Homework 8

Due Mar 14, 2025

Solutions and Rubric

Section 3.3 : 1., 4., 10.

Section 3.4 : 1., 2i, 2j, 5., 6., 13.

Grading Scheme

Total out of 35 (5 points per problem) :

- Section 3.3 : 1., 4., 10.
- Section 3.4 : 1., 5., 6., 13.

Question 1 (§ 3.3, nr. 1) (5 points)

a.) **False**, Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution

since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin \text{Range}(A)$.

b.) **False**, consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Observe that $L_A : \mathbb{R}^3 \to \mathbb{R}^2$

c.) **True**, 0 is always a solution to $Ax = 0$.

d.) **False**,

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n &= b_1 \\ &\vdots \\ a_{n1} x_1 + \cdots + a_{nn} x_n &= b \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{b}.$$

$L_A : F^n \to F^n$

- If L_A invertible, we have unique solution

- If L_A not invertible, then L_A both not injective and not surjective, by rank nullity.
- Thus $\exists\, b_1, b_2 \in F^n$ s.t. $Ax = b_1$ non unique solution by injectivity, And $Ax = b_2$ no solution by surjection.

e.) False, $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ induces a linear transformation $L_A : \mathbb{R}^3 \to \mathbb{R}^3$. But since $\text{Rank}(A) < 3$, we know L_A not surjective. Take $b \in \mathbb{R}^3 \setminus \text{Range}(A)$. Then $Ax = b$ has no solution by definition of $b \notin R(L_A)$.

f.) False, consider our example above with $Ax = b$, $\text{Rank}(A) < 3$, and $b \notin \text{Range}(A)$.

Then $Ax = b$ does not have a solution, but $Ax = 0$ has a solution when $x = 0$.

g.) True, if A invertible, and $Ax = b$, $b \neq 0$. then solution set is $\{A^{-1}b\} + \text{Ker}(A)$

$= \{A^{-1}b\}$, since $\text{Ker}(A) = \{0\}$

h.) False, Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

And $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We have 3 variables and 3 equations.

Then $Ax = b$ has solution $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus the solution set of $Ax = b$

is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} + \mathrm{Ker}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + v \mid Av = 0 \right\}$

But $Av = 0 \Rightarrow v = 0$ so solution set

is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, not a subspace of $\mathbb{R}^3$.

Question 2 (§ 3.3, nr 4.) (5 points)

(1) Compute A^{-1}

(2) Use A^{-1} to solve the system

a) Given $\begin{cases} x_1 + 3x_2 = 4 \\ 2x_1 + 5x_2 = 3 \end{cases}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Then $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, and

$$A^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

- We obtain a solution

$$x = A^{-1}b = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = (4)\begin{bmatrix} -5 \\ 2 \end{bmatrix} + (3)\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -20+9 \\ 8-3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 \\ 5 \end{bmatrix}$$

b.) Given a system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 4 \end{aligned}$$

which we rewrite

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

and set $A := \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ 1/9 & 1/3 & -2/9 \\ -4/9 & 2/3 & -1/9 \end{bmatrix}$

And the solution is

$$A^{-1}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ 1/9 & 1/3 & -2/9 \\ -4/9 & 2/3 & -1/9 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

Question 3 (§ 3.3, nr 10.) (5 points)

If the coefficient matrix of a system of m linear equations, and n unknowns have rank m, then the system has at least one solutions. True or false?

True, indeed since we have m equations and n unknowns we know that the coefficient matrix A is a $m \times n$ matrix. That is a linear map $A: F^n \to F^m$.
To say A has rank m, implies A is surjective, and hence for every $b \in F^m$, there exists $x \in F^n$ s.t $Ax = b$. Meaning the particular system $Ax = b$ will have a solution.

