpha_1, \alpha_2, \alpha_3$

$i=0$

s.t

$$\alpha_1(-1-x+2x^2)+\alpha_2(2+x-2x^2)+\alpha_3(1-2x+4x^2)=p \, ?$$

$$(-\alpha_1+2\alpha_2+\alpha_3)(1)+(-\alpha_1+\alpha_2-2\alpha_3)x+(2\alpha_1-2\alpha_2+4\alpha_3)x^2=\beta_0(1)+\beta_1x+\beta_2x^2$$

$$\begin{bmatrix}-\alpha_1+2\alpha_2+\alpha_3\\-\alpha_1+\alpha_2-2\alpha_3\\2\alpha_1-2\alpha_2+4\alpha_3\end{bmatrix}=\begin{bmatrix}\beta_0\\\beta_1\\\beta_2\end{bmatrix}$$

$$\begin{bmatrix}-1&2&0\\-1&0&-2\\2&-2&4\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\begin{bmatrix}\beta_0\\\beta_1\\\beta_2\end{bmatrix}.$$

So asking if $\mathrm{Range}(A)=\mathbb{R}^3$?

i.e surjective.

Since A invertible $\Leftrightarrow$ A injective & surject

we know $\mathrm{Range}(A)=\mathbb{R}^3 \Rightarrow$ the set spans $P_2(\mathbb{R})$.

b.)

c.)

Continue

d.)

e.)

Question 4 (FIS, 1.6, nr. 6) Not graded

Let F denote a field. Give 3 different bases for F^2 and $M_2(F)$.

Continue

Question 5 (FIS, 1.6, nr. 11)

Graded

Let V be a vector space with distinct vectors $u \neq v \in V$. Show that if $\{u, v\}$ is a basis for V, $a, b \in F \setminus \{0\}$, then

① $\{u+v, au\}$

② $\{au, bv\}$

are basis for V.

Proof: Need to show linear indep, and spanning set. First suppose

$$\alpha_1(u+v) + \alpha_2(au) = 0, \text{ some } \alpha_1, \alpha_2 \in F.$$

Well then $(\alpha_1 + a\alpha_2)u + \alpha_1 v = 0$, a trivial linear combination in V of the basis $\{u, v\}$.

Since $\{u, v\}$ LI $\Rightarrow$ $\alpha_1 + a\alpha_2 = 0$ and $\alpha_1 = 0$.

Thus $a\alpha_2 = 0$, and $a \neq 0 \Rightarrow \alpha_2 = 0$.

Hence $\{u+v, au\}$ LI.

Next to show $\text{span}\{u+v, au\} = V$, let $x = \alpha_1 u + \alpha_2 v$ arbitrary element in V.

Then $x = \alpha_1 u + \alpha_2 v + \alpha_2 u - \alpha_2 u$

$$= (\alpha_1 - \alpha_2) u + \alpha_2 (v+u)$$

$$= \left(\frac{\alpha_1 - \alpha_2}{a}\right) \cdot (a u) + \alpha_2 (v+u), \quad a \neq 0.$$

Is a linear combination of elements from $\{u+v, au\}$

Thus $x \in \text{span}\{u+v, au\} \Rightarrow \text{Span}(\{u+v, au\}) = V$.

Hence $\{u+v, au\}$ forms a basis.

A similar argument works for $\{au, bv\}$.

$\boxed{M}$

Rubric

- $2\frac{1}{2}$ pts showing linear independence
- $2\frac{1}{2}$ pts showing it generates.

Question 6 (FIS, 1.6, nr. 26)

Graded

Fix $n \in \mathbb{N}$, $a \in \mathbb{R}$. Determine the dimension of $W := \{ f \in P_n(\mathbb{R}) \mid f(a) = 0 \}$ as a subspace of $P_n(\mathbb{R})$.

Pf: Since W subspace of a $n+1$ dimensional space, we know $\dim(W) \leq n+1$.

Next observation is that W contains no constant polynomials. All elements of W have roots at a, and so we get multiplicites $1, 2, \ldots, n$, (max degree).

