Homework 5

Due Feb 19, 2025

Solutions and Rubric

2.3 : 1 2b, 3a, 4a, 9, 12

2.4 : 1, 2(e,f), 6, 9, 16, 20

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 25
- So final score is $\left(\frac{\text{score}}{50}\right) \cdot 8$

§2.3	1	Graded
	2b	Not graded
	3a	Completion
	4a	Not graded
	9	Completion
	12	Completion
§2.4	1	Graded
	2 e,f	Graded
	6	Not graded
	9	Graded
	16	Graded
	20	Not graded

Question 1 (§ 2.3, nr 1) Graded

Given vector space V, W, Z with

ordered (finite) bases α, β, γ,

and liner maps

$T: V \to W$ and $U: W \to Z$

a.) False :

Indeed, $UT: V \to Z$ and

$[UT]_\alpha^\gamma$ matrix with columns $[UT(v_i)]_\gamma$

when $v_i \in \alpha$.

But $[T]_\alpha^\beta$ is a $\dim(W) \times \dim(V)$

sized matrix, and $[U]_\beta^\gamma$ is a

$\dim(Z) \times \dim(W)$. So

$[T]_\alpha^\beta [U]_\beta^\gamma$ need not be defined.

b.) True:

Indeed, $T: V \to W$ and, $\beta = \{w_1, \dots, w_n\}$

let $\alpha = \{v_1, \dots, v_m\}$, and $v = \sum_{j=1}^{n} b_j v_j$

- $T(v_j) = \sum_{i=1}^{n} a_i^{(j)} w_i$

So $T(v) = \sum_{j=1}^{m} b_j T(v_j) = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} a_i^{(j)} w_i$

So $= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_i^{(j)} b_j \right) w_i$

$$[T(v)]_\beta = \begin{bmatrix} \sum_{j=1}^{m} a_1^{(j)} b_j \\ \vdots \\ \sum_{j=1}^{m} a_n^{(j)} b_j \end{bmatrix}$$

- Also

$$\Rightarrow [T(v_j)]_\alpha^\beta = \begin{bmatrix} a_1^{(j)} \\ \vdots \\ a_n^{(j)} \end{bmatrix}$$

$[T(v)]_\alpha^\beta$

$$\overset{n \times m}{\begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(m)} \\ \vdots & \vdots & & \vdots \\ a_n^{(1)} & a_n^{(2)} & & a_n^{(m)} \end{bmatrix}} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sum_{j=1}^{m} b_j a_1^{(j)} \\ \vdots \\ \sum_{j=1}^{m} b_j a_n^{(j)} \end{bmatrix}$$

c.) False

Indeed $u: W \to Z$, where α is a basis for V. So $[u]_\gamma^\beta$ not defined

d.) True

$I_V(v_j) = v_j$ so $[I_V(v_j)]_\alpha = e_j$.

And $[I_V]_\alpha = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

e) False

Indeed $T: V \to W$ so "$T(T(v))$" not defined.

f.) True

Indeed $A^2 = I$ implies

$$(A+I)(A-I) = A^2 - A + A - I = A^2 - I = 0.$$

Thus $A + I = 0$ or $A - I = 0$.

I.e $A = -I$ or $A = I$.

g.) False:

Since $T: V \to W$ abstract linear transf.

while $L_A: \mathbb{C}^n \to \mathbb{C}^m$, $\dim(V) = n$, $\dim(W) = m$.

h.) False:

Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i) True:

$$\left(L_{A+B}\right)\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{ij}+b_{ij} \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= [a_{ij}]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + [b_{ij}]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= L_A\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + L_B\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

j) True.

