Homework 9

Due Mar 26, 2025

Solutions and Rubric

4.1) 1, 3(a,b), 4(a,b), 8

4.2) 1, 2, 5, 13, 23, 26

Grading Scheme

5 points each

Score out of 50.

Question 1 : § 4.1, nr 1

a.) False, $\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b.) True, thm 4.1

c.) False, $\det(A) = 0 \iff A$ not invertible

d.) False, $\left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|$.

e) True

Question 2 (§4.1 , nr 3)

a.) $\det \begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$

$= (-1+i)(2-3i) - (1-4i)(3+2i)$

$= -2+3i+2i-3i^2 - (3+2i-12i-8i^2)$

$= -2+5i+3 - (3-10i+8)$

$= 1+5i - (11-10i)$

$= -10+15i$

b.) $\det \begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$

$= (5-2i)(7i) - (6+4i)(-3+i)$

$= 35i - 14i^2 - (-18+6i-12i+4i^2)$

$= 35i+14 - (-18-6i-4)$

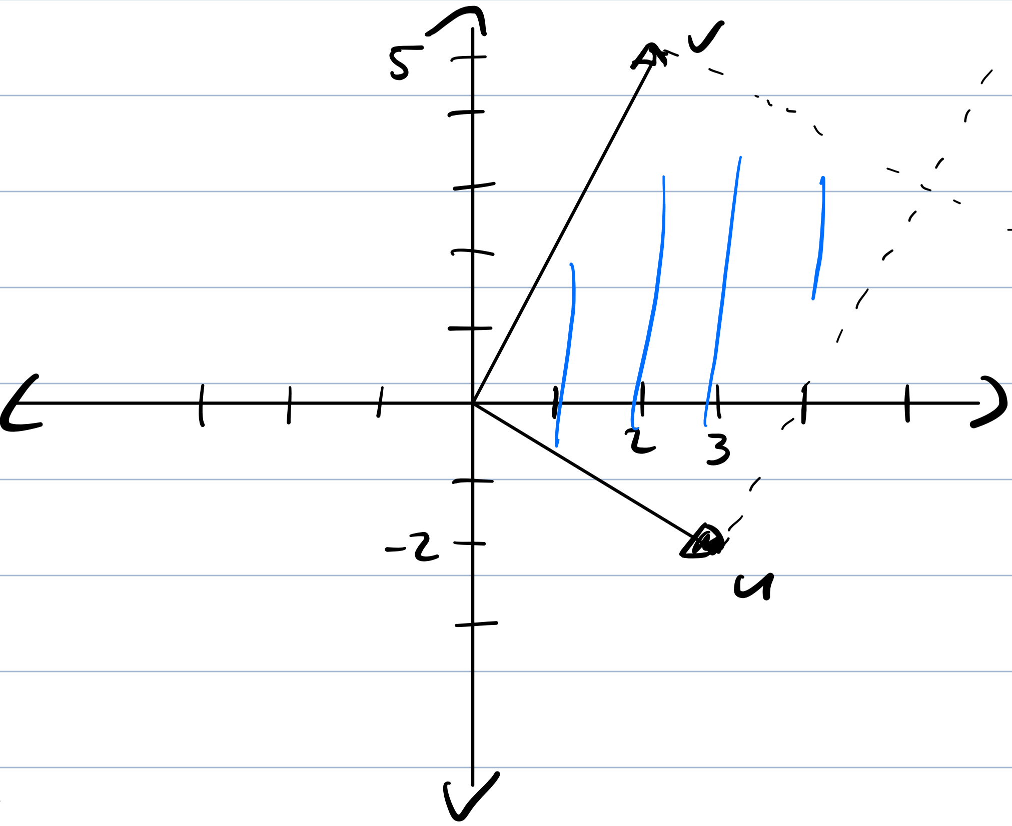
$= 35i+14+22+6i$

$= 36+41i$

Question 3 (§4.1, nr 4)

Calculate the area of the parallelogram determined by u, v:

a.) $u = (3,-2)$, $v = (2,5)$



Area

$$= \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} 3 & -2 \\ 2 & 5 \end{pmatrix}\right| = 15+4 = 19$$

b.) $u = (1,3)$, $v = (-3,1)$

$$\left|\det\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\right| = |(1)(1)-(3)(-3)| = 10$$