Observe $\{ x-a, (x-a)^2, \ldots, (x-a)^n \}$ is LI, and in W.

So $n \leq \dim(W) \leq n+1$.

Since $W \neq P_n(\mathbb{R}) \Rightarrow \dim(W) \neq n+1$

$\Rightarrow \dim(W) = n$.

(observe we have a basis) $\blacksquare$

Rubric

- $2\frac{1}{2}$ pts: justify $\dim(W) \geq n$
- $2\frac{1}{2}$ pts: justify $\dim(W) < n+1$.

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Question 7 (FIS, 2.1, nr 1.)

Graded

a) True

b) False, we need $T(\alpha v) = \alpha T(v)$

c) False, it's true for linear transformations but not general functions.

d) True, linear maps send additive identity to additive identity. Indeed
$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v) \Rightarrow T(0_v) = 0_w$

e) False, if $T: V \to W$ linear between finite dimensional vector spaces, then
$\dim(\ker(T)) + \dim(R(T)) = \dim(V)$

f) False, if T is not injective, then T does not preserve linear independence.

g) True, linear maps are completely determined by there action on a basis.

h) False, suppose $x_1 = 0$ but $y_1 \neq 0$.
We know $T(x_1) = 0$ for all linear transf. $T: V \to W$.

Question 8 (F.I.S 2.1, nr 3.)   Not graded

Given $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \longmapsto \begin{bmatrix} a_1 + a_2 \\ 0 \\ 2a_1 - a_2 \end{bmatrix}$$

Pf: let $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$.

Then $T\left(\alpha \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} \alpha a_1 + b_1 \\ \alpha a_2 + b_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (\alpha a_1 + b_1) + (\alpha a_2 + b_2) \\ 0 \\ 2(\alpha a_1 + b_1) - 2(\alpha a_2 + b_2) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha (a_1 + a_2) \\ 0 \\ \alpha \cdot 2(a_1 - a_2) \end{bmatrix} + \begin{bmatrix} b_1 + b_2 \\ 0 \\ 2(b_1 - b_2) \end{bmatrix}$$

$$= \alpha T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + T\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Basis for $N(T)$:

$$\ker(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 - x_2 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \right\}$$

Row reduce $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \right\}$$

$= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, so ϕ basis for $\ker(T)$.

Hence $\dim(N(T)) = \{0\}$

Basis for $R(T)$:

$$R(T) = \left\{ \begin{bmatrix} a_1 + a_2 \\ 0 \\ 2a_1 - a_2 \end{bmatrix} \middle| a_1, a_2 \in \mathbb{R} \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \middle| a_1, a_2 \in \mathbb{R} \right\}$$

We have a spanning set, and since $\begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ not scalar multiples, we know they are linearly independent. Hence $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ basis for $R(T)$, and $\text{Rank}(T) = 2$.

So T is one-to-one since $\ker(T) = \{0\}$ but not onto since $\dim(R(T)) \lneq \dim(\mathbb{R}^3)$.

Question 9 (F.I.S 2.1, nr 9.)

Not graded

State why the maps below are not linear.

(a) $T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ a_2 \end{bmatrix}$ not linear since $T(0) \neq 0$.

(b) $T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix}$ not linear since

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} + T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

but $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$.

c) $T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sin(a_1) \\ 0 \end{bmatrix}$ not linear since

$$T\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} + T\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

but $T\left(\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sin(\pi) \\ 0 \end{bmatrix} = 0$

d) $T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} |a_1| \\ a_2 \end{bmatrix}$, since $|-1| + |1| \neq |-1+1|$

e) $T\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1+1 \\ a_2 \end{bmatrix}$, since $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

§2.1, nr 6: (The trace is a linear map)

Recall that $tr : M_{n\times n}(F) \to F$ given by

$$tr(A) = \sum_{i=1}^{n} A_{i,i}.$$

Pf: $tr(\alpha A + B) = tr([\alpha a_{ij} + b_{ij}])$

$$= \sum_{i=1}^{n}(\alpha a_{ii} + b_{ii})$$

$$= \alpha \sum_i a_{ii} + \sum_i b_{ii}$$

$$= \alpha\, tr(A) + tr(B).$$

$\blacksquare$

Question 10 (F.I.S 2.1, nr 12.)