Question 2 (§ 2.3 2.6)

Not graded

Given $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{bmatrix}$

$C = [\,4 \;\; 0 \;\; 3\,]$.

$$A^t = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$$

$$\underset{2\times 3}{A^t}\,\underset{3\times 3}{B} = \begin{bmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{bmatrix}$$

$$\underset{3\times 3}{B}\,\underset{3\times 1}{C^t} = \begin{bmatrix} 12 \\ 16 \\ 29 \end{bmatrix}$$

$$\underset{1\times 3}{C}\,\underset{3\times 3}{B} = \begin{bmatrix} 27 & 7 & 9 \end{bmatrix}$$

$$\underset{1\times 3}{C}\,\underset{3\times 2}{A} = \begin{bmatrix} 20 & 26 \end{bmatrix}$$

Question 3 $\quad$ §2.3, 3(a.) $\quad$ Completion

Given $g(x) = 3+x$, $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, and $U: P_2(\mathbb{R}) \to \mathbb{R}^3$ linear maps,

- $T(f(x)) := f'(x)g(x) + 2f(x)$

- $U(a+bx+cx^2) := \begin{bmatrix} a+b \\ c \\ a-b \end{bmatrix}$,

and $\beta = \{1, x, x^2\}$, and $\gamma = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

a) Compute $[U]_\beta^\gamma$, $[T]_\beta^\beta$, $[UT]_\beta^\gamma$ directly, and compare theorem 2.11.

- $U(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow [U(1)]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

  $U(x) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow [U(x)]_\gamma = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

  $U(x^2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow [U(x^2)]_\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

So $[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

- For $\quad T(f(x)) := f'(x)\overset{x+3}{\overset{\shortparallel}{g(x)}} + 2f(x)$

$$T(1) = 0 \cdot g(x) + 2 \cdot 1 \Rightarrow [T(1)]_\beta = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} T(x) &= 1 \cdot g(x) + 2x \\ &= x+3 + 2x \\ &= 3 + 3x \end{aligned} \Rightarrow [T(x)]_\beta = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} T(x^2) &= 2x \cdot (x+3) + 2x^2 \\ &= 6x + 4x^2 \end{aligned} \Rightarrow [T(x^2)] = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$$

$$\Rightarrow [T]_\beta^\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

- check $\quad [u]_\beta^\gamma [T]_\beta^\beta$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$$

• Now $[UT]_\beta^\gamma$

$$UT(1) = U(2) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$UT(x) = U(3+3x) = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$UT(x^2) = U(6x+4x^2) = \begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$$

So $$[UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$$

So $[UT] = [U][T]$!

Question 4 (§ 2.3, 4 a.)) Not graded

Given $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $A \mapsto A^t$.

and $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Compute $[T(A)]_\alpha$ where $A = \begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix}$

Well $T(A) = \begin{bmatrix} 1 & -1 \\ 4 & 6 \end{bmatrix}$ so

$$[T(A)]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 6 \end{bmatrix}.$$

Question 5 (§ 2.3, nr 9.)

Completion

Define $T, U : F^2 \to F^2$ by

$$U\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x-y \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$$

One checks that U and T are linear maps.

Observe that $UT = 0$ but $TU \neq 0$.

Indeed

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{T}{\longmapsto} \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix} \overset{U}{\longmapsto} \begin{bmatrix} \frac{x+y}{2} - \left(\frac{x+y}{2}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \overset{U}{\longmapsto} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{T}{\longmapsto} \begin{bmatrix} \frac{1+0}{2} \\ \frac{1+0}{2} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observe

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad U\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad U\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad [U] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Then $[T][U] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

But $[U][T] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Question 6 (§ 2.3, nr 12)

Completion

Given V, W, Z vector spaces, T, U LT s.t.

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

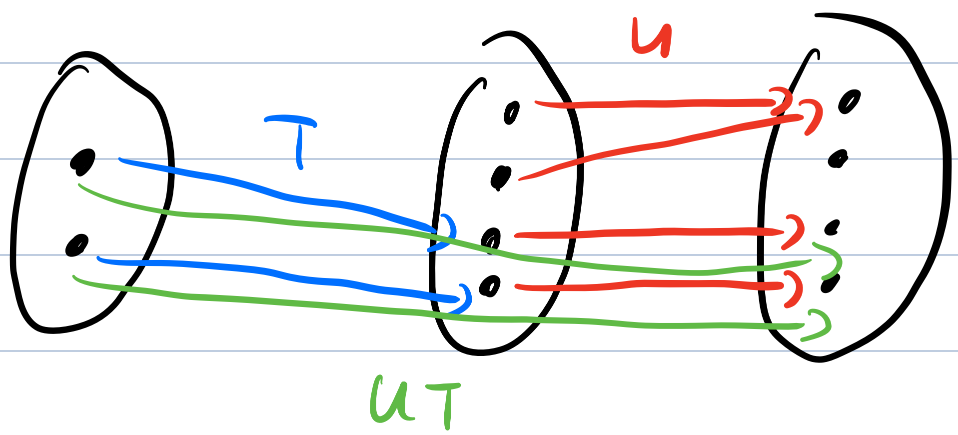
(a) UT one-to-one $\Rightarrow$ T one to one.

