Homework 6

Solutions and Rubric

Due Feb 26, 2025

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 25
- So final score is $\left(\frac{\text{score}}{50}\right) \cdot 8$

§2.5) nr. 1	Graded		
nr. 2	Graded		
nr. 3 a, d	Graded		
nr. 5	Graded		
nr. 7	Completeness		
§3.1) nr. 1	Graded		
nr. 4	Graded		
nr. 6	Completeness		

Question 1 (§ 2.5, nr 1)

Graded

a.) False:

Given $\beta = \{x_1, \ldots, x_n\}$, $\beta' = \{x_1', \ldots, x_n'\}$, ordered bases for a vector space V.

Then $[I_V]_{\beta'}^{\beta}$ has j^{th} column the vector $[x_j']_\beta$.

b.) True:

Its inverse is the change of co-ordinates back.

c.) True:

Given bases β, β' for V, and $Q = [I_V]_{\beta'}^{\beta}$

Then $Q^{-1} = [I_V]_{\beta}^{\beta'}$, and

$$[T]_\beta = [I_V]_{\beta'}^{\beta} [T]_{\beta'} [I_V]_{\beta}^{'}$$

d.) False,

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{but}$$

$$B = Q^t \quad A \quad Q$$

but $B \not\sim A$.

e.) True :

The inventable matrix realizing the similary is the change of basis.

Question 2 (§ 2.5, nr 2.)

Graded

Given ordered bases β, β', find the change of co-ordinate matrix changing β' co-ordinates into β.

a.) $\beta' = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\}$, $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Well $[I]_{\beta'}^{\beta} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

b.) $\beta' = \left\{ \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\}$, $\beta = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

Then $\begin{bmatrix} 0 \\ 10 \end{bmatrix} = (a) \begin{bmatrix} -1 \\ 3 \end{bmatrix} + (b) \begin{bmatrix} 2 \\ -1 \end{bmatrix} = (4) \begin{bmatrix} -1 \\ 3 \end{bmatrix} + (2) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{cases} -a + 2b = 0 \\ 3a - b = 10 \end{cases}$ $\Rightarrow 2b = a$

$\Rightarrow 3(2b) - b = 10$

$\Rightarrow 5b = 10$

$\Rightarrow b = 2 \Rightarrow a = 4$

And $\begin{bmatrix} 5 \\ 0 \end{bmatrix} = c \begin{bmatrix} -1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow \begin{matrix} -c + 2d = 5 \\ 3c - d = 0 \end{matrix} \Rightarrow d = 3c \Rightarrow 5c = 5$

$\Rightarrow c = 1$

$\Rightarrow d = 3$.

So $[I]_{\beta'}^{\beta} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

c.) $\beta' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\beta = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$

① $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

Solve $\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

② $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \beta_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \beta_2 \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

Solve $\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Solution:

$[I]_{\beta'}^{\beta} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$

d.) $\beta' = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$

$\beta = \left\{ \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

① $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

Solve $\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

② $\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \beta_1 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

Solve $\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$

Thus $[I]_{\beta'}^{\beta} = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}$

Question 3 (§ 2.5, nr 3 a,d)

Graded

Given ordered bases β and β' for $P_2(\mathbb{R})$. Find the change of coordinate matrix that change β'- coordinates into β-coordinates.

a.) $\beta = \{x^2, x, 1\}$ and

$$\beta' = \{a_2x^2 + a_1x + a_0, \; b_2x^2 + b_1x + b_0, \; c_2x^2 + c_1x + c_0\}$$

① $\underline{a_2}x^2 + \underline{a_1}x + \underline{a_0}\cdot 1$

$$[a_2x^2 + a_1x + a_0]_\beta = \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

② $\underline{b_2}x^2 + \underline{b_1}x + \underline{b_0}\cdot 1$

$$[b_2x^2 + b_1x + b_0]_\beta = \begin{bmatrix} b_2 \\ b_1 \\ b_1 \end{bmatrix}$$

③ $\underline{c_2}x^2 + \underline{c_1}x + \underline{c_0}\cdot 1$

$$[c_2x^2 + c_1x + c_0]_\beta = \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

Hence

$$[I]_{\beta'}^{\beta} = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_1 & c_0 \end{bmatrix}$$

Example:

$[I]_{\beta'}^{\beta}$ takes the polynomial $a_2x^2 + a_1x + a_0 \in P_2(\mathbb{R})$ from its coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\beta'}$ to $\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}_\beta$.

d.) $\beta' = \{x^2+x+4,\ 4x^2-3x+2,\ 2x^2+3\}$

$\beta = \{x^2-x+1,\ x+1,\ x^2+1\}$

① $x^2+x+4 = \alpha_1(x^2-x+1)+\alpha_2(x+1)+\alpha_3(x^2+1)$

$= (\alpha_1+\alpha_3)x^2 + (\alpha_2-\alpha_1)x + \alpha_1+\alpha_2+\alpha_3$

$\Rightarrow \begin{array}{l} \alpha_1 + \alpha_3 = 1 \\ -\alpha_1+\alpha_2 = 1 \\ \alpha_1+\alpha_2+\alpha_3 = 4 \end{array} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

② $4x^2-3x+2 = \beta_1(x^2-x+1)+\beta_2(x+1)+\beta_3(x^2+1)$

$\Rightarrow$ Solve $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

③ $2x^2+3 = \gamma_1(x^2-x+1)+\gamma_2(x+1)+\gamma_3(x^2+1)$

$\Rightarrow$ Solve $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution:

$[I]_{\beta'}^{\beta} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Question 4 (§ 2.5, nr 5)

Graded

Let T be the linear map defined on $P_1(\mathbb{R})$ by $T(p(x)) := p'(x)$.
Let $\beta = \{1, x\}$, and $\beta' = \{1+x, 1-x\}$ basis for $P_1(\mathbb{R})$.

Use the fact that $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ to find $[T]_{\beta'}$.

Well $[T]_{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta} [I_V]_{\beta'}^{\beta}$

where $I(1+x) = 1 \cdot 1 + 1 \cdot x$
and $I(1-x) = 1 \cdot 1 + (-1) \cdot x$

$$[I]_{\beta'}^{\beta} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad [I]_{\beta}^{\beta'} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x$
$T(x) = 1 = 1 \cdot 1 + 0 \cdot x$

$$[T]_{\beta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So $[T]_{\beta'} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Question 5 (§ 2.5, nr 7)

Completeness

Let $y = mx$, $m \neq 0$. Find expresion for $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

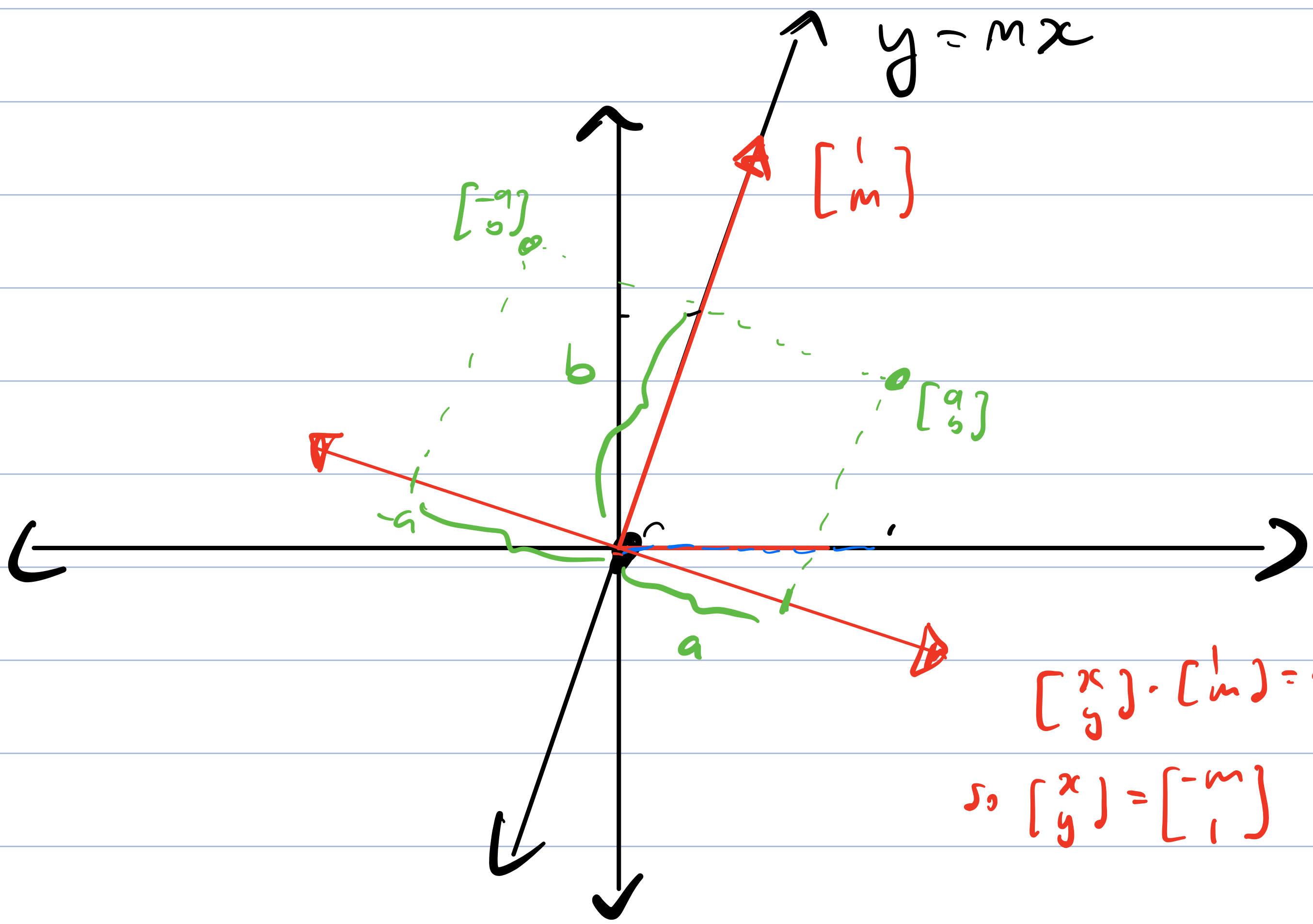
1.) T is reflection of $\mathbb{R}^2$ about $y = mx$

