Name: Solutions

MAC 2311 - Analytical Geometry and Calculus I
Quiz # 3, September 14, 2023

5 points

Problem 1 *Find the real number a such that the function f defined below is continuous on* $(-\infty, \infty)$

$$f(x) = \left\{ \begin{array}{ll} ax^2 + 2x, & \text{if } x < 2 \\ x^3 - ax, & \text{if } x \geq 2 \end{array} \right\} \tag{1}$$

Start by finding a such that

$$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$$

$f(2) = (2)^3 - 3(2) = 2$ (1 point)

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} ax^2 + 2x = a(2)^2 + 2(2) = 4a + 4$ (1 point) (1 point)

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 - ax = (2)^3 - a(2) = -2a + 8$ (1 point) (1 point)

We want a such that $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

Therefore we need a such that

$$4a + 4 = -2a + 8$$

$$\Rightarrow \quad 6a = 4$$

$$\Rightarrow \quad a = \frac{4}{6} = \frac{2}{3}$$

Then $a = \frac{2}{3}$ satisfy $\lim_{x\to 2} f(x) = f(a)$.

Problem 2 solution: (5 points)

Given $f(x) = \sqrt{x+1}$, find the derivative of f using the definition.

For $x > 1$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$ (1 point)

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h+1} - \sqrt{x+1}\right)}{h} \cdot \left(\frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}\right)$$ (1 point)

$$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$ (1 point)

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$ (1 point)

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}}$$ (1 point - evaluating limit)

$$= \frac{1}{2\sqrt{x+1}}$$

. Not on quiz but added for more practise.

Problem 2 *Let f be a differentiable function. Recall that the derivative of f at a point x in the domain of f is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Find the derivative of

$$f(x) = \sqrt{x+1} - x^2$$

using the definition of the derivative. Marks will not be allocated if differentiation rules are used.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)+1} - (x+h)^2 - \left(\sqrt{x+1} - x^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - (x^2 + 2xh + h^2) - \sqrt{x+1} + x^2}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+1} - 2xh + h^2 - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{-2xh + h^2}{h} + \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{(-2x+h)h}{h} + \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} (-2x+h) + \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \left(\frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}\right)$$

$$= \lim_{h \to 0} (-2x+h) + \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h \to 0} (-2x+h) + \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= -2x + \frac{1}{\sqrt{x+1} + \sqrt{x+1}}$$

$$f'(x) = -2x + \frac{1}{2\sqrt{x+1}}$$