(b) UT onto $\Rightarrow$ U onto.

(c) U, T onto and one to one $\Rightarrow$ UT also.

a.) Suppose $T(y) = T(x)$. Then $UT(y) = UT(x)$. By UT one-to-one we get $y = x$. Hence T one to one.

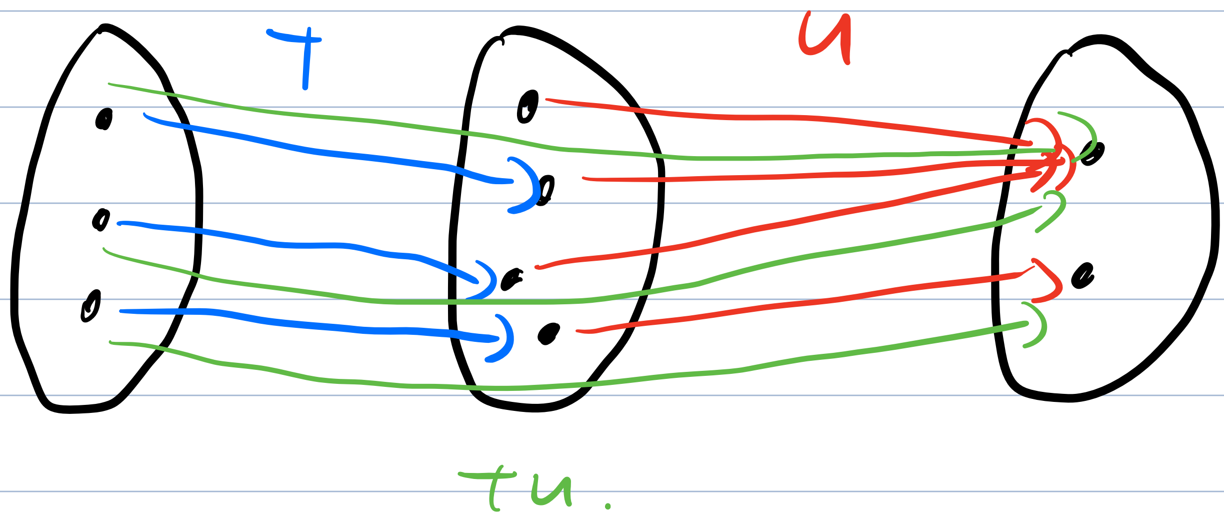
U need not be one-to-one on W, just on $R(T) \subseteq W$.



U not one to one.

b) let $z \in Z$. Since UT onto $\exists\, v \in V$ s.t $z = UT(v) = U(T(v))$ so U onto with $U(T(v)) = z$.

No $R(T) \subsetneq W$, and $R(U) = R(TU)$ is possible.



c.) Assume U and T are one-to-one and onto:

Recall

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

let $z \in Z$. Since U is onto $\exists\, w \in W$ s.t $U(w) = z$. Since T is onto $\exists\, v \in V$ s.t $T(v) = U(w)$.

Then $UT(v) = U(w) = z$.

Suppose $v \in \ker(UT)$. This means

$UT(v) = 0 \Rightarrow T(v) \in \ker(U)$.

Since U one-to-one we know $\ker(U) = \{0\}$,

and hence $T(v) = 0$, hence $v \in \ker(T)$.

Similarly, since T one-to-one, we get

$\ker(T) = \{0\} \Rightarrow v = 0$. As required

to show UT one-to-one.

