Discussion Class

October 12, 2023

Last time:

- Derivatives of trigonometric functions
- The Chain Rule

Today:

① Implicit differentiation,

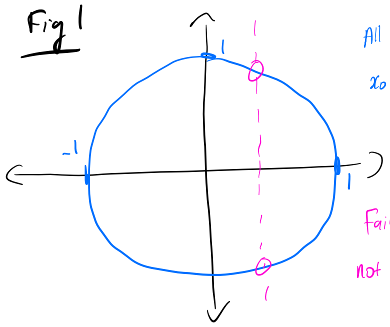
② Derivatives of inverse functions and logarithms.

① Implicit differentiation

- why do we need it?
- Why problem does it solve?

Example 1: $x^2+y^2=1$

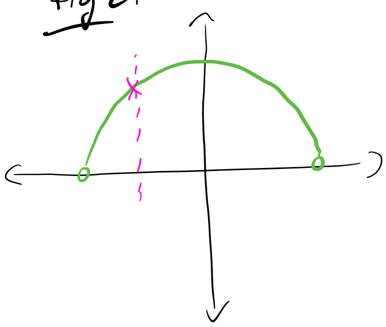
Fig 1



All the points (x_0, y_0) such that $x_0^2+y_0^2=1$ is drawn in blue.

Fails vertical line test, so it's not a graph of a function.

Fig 2:



All the points (x_0, y_0) s.t

- $y_0 > 0$ and
- $x_0^2+y_0^2=1$

is drawn on the graph here.

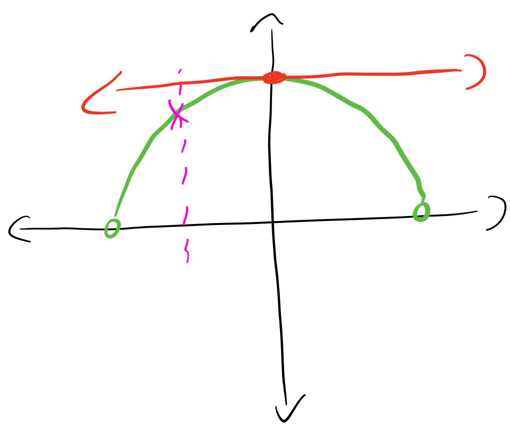
- This is the graph of some function since it satisfy the vertical line test.
- We want to find slopes of tangent line, so we want derivatives!
- Then we proceed:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(1)$$

$$2x+2y\cdot y' = 0$$

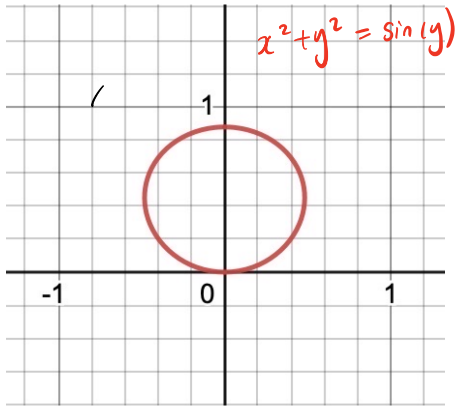
$$y' = \frac{-x}{y}.$$

What does $y'(0)$ mean?



In this example we can find y explicitly, but implicit differentiation comes to its own when we cannot find y explicitly.

Example: $x^2+y^2 = \sin(y)$



All points (x,y) that satisfy $x^2+y^2=\sin(y)$.

Then
$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(\sin(y))$$
$$2x + 2y\cdot y' = \cos(y)\cdot y'$$
$$2y\cdot y' - \cos(y)\cdot y' = -2x$$
$$y'(2y-\cos(y)) = -2x$$

$$\boxed{y' = \frac{-2x}{2y-\cos(y)}}$$

Example: (Implicit differentiation) Find y'.

