Homework 4 — Due Feb 12, 2025

Solutions and Rubric

2.2: 1, 2(a), 5(c), 10, 17

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 25
- So final score is $\left(\frac{\text{score}}{50}\right) \cdot 8$

Q1.) 1	Graded		
Q2.) 2(a)	Graded		
Q3.) 5(c)	Graded		
Q4.) 10	Graded		
Q5.) 17	Graded		

Question 1 (FIS, §2.2, nr 1)

Given a vector space V, W, both finite dimensional with ordered bases β and γ.
And $T, U : V \to W$ linear transformations.

a) True, $aT + U : V \to W$ forms a linear transformation

b) True, since $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies T and U agree on a basis for V.

c) False,

$$\underbrace{\begin{bmatrix} | & & | \\ T(v_1) & \cdots & T(v_m) \\ | & & | \end{bmatrix}}_{n \times m}$$

d) True, $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

e) True, $\mathcal{L}(V, W)$ is a vector space

f) False, $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3) \neq \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

Question 2 (§2.2, nr 2)

Calculate matrices for linear transformation wrt standard bases β and γ for $\mathbb{R}^n$, $\mathbb{R}^m$.

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mapsto \begin{bmatrix} 2a_1 - a_2 \\ 3a_1 + 4a_2 \\ a_1 \end{bmatrix}$

well

$$\begin{bmatrix} 2a_1 - a_2 \\ 3a_1 + 4a_2 \\ a_1 + 0a_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

Question 3 (FIS, §2.2, 5)

c) Given $tr : M_{2\times 2}(F) \to F$. Compute $[tr]$.

Recall that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(F)$ and $\{1\}$ forms a basis for F.

Then

$$\left[tr\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; tr\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; tr\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \; tr\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

d) Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]$ with respect to $\{1, x, x^2\}$ and $\{1\}$. Note $[T]$ must be 1×3.

Well $\begin{bmatrix} T(1) & T(x) & T(x^2) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$

Question 4 (FIS, §2.2, nr 10)

Given a vector space V with basis $\beta = \{v_1, \dots, v_n\}$, Define $v_0 := 0$

Recall that there exists a unique linear $T: V \to V$ s.t. $T(v_j) = v_j + v_{j-1}$. Find $[T]_\beta$.

Well
$$\begin{bmatrix} | & | & & | \\ [T(v_1)] & [T(v_2)] & \cdots & [T(v_n)] \\ | & | & & | \end{bmatrix}$$

Well
$$T(v_1) = v_1 + v_0 = v_1 = v_1 + 0v_2 + \dots + 0v_n$$
$$T(v_2) = v_2 + v_1 = 1v_1 + 2v_1 + 0v_3 + \dots + 0v_n$$
$$\vdots$$
$$T(v_n) = v_n + v_{n-1} = 0v_1 + \dots + v_{n-1} + v_n$$

So
$$[T] = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & & \\ \vdots & \vdots & 0 & & \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Question 5 (FIS: §2.2, nr 17)

Given finite dimensional vector spaces V, W s.t. $\dim(V) = \dim(W)$, and $T: V \to W$ a linear transformation.

Find bases β and γ for V, W respectfully s.t. $[T]_\beta^\gamma$ is diagonal.

Pf: Take $\beta = \{v_1, \ldots, v_n\}$ a basis for V.

Then $\{T(v_1), \ldots, T(v_n)\}$ is a generating set for $R(T)$. By the replacement applied to $G = \{T(v_1), \ldots, T(v_n)\}$, and $L = \phi$ (the empty set is l.i.) we get a subset of G that forms a basis for $R(T)$.

By relabelling, let $\{T(v_1), \ldots, T(v_k)\}$, $k \leq n$ be the basis obtained from above.

Now extend $\gamma := \{T(v_1), \ldots, T(v_k), w_{k+1}, \ldots, w_n\}$ to a basis for W.

Now we need to change the basis for V.