Question 7 (§ 2.4 , nr 1)

Graded

Given $T: V \to W$ linear transformation between finite dim. vspaces V and W, each with an ordered basis α and β, respectfully. A, B matrix.

a) False, T need not be invertable.

b) True, holds for general functions.

c.) False, as functions T and L_A are not equal since act between possibly diff spaces. (see page 93.)

d.) False, $M_{2\times 3}(F) \cong F^6$ since isom. iff dim eq.

e) True, $P_n(\mathbb{C}) \cong P_m(\mathbb{C})$ iff $n=m$ since dimension are $n+1$ and $m+1$.

f) False, take $\underset{A}{(1\ 0\ 0)} \underset{B}{\begin{pmatrix}1\\0\\0\end{pmatrix}} = 1$

where A a 1×3 matrix and B a 3×1 matrix

g.) True, suppose A invertible. Then
$$(A^{-1})A = A(A^{-1}) = I$$
Thus $(A^{-1})^{-1} = A$.

h) True, supp. A invertible.
Then $L_A \circ L_{A^{-1}} = L_{AA^{-1}} = L_I$
and $L_{A^{-1}} \circ L_A = L_{A^{-1}A} = L_I$
Similarly, if L_A invertible, then $[L_A]$ inverse for A.

i) True, since A invertible induces a invertible LT $L_A : \mathbb{C}^n \to \mathbb{C}^m$.
Hence $\dim(\mathbb{C}^n) = \dim(\mathbb{C}^m)$
$\Rightarrow$ $m = n$.

Question 8 (§ 2.4 , nr 2 (e), (f))

Graded

Determine invertability of the linear maps:

e.) Given $T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$

by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} := a + 2bx + (c+d)x^2.$

Not invertable, since $\dim(M_{2\times2}(\mathbb{R})) = 4$ and $\dim(P_2(\mathbb{R})) = 2+1 = 3.$

Since T is linear, its sufficient to test invertability by looking at dimension.

f.) $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a+b & a \\ c & c+d \end{bmatrix}$$

What is $\ker(T)$?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Ker(T)$

$$\left.\begin{aligned} a+b &= 0 \\ a &= 0 \\ c &= 0 \\ c+d &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 0 \\ c &= 0 \end{aligned} \Rightarrow \begin{aligned} b &= 0 \\ c &= 0. \end{aligned}$$

Thus $Ker(T) = \{0\} \Rightarrow$ injective.

Since T maps between same space,

we get $R(T) = M_{2\times 2}(\mathbb{R}) \Rightarrow$ surjective.

Hence T is invertible.

Question 9 (§ 2.4 nr 6)

Not graded

Show that if A is an invertable $n \times n$ matrix, and $AB = 0$, then $B = 0$.

Indeed $A^{-1}(AB) = A^{-1}0 = 0$

$\Rightarrow B = 0$.

Question 10 (§ 2.4, nr. 9)

Graded

Given $n \times n$ matrices A, B. Show

a) If $AB = I$, then A and B are invertible.

well consider $L_A, L_B : F^n \to F^n$.

We know $L_A \circ L_B = L_{AB} = L_I = I_{F^n}$.
Showing that $L_A L_B$ is one-to-one.

By exercise 12 a), we get L_B is one-to-one. Then $L_B : \mathbb{C}^n \to \mathbb{C}^n$ invertible, by rank-nullity. Hence $[L_B] = B$ invertible as matrix.

Similar idea when using $L_A L_B$ onto to get L_A onto $\Rightarrow$ L_A invertible by rank nullity. Hence $[L_A] = A$ invertible.

Hence A^{-1}, and B^{-1} exists and
$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I$
to show $(AB)^{-1} = B^{-1}A^{-1}$.

b) Take $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$

$A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$AB = I_{1\times 1}.$

a.) 4 points

b.) 1 point

Question 11 (§2-4, nr 16) Graded

Let $B \in M_n(\mathbb{C})$ and invertible.

Define $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\Phi(A) := B^{-1}AB.$$

Show that Φ is an isomorphism of vector spaces.

Proof:

To show linearity observe that

$$B^{-1}(\alpha A + C)B = (\alpha B^{-1}A + B^{-1}C)B$$
$$= \alpha B^{-1}AB + B^{-1}CB$$

And for invertability, observe that $\Phi^{-1}(A) = BAB^{-1}$.
Indeed since $B^{-1}(BAB^{-1})B = A$.