$x^2+xy-y^2=4$, $y>2$ so that y is a well-defined function

$$\frac{d}{dx}(x^2+xy-y^2) = \frac{d}{dx}(4)$$
$$2x + y + xy' - 2y\cdot y' = 0$$
$$(x-2y)y' = -2x-y$$

$$y' = \frac{-2x-y}{x-2y}$$

Example 2: $y\cos x = x^2+y^2$, $y > \frac{1}{2}$

Find y'.

$$\frac{d}{dx}(y\cos(x)) = \frac{d}{dx}(x^2+y^2)$$

$$y'\cos(x) - y\sin(x) = 2x + 2y\cdot y'$$

$$y'\cos(x) - 2y\cdot y' = 2x + y\sin(x)$$

$$y'(\cos(x) - 2y) = 2x + y\sin(x)$$

$$\boxed{y' = \frac{2x+y\sin(x)}{\cos(x)-2y}}$$

Example 3: $e^{x/y} = x-y$, for all $x<0$

Find y'.

$$\frac{d}{dx}\left(e^{x/y}\right) = \frac{d}{dx}(x-y)$$

$$e^{\frac{x}{y}}\cdot\frac{d}{dx}\left(\frac{x}{y}\right) = 1-y'$$

$$e^{\frac{x}{y}}\cdot\left(\frac{1\cdot y - x\cdot y'}{y^2}\right) = 1-y'$$

$$e^{x/y}\cdot\frac{1}{y} - e^{\frac{x}{y}}\cdot x\cdot y' = 1-y'$$

$$e^{x/y}\cdot\frac{1}{y} - 1 = e^{x/y}\cdot x\cdot y' - y'$$

$$e^{x/y}\cdot\frac{1}{y} - 1 = \left(e^{\frac{x}{y}}\cdot x - 1\right)y'$$

$$y' = \frac{e^{\frac{x}{y}}\cdot\frac{1}{y} - 1}{e^{\frac{x}{y}}\cdot x - 1}$$

$$\boxed{y' = \frac{e^{\frac{x}{y}} - y}{y\left(e^{\frac{x}{y}}\cdot x - 1\right)}}$$

Example 4: (Inverse trigonometric functions)

- $\sin^{-1}(x)$
- $\cos^{-1}(x)$
- $\tan^{-1}(x)$
- $\csc^{-1}(x)$
- $\sec^{-1}(x)$
- $\cot^{-1}(x)$

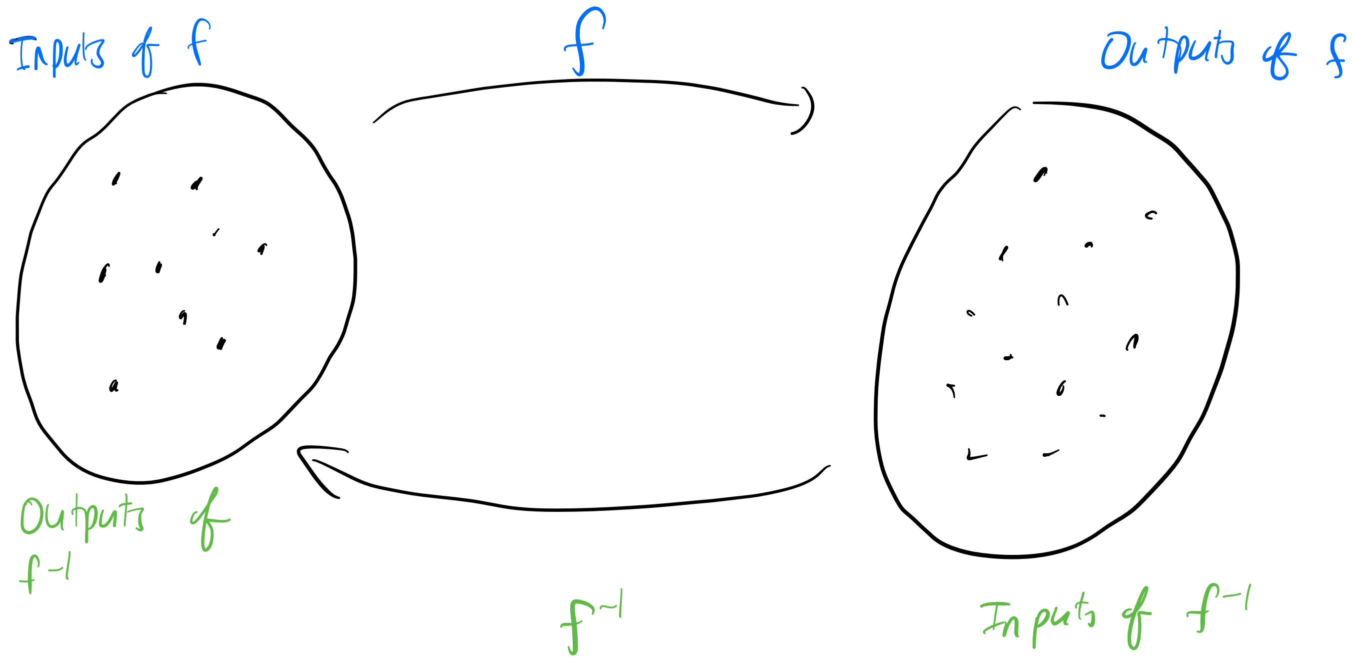
Let's find $\frac{d}{dx}\left(\cos^{-1}(x)\right)$