Question 4 (§ 4.1, nr 8)

Observe that

$$\det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad \quad , \text{ as required.}$$

$\blacksquare$

Question 5 (§4.2, nr 1)

a.) False

b.) True, thm 4.4

c.) True, cor page 215

d.) True, page 217

e.) False, $\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2 \neq 1 = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

f.) False, page 217

g.) False, A is invertable by rank-nullity $\Rightarrow \det(A) \neq 0$.

h.) True

Question 6 (4.2, nr 2)

Find the value k s.t

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Recall that we know how det. are changed by elementary row operations. Since the left handside matrix is obtained by 3 elementary operations, each scaling a row by 3 we get

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = 3\det\begin{pmatrix} a_1 & a_2 & a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix}$$

$$= 3^2 \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix}$$

$$= 3^3 \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$\Rightarrow k = 27$

Question 7 (§4.2, nr 5)

Given $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ (along first row)

$$\det(A) = \sum_{j=1}^{3} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j})$$

$$= 0 + (-1)^{1+2} \cdot 1 \cdot \det\begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} + (-1)^{1+3} \cdot 2 \cdot \det\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$= (-1)\left(0 - (-3)(2)\right) + (2)(-3 - 0)$$

$$= -12.$$

Question 8 (§4.2, nr. 13)

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\det(A) = \sum_{j=1}^{3} (-1)^{1+j} A_{ij} \det(\tilde{A}_{ij})$$

$$= 0 + 0 + (-1)^{1+3} \cdot 1 \cdot \det\begin{pmatrix} 0 & 2 \\ 4 & 5 \end{pmatrix}$$

$$= 0 + (0)(5) - (2)(4)$$

$$= -8$$

Question 9 (§4.2, nr 23)

Let $A \in M_n(F)$ an upper triangular matrix,
Show that

$$\circledast \quad \det(A) = \prod_{i=1}^{n} A_{ii}$$

(product of main diag. entries)

Proof: (via induction)

- **Base step** : $n = 2$;

We know for $n=2$, $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $\det(A) = ac$.

- Inductive step : Suppose that for any $n \times n$ upper triangular matrix $\circledast$ holds.

Now consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ 0 & a_{22} & & & \vdots \\ \vdots & 0 & a_{33} & & \vdots \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{n+1}a_{n+1} \end{bmatrix}$$

The key observation is to use co-factor

expansion along the n+1 row.

Then $\det(A)$

$$= \sum_{j=1}^{n+1} (-1)^{(n+1)+j} \cdot A_{n+1,j} \cdot \det(\tilde{A}_{n+1,j})$$

$$= 0 + \dots + (-1)^{(n+1)+(n+1)} A_{n+1,n+1} \det(\tilde{A}_{n+1,n+1})$$

Where $(-1)^{(n+1)+(n+1)} = (-1)^{2n+2} = 1$.

Now since $\tilde{A}_{n+1,n+1}$ the $n \times n$ matrix obtained from A by removing row $n+1$, and col. $n+1$, we know its upper triangular with diagonal entries $A_{11}, \dots, A_{nn}$. Hence inductive hypothesis apply, as needed to get

$$\det(A) = A_{n+1,n+1} \det(\tilde{A}_{n+1,n+1})$$
$$= A_{n+1,n+1} \prod_{i=1}^{n} A_{i,i}$$
$$= \prod_{i=1}^{n+1} A_{i,i}.$$

$\blacksquare$

Question 10 (§ 4.2, nr 26)

Under what conditions on $M_n(F)$ do we have $\det(-A) = \det(A)$ for all $A \in M_n(F)$?

Proof: If $n = 2k$, then $\det(-A) = (-1)^{2k} \det(A)$.

- If F has characteristic 2. (Not needed for full credit)

§ 4.2, nr 27

Show that if $A \in M_n(F)$ has two identical columns, then $\det(A) = 0$.

Pf: If A has 2 identical columns, then $\operatorname{rank}(A) < n$

$\Rightarrow \det(A) = 0$

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