Indeed $T(v_j)$ might not be zero for $k < j \leq n$ But we have the following:

Fix $k < j \leq n$, and consider $T(v_j)$. Since $T(v_j) \in \operatorname{span}\{T(v_1), \dots, T(v_k)\}$ we know $\exists$ scalars $\alpha_1^{(j)}, \dots, \alpha_k^{(j)}$ s.t

$$T(v_j) = \sum_{i=1}^{k} \alpha_i^{(j)} T(v_i)$$

$$\Rightarrow T\left(v_j - \sum_{i=1}^{k} \alpha_i^{(j)} T(v_i)\right) = 0.$$

Call $\tilde{v}_j := v_j - \sum_{i=1}^{k} \alpha_i^{(j)} v_i$

Now consider a new basis for V namely

$$\tilde{\beta} := \{v_1, v_2, \dots, v_k, \tilde{v}_{k+1}, \dots, \tilde{v}_n\}.$$

To show our new set $\tilde{\beta}$ is linearly independent, observe

$$a_1 v_1 + \dots + a_k v_k + a_{k+1} \tilde{v}_{k+1} + \dots + a_n \tilde{v}_n = 0$$

we can rewrite this as a linear combination of the original $\{v_1, \dots, v_n\}$ and apply linear independence to get $a_i = 0 \ \forall i$.

And hence we have a set of n linearly independent vectors in an n-dim space, thus its spans the space.

So $\tilde{\beta}$ is a basis for V where

$$T(v_1) = 1 \cdot T(v_1) + 0\, T(v_2) + \dots + 0 w_n$$

$$\vdots$$

$$T(v_k) = 0 \cdot T(v_1) + \dots + 1 \cdot T(v_k) + \dots + 0 w_n$$

$$\tilde{T}(v_k) = 0 = 0\, T(v_1) + \dots + 0 \cdot w_n$$

s.t

$$[T]_{\tilde{\beta}}^{\beta} = \begin{bmatrix} 1 & 0 & & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & 0 & & 0 \\ 0 & 0 & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & 1 & 0 & & \vdots \\ \vdots & & & 0 & & & \\ & & 1 & \vdots & & 0 & \\ 0 & & & 0 & & & \end{bmatrix}$$

Rubric:

2 points for basis on V

2 point for basis on W

1 point writing matrix of the linear transf.

Method 2 :

Given a linear transformation $T: V \to W$ between finite dimensional vector spaces $\dim(V) = \dim(W)$.

Let $k = \dim(\ker(T))$, and $\{v_1, .., v_k\}$ basis for $\ker(T)$. By the replacement theorem we can extend to a basis $\beta := \{v_1, .., v_n\}$ for V.

Claim: $\{T(v_{k+1}), .., T(v_k)\}$ is a basis for $R(T)$.

First observe that since $\{v_1, .., v_n\}$ span V, we know $\text{span}\{T(v_1), ..., T(v_n)\} = R(T)$. Since $T(v_1) = ... = T(v_k) = 0$ we get $\text{Span}\{T(v_{k+1}), ..., T(v_n)\} = R(T)$.

For linear independence suppose

$$\sum_{i=k+1}^{n} \alpha_i T(v_i) = 0.$$

$\Rightarrow T\left(\sum_{i=k+1}^{n} \alpha_i T(v_i)\right) = 0$

$\Rightarrow \sum_{i=k+1}^{n} \alpha_i v_i \in Ker(T)$. Since $\{v_1, \dots, v_k\}$

basis for $Ker(T)$ we get

$$\sum_{i=k+1}^{n} \alpha_i v_i = \sum_{i=1}^{k} \beta_i v_i .$$

$$\Rightarrow -\sum_{i=1}^{k} \beta_i v_i + \sum_{i=k+1}^{n} \alpha_i v_i = 0.$$

Apply linear independence of $\{v_1, \dots, v_n\}$
to conclude $\alpha_i = 0 \ \forall i = k+1, \dots, n$,
as needed to conclude $\{T(v_{k+1}), \dots, T(v_n)\}$
basis for the $R(T)$.

Extend $\gamma := \{T(v_{k+1}), \dots, T(v_n), w_1, \dots, w_k\}$ to
basis for W.

Then
$$\begin{aligned}
T(v_1) &= 0 = 0\,T(v_1) + \dots + 0 w_1 + \dots + 0 w_n \\
&\vdots \\
T(v_k) &= 0 \\
T(v_{k+1}) &= T(v_{k+1})
\end{aligned}$$

$\vdots$

$$T(v_n) = T(v_n)$$

Then
$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & & 0 & & 0 \\ 0 & 0 & \ddots & 0 & & 0 \\ \vdots & \vdots & & \vdots & & \\ & & & T(v_{k+1}) & & \\ & & & 0 & \ddots & 0 \\ 0 & & \cdots & 0 & & T(v_n) \end{bmatrix}$$