Homework Solutions and Rubric

Due Mar 5, 2025

Grading Scheme

- HW counts 8 marks on Canvas
- 5 points per problem
- Total 25
- So final score is $\left(\frac{\text{score}}{50}\right) \cdot 8$

Question 1 (§3.2 , exercise 1)

Graded

a) False, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has rank 1.

b.) False, $\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Product has rank 0, but both matrics A, B have rank 1.

c.) True, any m×n matrix with at least 1 non zero entry has rank ≥ 1.

d.) True.

e.) False, elementary column operations can be performed by multiplying an elementary matrix (hence inv) on the right.

f.) True, since the rank of a matrix A is equal to its transpose.

g.) True, provided the matrix is invertible.

h.) True, since $L_A: \mathbb{C}^n \to \mathbb{C}^n$ and the dimension of the co-domain is an upper bound for the dimension of the range of L_A.

i.) True, because $L_A: \mathbb{C}^n \to \mathbb{C}^n$ having rank n implies L_A is onto. By rank nullity we know its one-to-one. Hence invertable. Thus A is invertable.

Question 2, §3.2, nr 2

Graded

Find the rank of the following matrices:

Idea: Given a $m \times n$ matrix A we can find a finite sequence of elementary matrices to turn A into an upper triangular matrix. Where we mean all entries below the main diagonal are zero. Then the # of non zero entries along the main diagonal is the # of L.I. columns. Since elementary operations preserve rank, we can read of the rank.

a.) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, Rank 2 since all columns are linearly independent.

b.) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, Rank 3 c.) $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$, Rank 2, since rows are L.I.

d.) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$, Rank 1, only 1 linearly indep. col.

e.) $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, Rank 3

f.) $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$, Rank 3

g.) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$, Rank 1 since only 1 linearly independent column.

Question 3 (§ 3.2, nr. 5)

Compute 1.) Inverses with $(A\ I)$.
2.) Rank

a.) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Rank 2 with inverse

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1-2}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

b.) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. Rank 1 since linearly indep columns. Also not invertable.

c.) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$. Rank 2 and hence not invertable.

d.) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$. Rank 3 so invertable and inverse

$$\begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \cdots \end{pmatrix}$$

$$\begin{pmatrix} 3/2 & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

e.) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$. Rank 3 so invertable with inverse

$$\begin{pmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{pmatrix}$$

f.) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Thus rank 2 and not invertable.

g.) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$. Rak 4 with inverse

$$\begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}$$

h.) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$. Rak 3 ; so not invertable

(§ 3.2, nr 14.)

Let $T, U : V \to W$ be linear transformations. Show

a.) $R(T+U) \subseteq R(T) + R(U)$

b.) If $\dim(W) < +\infty$, then
$\operatorname{rank}(T+U) \le \operatorname{Rank}(T) + \operatorname{Rank}(U)$

c) Deduce from b.) that
$\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$
$\forall$ $m \times n$ matrices A, B.

Pf: a.) $R(T+U) = \{ T(v) + U(v) \mid v \in V \}$

$\subseteq \{ T(v) + U(w) \mid v, w \in V \}$ (2)

$= R(T) + R(U).$

b.) $\dim(R(T+U)) \le \dim(R(T) + R(U))$

(2) $\le \dim(R(T)) + \dim(R(U)).$

c.) Apply above to $L_A, L_B : \mathbb{C}^m \to \mathbb{C}^n.$
(1)

Question 5 $\S$ 3.2, nr 21

Suppose A is a $m \times n$ matrix of rank m. Show A has a right inverse. That is show that there exists a $n \times m$ matrix B such that $AB = I_m$.

<u>Proof</u>: Observe that since $\text{Rank}(A) = m$, we know $n \geq m$, since $\min\{m, n\}$ is an upper bound for the rank.

$$m \left[\quad \underset{n}{A} \quad \right]$$

We know $\exists$ finitely many elementary matrices $E_1, \dots, E_k$ of size $n \times n$ s.t

$$A E_1 \dots E_k = \underset{m \times n}{\begin{bmatrix} I_m & O_{n-m} \end{bmatrix}}$$

Then

$$\underset{m \times n}{A} \underset{n \times n}{E_1 \dots E_k} \underset{n \times m}{\begin{bmatrix} I_m \\ 0 \end{bmatrix}} = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix} = I_m.$$

Chose $B = E_1 \dots E_k \begin{bmatrix} I_m \\ 0 \end{bmatrix}$ size $n \times m$

$n \times n$ $\quad$ $n \times m$

§3.2, nr. 3

Show that the only matrix A with rank 0 is the zero matrix.

