Discussion Class

October 19, 2023

L18 and L19 Material

Last time

- Implicit Differentiation
- Derivatives of inverses and using logarithms to calculate derivatives

Today:

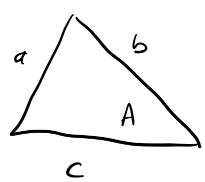
- Related Rates
- Linear approximations and differentials.

1. Related Rates

1. Identify two quanties (variables) that change with respect to time.
2. Find a relationship between them.
3. Differentiate with respect to time.
4. Solve the unknown at a specific point in time.

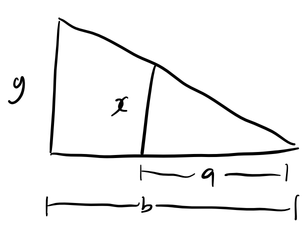
Common relationships :

1. • Pythagorean identity : $a^2+b^2=c^2$
2. • Trigonometric relationship: $\sin, \cos, \tan, \ldots$
   (angle and side) $\arcsin, \arccos, \ldots$
3. • Law of cosine :



$a^2 = b^2+c^2-2bc\cos(A)$

4. • Similar triangles



$\frac{x}{a} = \frac{y}{b}$

5. Volumes | areas :

$V = \frac{\pi r^2 \cdot h}{3}$ (cone)

Example 1 ; (Area example - 5)

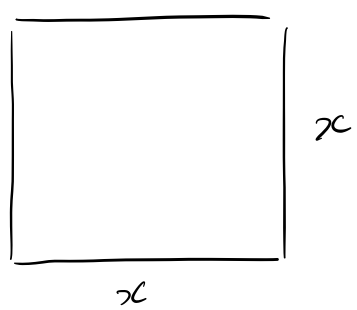
Q: 13. Consider a square with side lengths x measured in centimeters. If the side lengths of the square are increasing at a rate of 2 centimeters per second, then at what rate is the area, A, of the square increasing when sides have length 10 centimeters?

1 two quantities : A - area

x - side length

2 relationship.

$A = x^2$



constant rate

Given $\frac{dx}{dt} = 2 \frac{cm}{sec}$, Find $\left.\frac{dA}{dt}\right|_{x=10}$.

3

So $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$, $\frac{dA}{dx} = 2x$.

And at the point in time when $x = 10\,cm$,

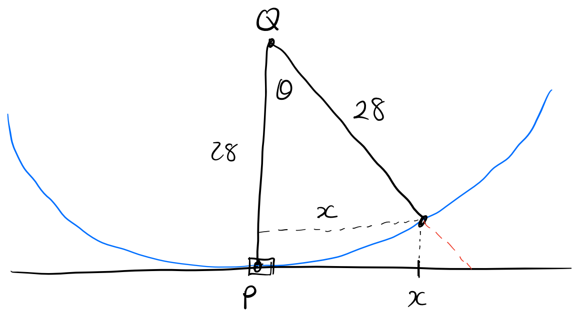
we know $\left.\frac{dA}{dx}\right|_{x=10} = 20$. 4

So $\left.\frac{dA}{dt}\right|_{x=10} = 20 \cdot 2 = \boxed{40} \frac{cm^2}{sec}$

Example 2 : (Side lengths and angles)

Q : Given a swing with a 26 feet rope, denote the bottom by point P and top by point Q. Some pushes the swing from its dead hang position to the right at a constant rate of $10\,\frac{ft}{sec}$. What is the angular velocity at time $t=1$?

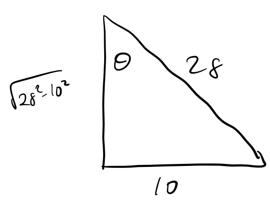
Consider the following picture



Given $\frac{dx}{dt} = 10$, find $\frac{d\theta}{dt}$ at time $t=1$.

1. quantities : θ and x
2. Relationship : $\sin\theta = \frac{x}{28}$

Then
$$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{x}{28}\right)$$
$$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{28}\frac{dx}{dt}$$



$$\cos\theta = \frac{\sqrt{28^2-10^2}}{28}$$

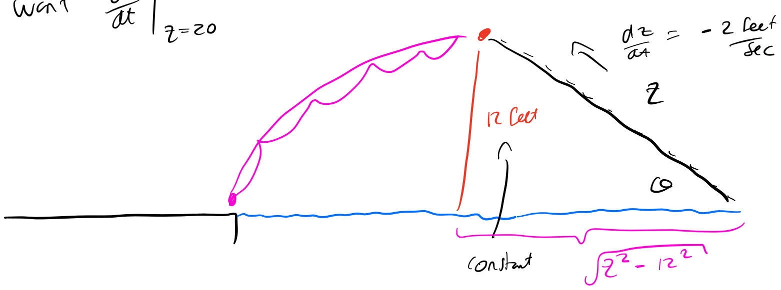
$$\left.\frac{d\theta}{dt}\right|_{t=1} = \frac{1}{\cancel{28}} \cdot 10 \cdot \frac{\cancel{28}}{\sqrt{28^2-10^2}}$$

$$\left.\frac{d\theta}{dt}\right|_{t=1} = \boxed{\frac{10}{\sqrt{28^2-10^2}}}$$

Example 3 ; (Exam 3A, Fall 2021)