Not graded

Is there a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ s.t

$$T\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad T\begin{bmatrix}-2\\0\\-6\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}.$$

Sol: No, since suppose T is linear and $T\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$. Then $T\begin{bmatrix}-2\\0\\-6\end{bmatrix} = -2\,T\begin{bmatrix}1\\0\\3\end{bmatrix} = -2\begin{bmatrix}1\\1\end{bmatrix}$

$$\neq \begin{bmatrix}2\\1\end{bmatrix}.$$

Question 11 (F.I.S 2.1, nr 13.)

Not graded

Let V, W vector spaces, $T: V \to W$ linear map and $\{w_1, \dots, w_k\}$ linearly independent in $R(T)$. Show that if $v_i \in V$ is chosen s.t. $T(v_i) = w_i$ for all i, then $\{v_1, \dots, v_k\}$ is linearly independent in V.

Pf: Suppose $\sum_{i=1}^{k} \alpha_i v_i = 0$ for $\alpha_i \in F$.

Then $T(\sum \alpha_i v_i) = 0$ i.e. $\sum \alpha_i T(v_i) = 0$

$\Rightarrow$ $\sum \alpha_i w_i = 0$. Since $\{w_i\}_{i=1}^{k}$ linearly independent in W we know $\alpha_1 = \dots = \alpha_k = 0$.

Thus $\{v_1, \dots, v_k\}$ linearly independent in V.

$\boxtimes$

This says that a linearly independent set for the range of T can induce a linearly independent set with same # of elements in V.

§2.1, nr. 14

Recall Thm 2.2 that says linear maps maps bases for the domain to spanning sets for the range. We might lose linear independence.

Below we see injectivity is what we need.

Let V and W be vector spaces and $T: V \to W$ linear. Show

a) T is one to one iff T carries L.I. sets in V to L.I. sets in W.

b) Suppose T is one-to-one, $S \subseteq V$. Then S is LI iff $T(S)$ is LI.

c) Show that if T is one-to-one and onto, then $\forall$ basis $\beta = \{v_1, \ldots, v_n\}$ for V we have $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

Pf a) : Suppose T is one-to-one, and $\{v_i\}_{i=1}^{k}$ L.I. in V. Then let $\sum \alpha_i T(v_i) = 0$.

Observe that $T(\sum \alpha_i v_i) = 0$. Since T one-to-one we know $\ker(T) = 0$. Hence $\sum \alpha_i v_i \in \ker(T)$ $\Rightarrow$ $\sum \alpha_i v_i = 0$. Since $\{v_i\}$ L.I we get $\alpha_i = 0$ $\forall i$. Thus $\{T(v_i)\}_{i=1}^{k}$ is linearly independent.

Conversely, suppose $\{T(v_i)\}_{i=1}^{k}$ is L.I. in W for all $\{v_i\}_{i=1}^{k}$ L.I. in V.

To show $\ker(T) = 0$, suppose $v \in \ker(T)$. Then $v = \sum \alpha_i v_i$, where $\{v_1, \ldots, v_k\}$ basis for V.

So $\sum \alpha_i T(v_i) = 0$. But since $\{T(v_i)\}_{i=1}^{k}$ L.I in W $\Rightarrow$ $\alpha_i = 0$ for all i. Thus $v = \sum \alpha_i v_i = 0$. Hence $\ker(T) = 0$ $\Leftrightarrow$ T is one to one.

Pf b : Complete

Pf c : Complete.

Question 12 (F.I.S, 2.1, nr 15.)

Graded

Show that $T: P(\mathbb{R}) \to P(\mathbb{R})$ given by

$T(p(x)) := \int_0^x p(t)\,dt$, is

linear, one to one, and not onto.