Question 4. (§ 3.4, nr 1.) (5 points)

a.) False

b.) True

c.) True

d.) True

e.) False

f.) True

g.) True

Question 5. (§ 3.4, nr 2.)

i) Given a system of equations

$$\begin{aligned} 3x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 2 \\ x_1 - x_2 + 2x_3 + 3x_4 + x_5 &= -1 \\ 2x_1 - 3x_2 + 6x_3 + 9x_4 + 4x_5 &= -5 \\ 7x_1 - 2x_2 + 4x_3 + 8x_4 + x_5 &= 6 \end{aligned}$$

Form the augmented system $[A|b]$

$$\left[\begin{array}{ccccc|c} 3 & -1 & 2 & 4 & 1 & 2 \\ 1 & -1 & 2 & 3 & 1 & -1 \\ 2 & -3 & 6 & 9 & 4 & -5 \\ 7 & -2 & 4 & 8 & 1 & 6 \end{array}\right]$$

Performing elementary operations does not change the solution set. We compute the reduced row echelon form

RREF $\sim$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From this we know

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 - x_5 \\ x_2 - 2x_3 + 4x_5 \\ x_4 + 2x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Set $t = x_5$, and $s = x_3$ parameters that vary. Then solution takes the form

- $x_1 = 2 + t$
- $x_2 = 2s - 4t$
- $x_3 = s$
- $x_4 = -1 - 2t$
- $x_5 = t$

That is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + (t) \begin{bmatrix} 1 \\ -4 \\ 0 \\ -2 \\ 1 \end{bmatrix} + (S) \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

j) Given a system of equations

$$\begin{aligned}
2x_1 + 0x_2 + 3x_3 + 0x_4 - 4x_5 &= 5 \\
3x_1 - 4x_2 + 8x_3 + 3x_4 + 0x_5 &= 8 \\
x_1 - x_2 + 2x_3 + x_4 - x_5 &= 2 \\
-2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 &= -8
\end{aligned}$$

By similar computation as above one obtains a solution set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + (s) \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}$$

Question 6. (§ 3.4, nr 5.) (5 points)

Given

$$\tilde{A} = \begin{array}{c} \begin{matrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 & \tilde{a}_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} \end{array}$$

the reduce-row-echelon form of A, and

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & \times & 1 & \times \\ -1 & -1 & \times & -2 & \times \\ 3 & 1 & \times & 0 & \times \end{pmatrix} \end{array}$$

Find the missing columns of A.

• Sol: The observation one needs to make is that there exists a product of elementary matrices C s.t $C\tilde{A} = A$ which implies $C \cdot \hat{a}_i = a_i$ where $a_i, \hat{a}_i$ column i of A and $\hat{A}$.

By looking at $\tilde{A}$ one observes

$$\tilde{a}_3 = 2\tilde{a}_1 - 5\tilde{a}_2 \quad \text{and} \quad \tilde{a}_5 = -2\tilde{a}_1 - 3\tilde{a}_2 + 6\tilde{a}_5$$

Then left multiply by C to get

$$a_3 = 2a_1 - 5a_2 \quad \text{and} \quad a_5 = -2a_1 - 3a_2 + 6a_5$$

Which implies

$$a_3 = 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

and

$$a_5 = (-2)\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -7 \\ -9 \end{bmatrix}$$

And we obtain

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$

Question 7. (§3.4, nr 6) (5 points)

Given
$$\tilde{A} = \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
the reduce-row-echelon form of A,

and
$$A = \begin{pmatrix} 1 & x & -1 & x & x & 3 \\ -2 & x & 1 & x & x & -9 \\ -1 & x & 2 & x & x & 2 \\ 3 & x & -4 & x & x & 5 \end{pmatrix}$$

Find columns 2, 4, 5 of A.

Combining all row operations into invertible C we get $CA = \tilde{A}$

$$\underbrace{\begin{bmatrix} — c_1 — \\ — c_2 — \\ — c_3 — \\ — c_4 — \end{bmatrix}}_{4\times 4} \underbrace{\begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{matrix} \\ \begin{bmatrix} 1 & x & -1 & x & x & 3 \\ -2 & x & 1 & x & x & -9 \\ -1 & x & 2 & x & x & 2 \\ 3 & x & -4 & x & x & 5 \end{bmatrix} \end{array}}_{4\times 6} = \underbrace{\begin{array}{c} \begin{matrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 & \tilde{a}_4 & \tilde{a}_5 & \tilde{a}_6 \end{matrix} \\ \begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}}_{4\times 6}$$

So $C \cdot a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C \cdot a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $C \cdot a_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$C \cdot a_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $C \cdot a_4 = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $C \cdot a_6 = \begin{bmatrix} 5 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

- <u>Key observation</u>: Ca_2 and Ca_3 scalar multiples.