Pf: Clearly the 0 matrix has rank 0.
To show a rank 0 matrix is the zero matrix, lets prove the converse.
Suppose A is not the zero matrix.
That is A is a $m \times n$ matrix with at least one non-zero entry.

By thm 3.6, we can find a finite sequence of elementary operations to transform A into $\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$ where $0 \le r \le m$.

Since row operations only change one row at a time, and cannot produce a zero row, when only one non-zero row is left, we know $\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$ has at least one non-zero row $\Rightarrow$ non-zero rank
Since elementary operation leave rank unchanged, we know A had non-zero rank.

$\square$

FIS § 3.2, nr. 17

Show that if B is a 3x1 matrix, and C is a 1x3 matrix, then the 3x3 matrix BC has rank 1.

Conversely if A is a 3x3 matrix of rank 1, then A factors as BC where B a 3x1 and C a 1x3 matrix.

Proof: Recall that rank of BC is defined as the dimension of the range of $L_{BC}: \mathbb{C}^3 \to \mathbb{C}^3$.

Also $L_{BC} = L_B \circ L_C$. And from thm 3.7 we know

$$\text{Rank}(L_B L_C) \leq \text{Rank}(L_C)$$

where since $L_C: \mathbb{C}^3 \to \mathbb{C}^1$, and since codomain one dimensional, we know $\text{Rank}(C) \leq 1$.

Next suppose A is a 3×3 matrix of rank 1.

Then consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and we know $\dim(R(L_A)) = 1$.

Since $A \neq 0$, we have at least one column with a non-zero entry.

WLOG say $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Then $R(L_A) = \operatorname{span}\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \right\}$, and

since the other two columns are in the range of L_A we know

$$\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \alpha_2 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

So $A = \begin{bmatrix} a_{11} & \alpha_1 a_{11} & \alpha_2 a_{11} \\ a_{21} & \alpha_1 a_{21} & \alpha_2 a_{21} \\ a_{31} & \alpha_1 a_{31} & \alpha_2 a_{31} \end{bmatrix}$

$$= \underset{3\times 1}{\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \end{bmatrix}$$

Fis, § 3.2, nr 18

Given a $m \times n$ matrix A, and $n \times p$ matrix B. Show that AB can be written of a sum of n matrices of rank at most one.

$$\underset{m \times n}{\begin{bmatrix} \\ & A & \\ \\ \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} & B & \end{bmatrix}}$$

Let A_i denote the $m \times n$ matrix with the same i^{th} column of A, and the rest all zero entries. Then $A = \sum_{i=1}^{n} A_i$.

And $AB = \left(\sum A_i\right) B = \sum_{i=1}^{n} A_i B$, where

each $\operatorname{Rank}(A_i B) \leq \operatorname{Rank}(A_i) \leq 1$,

since at most one non-zero column in A_i

§ 3.2, exercise 22

Let B be a $n \times m$ matrix with rank m. Show that there exists a $m \times n$ matrix A s.t $AB = I_m$.

<u>Proof</u>: Since $Rank(B) = m$ we know $n \geq m$ since $\min\{n, m\}$ is an upper bound for the rank.

So

$$n \left[\quad B \quad \right]_{m}$$

We can find finite number of elementary matrices $E_1\, E_2 \ldots E_k$ size $n \times n$ s.t

$$\underset{n \times n}{E_k \cdots E_1} \underset{n \times m}{B} = \begin{bmatrix} I_m \\ O_{n-m} \end{bmatrix}$$

Then $\begin{bmatrix} I_m & 0_{n-m} \end{bmatrix} \underbrace{E_k \cdots E_1}_{n\times n} \underbrace{B}_{n\times m}$

$$= \begin{bmatrix} I_m & 0_{n-m} \end{bmatrix} \begin{bmatrix} I_m \\ 0_{n-m} \end{bmatrix}$$

$$= I_m$$

So set $A = \begin{bmatrix} I_m & 0_{n-m} \end{bmatrix} \underbrace{E_k \cdots E_1}_{n\times n}$.