2.) T is orthogonal projection onto L.

Idea: We know in the standard basis

1.) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ is reflection accross $y = 0$

2.) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ is orthogonal projection onto $y = 0$.



$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ m \end{bmatrix} = 0$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -m \\ 1 \end{bmatrix}$

If $[v]_{\beta'} = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\beta' = \left\{ \begin{bmatrix} -m \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ m \end{bmatrix} \right\}$,

Reflection accross $y = mx$ is just $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ -b \end{bmatrix}$

- Standard basis $\underset{\beta}{}$ $\rightarrow$ β' $\rightarrow$ reflect $\rightarrow$ Back to stand.

- $Q^{-1} = [I]_{\beta'}^{\beta} = \begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

$\begin{bmatrix} -m \\ 1 \end{bmatrix} = (-m) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} 1 \\ m \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (m) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $Q = [I]_{\beta}^{\beta'} = (Q^{-1})^{-1} = \frac{-1}{m^2+1} \begin{bmatrix} m & -1 \\ -1 & -m \end{bmatrix}$

$$= \frac{1}{m^2+1} \begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}$$

So $[T]_{\beta} = Q^{-1} [T]_{\beta'} Q$, $[T]_{\beta'} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \frac{1}{m^2+1} \begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}$$

$$= \frac{1}{m^2+1} \begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix} \begin{bmatrix} m & -1 \\ 1 & m \end{bmatrix}$$

$$= \frac{1}{m^2+1}\begin{bmatrix} -m^2+1 & 2m \\ 2m & -1+m^2 \end{bmatrix}$$

$$= \frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

For projection onto line $\perp$ to $y=mx$, use $[T]_{\beta'} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$[T]_\beta = Q^{-1}[T]_{\beta'}Q$$

$$= \frac{1}{m^2+1}\begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}$$

$$= \frac{1}{m^2+1}\begin{bmatrix} -m & 1 \\ 1 & m \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & m \end{bmatrix}$$

$$= \frac{1}{m^2+1}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

Question 6: (§ 3.1, nr 1) Graded

a.) True, sine elementry operations does not change the size of the matrix

b.) False, multiplying by any non-zero scalar is also an elementry operation.

c.) True, for example it is the result of scaling row 1 by 1.

d.) False, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, which is not an elementry matrix.

e.) True, see theorem 3.2

f.) False, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ not elementry matrix.

g.) True, suppose E is obtained via an elementry row operation on I. The E^t is obtained by the same operation as a column operation.

h.) False, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and

$B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. We can obtain B from A

by $A \xrightarrow{R_2 + R_1} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. But any column operation

on A will not change the fact that the

bottom rows stay zero.

i) True, if we can obtained B from A

with a single elementary row operation, then

we can also obtain A from B with a

single elementary row operation, since

elementary operations are invertable.