Solve.

More Implicit Differentiation:

- $e^y \cos x = 1 + \sin(xy)$
- $x \sin y + y \sin x = 1$
- $\tan(x-y) = \frac{y}{1+x^2}$

② Derivatives of inverses



By definition of being an inverse we know

- $f^{-1}(f(x)) = x$
- $f(\underbrace{f^{-1}(x)}_{\text{Apply } f^{-1} \text{ to } x}) = x$

Apply f to the point $f^{-1}(x)$.

So we know: $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$

- f^{-1} : inverse of f
- f' : derivative of f

plug in $f^{-1}(x)$ into $f'(x)$

evaluate f^{-1} at x

Example: Given $f(x) = x^5 + 3x^2 - x + 2$

and g as the inverse of f.

Find $g'(5)$.

We know $g'(5) = \dfrac{1}{f'(g(\ ))}$

And $g(5) = y$ iff $f(y) = 5$ (inverses of each other)

And $y^5 + 3y^2 - y + 2 = 5$

With guess and check $y = 1$ works.

Hence $y = 1$ and so $g(5) = 1$

So $g'(5) = \dfrac{1}{f'(1)}$, where $f'(1)$

$= 5(1)^4 + 6(1) - 1$

$= 10$

$$\boxed{g'(5) = \frac{1}{10}}$$

② Using logarithms to find derivatives.

Example :

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Take logarithm on both sides:

$$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$\ln y = \ln(x^{3/4}) + \ln\left((x^2+1)^{\frac{1}{2}}\right) - \ln\left((3x+2)^5\right)$$

$$\ln y = 3/4 \ln(x) + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Take derivative on both sides with respect to x

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(3/4\ln(x) + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right)$$

↓ implicit diff

$$\frac{1}{y}\cdot y' = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{3x+2}\cdot 3$$

$$y' = \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)\cdot y$$

$$y' = \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

Example: Differentiate $y = x^{\sqrt{x}}$ (*)

Take natural logarithm on both sides

$$\ln(y) = \ln(x^{\sqrt{x}})$$

$$\ln(y) = \sqrt{x}\,\ln(x)$$

Take derivative on both sides

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x^{\frac{1}{2}} \cdot \ln(x) \right)$$

implicit differentiation

product rule

$$\frac{1}{y} \cdot y' = \frac{1}{2} x^{-\frac{1}{2}} \cdot \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x}$$

$$y' = \left(\frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} \right) y$$

$$y' = \frac{\ln x + 2}{2\sqrt{x}} \cdot y$$

$$y' = \frac{\ln x + 2}{2\sqrt{x}} \cdot x^{\sqrt{x}}$$