$\boxtimes$

Question 12 (§ 2.4, nr 20) Not graded

let $T: V \to W$ linear transf. between vector spaces of size n and m, and ordered basis β and γ, respectively.

Show that for $A = [T]_\beta^\gamma$ that

1.) $\mathrm{Rank}(T) = \mathrm{Rank}(L_A)$

2.) $\dim(\mathrm{Ker}(T)) = \dim(\mathrm{Ker}(L_A))$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi_\beta := [\cdot]_\beta"'] & W \arrow[d, "\Phi_\gamma \; [\cdot]_\gamma"] \\ F^n \arrow[r, "L_A"] & F^m \end{tikzcd}$$

where $\Phi_\beta : V \to F^n$ $\sum \alpha_i v_i \mapsto \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ isomorphism.

We know $L_A \circ \Phi_\beta = \Phi_\gamma \circ T$ (✳)

$\Rightarrow L_A = \Phi_\gamma \circ T \circ \Phi_\beta^{-1}$

① Show $\dim(R(L_A)) = \dim(R(T))$.

Well observe that $F^n = \Phi_B^{-1}(V)$.

So $L(F^n) = L(\Phi_B^{-1}(V))$.

So $\operatorname{rank}(L) = \operatorname{rank}(L \circ \Phi_B^{-1})$ ⊛

And since Φ_γ isomorphis, we get

$\dim(L \circ \Phi_B^{-1}(V)) = \dim(\Phi_\gamma \circ L \circ \Phi_B^{-1}(V))$

$= \dim(T(V))$

$\Rightarrow \operatorname{rank}(L \circ \Phi_B^{-1}) = \operatorname{rank}(T)$,

as requed.

② Next observe that

1. $\ker(L_A) = \Phi_B(\ker(T))$ and

2. Also $\dim(\Phi_B(\ker(T))) = \dim(\ker(T))$

since Φ_β isomorphic.

For 1,

$x \in \Phi_\beta(\ker(T)) \Rightarrow x = \Phi_\beta(v)$ s.t. $T(v) = 0$.

then $L_A(x) = L_A \circ \Phi_\beta(v)$

$= \Phi_\gamma \circ T(v)$

$= 0$

$\Rightarrow x \in \ker(L_A)$.

Conversely, if $x \in \ker(L_A)$,

then set $v := \Phi_\beta^{-1}(x)$, and

observe $\Phi_\gamma \circ T(v) = L_A \circ \Phi_\beta(v)$

$= L_A(x)$

$= 0$

$\Rightarrow \Phi_\gamma(T(v)) = 0$

$\Rightarrow$ $T(v) = 0$ sin Φ_γ inj.

Th $x = \Phi_\beta(\Phi_\beta^{-1}(x))$, wh $v = \Phi_\beta^{-1}(x) \in \ker(T)$

$\blacksquare$

This Result says that isomorphism leave dimensions unchanged.

§2.4, nr 17:

Let $T: V \to W$ a linear map between finite dimensional vector space, and $V_0 \subseteq V$.

If T is an isomorphism, then

$$\dim(T(V_0)) = \dim(V_0).$$

Proof: Let $v_1, \ldots, v_n$ be a basis for V_0. We claim $\{T(v_1), \ldots, T(v_n)\}$ is a basis for $T(V_0)$.

- Indeed, suppose

$$\sum_{i=1}^{n} \alpha_i T(v_i) = 0 \Rightarrow T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = 0$$

By injectivy.

$$\Rightarrow \sum_{i=1}^{n} \alpha_i v_i \in \operatorname{Ker}(T) = \{0\}$$

$$\Rightarrow \sum \alpha_i v_i = 0$$

$$\Rightarrow \alpha_1 = \ldots = \alpha_n = 0,$$ by LI of

$\{v_1, \dots, v_n\}$.

Hence $\{T(v_1), \dots, T(v_n)\}$ is lineary independent.

- Futhermore let $y \in T(V_0)$. By surjectivity, $\exists$ $v \in V_0$ s.t.

$$y = T(v) = T(\sum \beta_i v_i) = \sum \beta_i T(v_i),$$

as required to show $\{T(v_1), \dots, T(v_n)\}$ spans.

$\blacksquare$