Question 7 (§3.1, nr. 4) Graded

We can obtain an elementary matrix E_n in at least two distinct ways! (non-uniqueness)

Suppose E_n is an $n \times n$ elementary matrix:

① If E_n was obtained by scaling the j^{th} row of I_n, then scaling the j^{th} column of I_n gives you E_n.

② If E_n was obtained by swapping row i with row j, then we can obtain E_n from I_n by swapping column i with column j.

③ If E_n was obtained by $I_n \xrightarrow{R_i + \alpha R_j} E_n$. Then

$I_n \xrightarrow{C_j + \alpha C_i} E_n$

Example: $n=3$, $E_3 = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} \xrightarrow{R_1 + 5R_2} \underbrace{\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3}$$

$$I_3 \ \Big\downarrow\ C_2 + 5C_1$$

$$\underbrace{\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3}$$

Question 8 (§3.1, nr 6.)

Completeness

Let A be a $m \times n$, and B a $m \times n$ matrix obtained by performing elementary row operations on A.
Show that B^t can be obtained by performing elementary column operations.

<u>Proof</u>: To say B is obtained from A by performing row operations, means $\exists$ elementary matrices $E_1, \ldots, E_k$

s.t. $\underset{m \times n}{B} = \underset{m \times m}{E_k} E_{k-1} \ldots \underset{m \times m}{E_1} \underset{m \times n}{A}$. (multiply on left)

Where A a $m \times n$ matrix so E_i $m \times m$.

Now applying transposes, we get

$$\underset{n \times m}{B^t} = (E_k \ldots E_2 E_1 A)^t$$
$$= A^t (E_k \ldots E_2 E_1)^t$$
$$= A^t E_1^t (E_k \ldots E_2)^t$$
$$= \ldots$$
$$= A^t E_1^t E_2^t \ldots E_k^t.$$

$n \times m$ $m \times m$

From previous exercise recall that E_i^t still elementary matrices. And multiplying on the right is equivalent to performing to column operation.

Hence B^t is obtained by performing elementary column operations on A^t.

(Same idea for showing if $B = AE$, then $B^t = E^t A^t$)

Exercise:

1.) Two projections are similar iff the have the same rank.

(*) Trace of similar matrices are equal.

2.) For every projection $P: V \to V$ finite dimensional V, $\exists$ an ordered basis for V s.t $[P]_\gamma$ is diagonal with only 1's and zeros along the diagonal.

Pf 2: Well recall that

① $V = R(T) \oplus \ker(T)$.

Indeed, for every $v \in V$ write $v = Pv + (v - Pv)$, where $P(v - Pv) = Pv - P^2 v$
$= 0$.

And if $v \in R(T) \cap \ker(T)$, then $P(v) = 0$ and $Pv = v \Rightarrow v = 0$.

② let $\{v_1, .., v_k\}$ be a basis for $R(T)$, $\{v_{k+1}, .., v_n\}$ a basis for $\ker(T)$.

One checks $\gamma := \{v_1, .., v_n\}$ a basis for V.

Then $Pv_1 = v_1 \longrightarrow [Pv_1]_\gamma = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

$Pv_k = v_k \longrightarrow [Pv_k]_\gamma = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← kth row.

$Pv_j = 0, \ \forall j \in \{k+1, \dots, n\}$.

So $[P]_\gamma = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$

Now suppose we where given a basis $\beta = \{w_1, \dots, w_n\}$ where $[P]_\beta$ was not diagonal. Then take

$$[P]_\gamma = [\quad]_\beta^\gamma [P]_\beta [I_v]_\gamma^\beta$$

$$= Q^{-1} [P]_\beta Q$$

Where $Q = [I_v]_\gamma^\beta = \begin{bmatrix} | & & | \\ [v_1]_\beta & \cdots & [v_n]_\beta \end{bmatrix}$

And $Q^{-1} = \begin{bmatrix} | & & | \\ [w_1]_\gamma & \cdots & [w_n] \\ | & & | \end{bmatrix}$.