Pf: Well for $k \geq m$

Well $T\left(\alpha \sum_{n=0}^{k} a_n x^n + \sum_{n=0}^{m} b_n x^n\right)$, when we add zeros s.t sum match

$$= T\left(\sum_{n=0}^{m} (\alpha a_n + b_n) x^n\right)$$

, $a_i = 0$, $i > k$

$$= \int_0^x \sum_{n=0}^{m} (\alpha a_n + b_n) t^n \, dt$$

Finite sum

& property of integrals

$$= \alpha \int_0^x \sum a_n t^n \, dt + \int_0^x \sum b_n t^n \, dt$$

$$= \alpha\, T\left(\sum a_n x^n\right) + T\left(\sum b_n x^n\right) dt$$

To show $\ker(T)=0$, suppose

$$\int_0^x \sum a_n t^n \, dt = 0$$

$$\Rightarrow \sum \left(\frac{a_n}{n+1}\right) x^{n+1} = 0 \quad \Rightarrow \frac{a_n}{n+1} = 0 \ \forall n$$

$$\Rightarrow a_n = 0 \ \forall n$$

$$\Rightarrow \sum a_n x^n = 0.$$

To show not surjective, observe that if we cannot obtain non-zero constant polynomials. Indeed if $p(x) \in P(\mathbb{R})$, and non zero, then $T(p(x))$ has degree at least one.

Rubric:

- 1 point linearity
- 2 points injectivity
- 2 points not surjectivity.

Question 13 (F.I.S, 2.1, nr 16.)

Completion

Let $T: P(\mathbb{R}) \to P(\mathbb{R})$ given by $T(f(x)) = f'(x)$. Prove that T is onto but not one-to-one. Why does this not contradict rank-nullity?

Pf: Let $p(x) = \sum_{n=0}^{k} a_n x^n \in P(\mathbb{R})$. Then

let $\hat{p}(x) := \sum_{n=0}^{k} \left(\frac{a_n}{n+1}\right) x^{n+1} \in P(\mathbb{R})$ and

$$T(\hat{p}(x)) = \sum_{n=0}^{k} \frac{a_n}{n+1} T(x^{n+1}) = \sum_{n=0}^{k} a_n x^n = p$$

as required to conclude surjectivity.

T is not one-to-one since all non-zero constant polynomials are in the kernel of T.

$\boxtimes$

Observe that integration was injective but not surj. while differentiation is surj. but not injective.

(Question 14, §2.1, nr. 17) Completion

Let V, W be finite dimensional vspaces, and $T: V \to W$ a L.T. Show that

(a) If $\dim(V) < \dim(W)$, then T cannot be onto (surj)

(b) If $\dim(V) > \dim(W)$, then T cannot be one-to-one (inj).

Pf (a): Well let $\{b_1, \dots, b_n\}$ be a basis for V. By Thm 2.2, pg 69, we know

$\mathrm{Span}(\{T(b_1), \dots, T(b_n)\}) = R(T)$

Thus in linearly independent set can have at most n-elements. Thus $\dim(R(T)) \leq n$.
But then $\dim(R(T)) \leq \dim(V) < \dim(W)$.

Hence $R(T) \subsetneq W \Rightarrow T$ not onto.

Pf (b): Again take a basis $\{b_1, .., b_n\}$ for V.

Then $span(\{T(b_1), .., T(b_n)\}) = R(T) \subseteq W$.

Since $R(T)$ is a subspace of W, we know $dim(R(T)) \leq dim(W)$.

So $dim(R(T)) \leq dim(W) < dim(V) = n$.
Hence $\{T(b_1), .., T(b_n)\}$ cannot be L.I.
$\Rightarrow \exists\ \alpha_1, .., \alpha_n \in F$ not all zero s.t

$$\sum_{i=1}^{n} \alpha_i T(b_i) = 0$$

linear

$$\Rightarrow T\left(\sum_{i=1}^{n} \alpha_i b_i\right) = 0.$$

But know since $\{b_1, .., b_n\}$ is a basis for V, we know every element has a unique expression as a L.C. from $\{b_1, .., b_n\}$

The zero vector has expression when all coeff are zero. Since some α_i's non-zero, we know $\sum_{i=1}^{n} \alpha_i b_i \neq 0$.

Hence $\ker(T) \neq \{0\}$

$\Rightarrow$ T is not injective

$\square$