So $C \cdot a_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $C \cdot a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

So $Ca_2 = -3C \cdot a_1$

$\Rightarrow a_2 = -3a_1$ since C^{-1} exists

$\Rightarrow a_2 = -3 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 3 \\ -9 \end{bmatrix}$

- Next to obtain a_4, we observe

$$\tilde{a}_4 = 4\tilde{a}_1 + 3\tilde{a}_3$$

} since $\tilde{a}_i = Ca_i$

$\Rightarrow Ca_4 = 4Ca_1 + 3Ca_3$ } since C^{-1} exists

$\Rightarrow a_4 = 4a_1 + 3a_3$

$$= 4 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -5 \\ 2 \\ 0 \end{bmatrix}$$

- And lastly to obtain a_5, one observes

$$\tilde{a}_5 = -\tilde{a}_6 + 5\tilde{a}_1 - 2\tilde{a}_2$$

$$\Rightarrow \quad C\tilde{a}_5 = -C\tilde{a}_6 + C5\tilde{a}_1 - 5C\tilde{a}_2$$

$$\Rightarrow \quad a_5 = -a_6 + 5a_1 + 2a_3$$

$$= (-1)\begin{bmatrix} 3 \\ -9 \\ 2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

Thus

$$A = \begin{pmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{pmatrix}$$

Question 8. (§ 3.4, nr 13.) (5 points)

Let V denote the solution set to the system

$$\begin{aligned} x_1 \quad -x_2 \qquad\quad\; + 2x_4 - 3x_5 + x_6 &= 0 \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 &= 0 \end{aligned}$$

Given $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

a.) Show S is linearly independent

b.) Extend S to a basis for V.

Sol: Since the vectors are not scalar multiples, we see S is linearly independent. And by checking both vectors satisfy the system one can conclude $S \subseteq V$ too.

First observe

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{bmatrix} : \mathbb{R}^6 \to \mathbb{R}^2$$

and $V = \ker(A)$. By rank-nullity, and since

$\text{Rank}(A) = 2$ (· 2 LI coln) we see $\dim(V) = \dim(\ker(A))$

$= 4$.

Step 1: Find a basis for $\ker(A)$, and use

it to extend the basis of S.

Solve x_1 and x_2 i.t.o. x_3, x_4, x_5, x_6

to get

$$x_1 = x_3 - x_4 + x_5 - 3x_6 \quad (*)$$

$$x_2 = x_3 + x_4 - 2x_5 - 2x_6 \quad (**)$$

Set $x_3 = s$, $x_4 = t$, $x_5 = r$, $x_6 = v$ parameters

Then

$$x_1 = s - t + r - 3v$$

$$x_2 = s + t - 2r - 2v$$

$$x_3 = s$$

$$x_4 = t$$

$$x_5 = r$$

$$x_6 = v$$

And hence

$$\ker(A) = \left\{ (s)\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix} + (t)\begin{bmatrix}-1\\1\\0\\1\\0\\0\end{bmatrix} + (r)\begin{bmatrix}1\\-2\\0\\0\\1\\0\end{bmatrix} + (v)\begin{bmatrix}-3\\-2\\0\\0\\0\\1\end{bmatrix} \;\middle|\; \begin{matrix}s,t,r,\\ v \in \mathbb{R}\end{matrix} \right\}$$

Where we have a basis for V, but we want one that extends S.

Step 2: Place all columns in a 6×6 matrix with the vectors in S in the first two colums

$$B := \begin{bmatrix} 1 & 0 & 1 & -1 & 1 & -3 \\ 0 & 2 & 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: \tilde{B}$$

Then a basis for V that extends S is

$$\left\{ \begin{bmatrix}1\\0\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\-2\\0\\0\\0\\1\end{bmatrix} \right\}$$

NB! Put colums of S in first two cols so they appear in RREF as L2.