Discussion Notes 4

February 06, 2024

Last Time:

- L9 - The derivative
  - Instantaneous rate of change
  - Slope of a tangent line at a point
  - velocity
- L10 - The derivative as a function.
  - Differentiability
  - Sketching functions

Today:

- L11 - Calculating derivatives
  - Power rule.
  - Exponential rule.
  - Applications.

First recall how to work with exponents:

- For x, y, a, b real numbers, $x \neq 0$.

1) $y^0 = 1$

2) $\frac{1}{x^a} = x^{-a}$

3) $y^a y^b = y^{a+b}$

4) $\frac{x^a}{x^b} = x^{a-b}$

5) $(x^a)^b = x^{ab}$

6) $\sqrt{x} = x^{\frac{1}{2}}$

7) $\sqrt[n]{x} = x^{\frac{1}{n}}$, $n = 1, 2, 3, \ldots$

Differentiation Rules

Constant Rule

$$\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$$

Sum Rule

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f + \frac{d}{dx} g$$

Power Rule

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

Exponential rule

$a > 0$ and $a \neq 1$.

$$\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$$

Example: (Power & sum Rule)

- a) $f(x) = 15x^{100} - 13x^{12} + 5x - 46$
- b) $g(x) = 8x^3 - \frac{1}{3x^5} + x - 23$
- c) $T(y) = \sqrt{y} + 9\sqrt[3]{y^7} - \frac{2}{\sqrt[5]{y^2}}$
- d) $h(z) = z^{\pi} - x^{\sqrt{2}}$
- e) $k(t) = \sqrt[3]{t^2}(2t - t^2)$
- f) $p(x) = \dfrac{2t^5 + t^2 - 5}{t^2}$

Examples (Exponential Rule) $e \approx 2.71\ldots$

1) $f(x) = e^x$
2) $g(x) = e^{x+5}$
3) $h(x) = e^2$
4) $k(t) = \sqrt{et}$
5) $h(t) = t^e$

Example: Find the equation of the tangent line to the graph of k where

- $k(t) = \sqrt[3]{t^2}\,(2t - t^2)$

at the point $t = 1$.

Solution: Step 1: Find the slope of the tangent line:

- $k(t) = t^{\frac{2}{3}} \cdot (2t - t^2) = 2t^{5/3} - t^{\frac{8}{3}}$

$$\frac{dk}{dt} = 2 \cdot \frac{5}{3} t^{\frac{2}{3}} - \frac{8}{3} t^{5/3}$$

Then
$$\left.\frac{dk}{dt}\right|_{t=1} = 2 \cdot \frac{5}{3}(1)^{\frac{2}{3}} - \frac{8}{3}(1)^{5/3}$$
$$= \frac{10}{3} - \frac{8}{3}$$
$$= 2/3 \leftarrow \text{slope}$$

Step 2: Find a point (x_1, y_1) on the tangent line.

- Next we know that the tangent line must pass through the point $(1, k(1))$.
- Meaning $y - k(1) = \frac{2}{3}(t - 1)$ is the eq.

Find $k(1) = 1 \cdot (2 - 1) = 1 \quad (y_1)$

So $y - 1 = \frac{2}{3}(t-1) = \frac{2}{3}t - \frac{2}{3}$

$$\Rightarrow \boxed{y = \frac{2}{3}t + \frac{1}{3}}$$

Applications:

Question 1 (Slopes of tangent lines)

Find the equation of the tangent line to

$$f(x) = 4x - 8\sqrt{x} \quad \text{at } x = 16$$

Question 2: Is $f(x) = 2x^3 + \frac{300}{x^3} + 4$ increasing, decreasing, or neither at $x = -2$?