3. Suppose someone who is fishing is standing on the edge of a dock using a fishing pole to catch a fish. If the tip of the fishing pole is 12 feet above the water and the line is being reeled in at a rate of 2 feet per second, then at what rate is the angle θ between the line and the water changing when there are 20 feet of line out?

we want $\left.\frac{d\theta}{dt}\right|_{z=20}$



- $\sin\theta = \frac{12}{z}$ (Relationship between θ and z)

well $\frac{d}{dt}(\sin\theta) = \frac{d}{dt}(12z^{-1})$

$$\cos\theta \cdot \frac{d\theta}{dt} = -12z^{-2}\frac{dz}{dt} = \frac{-12}{z^2}\frac{dz}{dt}$$

$$\frac{d\theta}{dt} = \frac{-12}{z^2\cos\theta}\cdot\frac{dz}{dt}$$

And at $z=20$, we know $\cos(\theta) = \frac{\sqrt{20^2-12^2}}{20}$

So $\left.\frac{d\theta}{dt}\right|_{z=20} = \frac{-12}{20^2}\cdot\frac{20\cdot(-2)}{\sqrt{20^2-12^2}} = \frac{-12}{20^2}\cdot\frac{20}{16}\cdot -2 \ \frac{\text{rad}}{\text{sec}}$

$$= \frac{24}{16\cdot 20}$$

$$= \frac{6}{16\cdot 5}$$

$$= 3/40\ \frac{\text{rad}}{\text{sec}}.$$

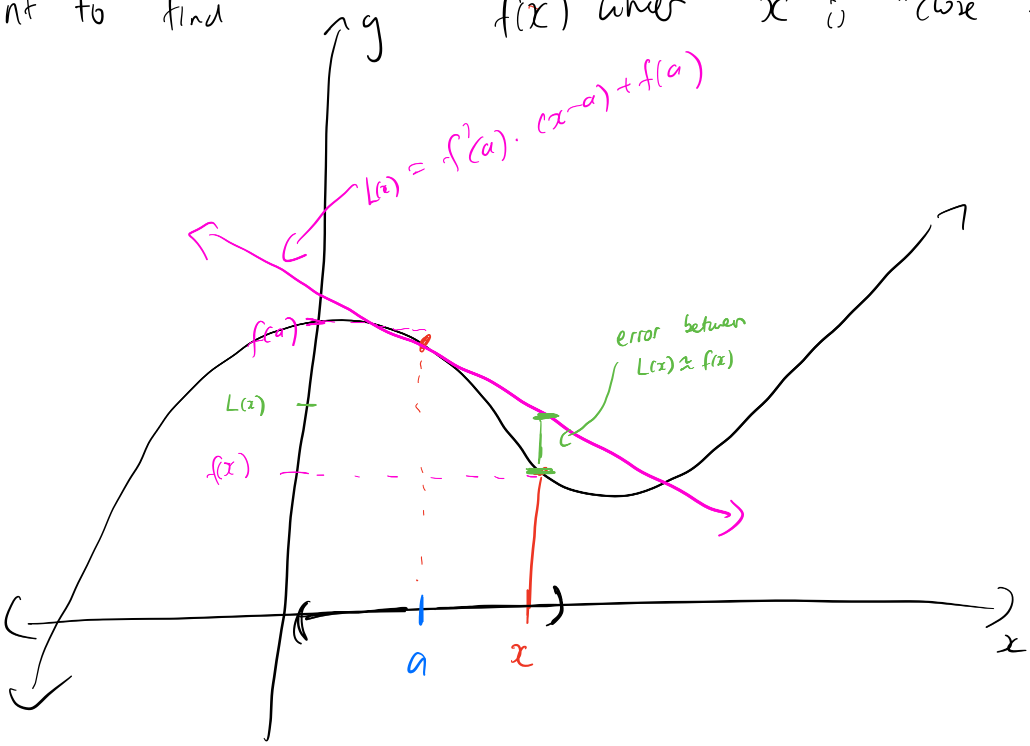
② Linear Approximation

Main Idea:

Suppose we are given a differentiable function $f: \mathbb{R} \to \mathbb{R}$ and a value a. And we want to find $f(a)$. Sometimes $f(a)$ is difficult to calculate but $f(a) \approx f(a+h)$ for some h small, and $f(a+h)$ is easy to find. Think of $\ln(0.95)$ and $\ln(1)$ or

$$\sin\left(\frac{109\pi}{108}\right) \approx \sin(\pi).$$

Say we have f, and we know what $f(a)$ is. We want to find $f(x)$ when x is "close" to a.



We know $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

So if x is "close enough" to a then

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

$$\Rightarrow f'(a) \cdot (x-a) \approx f(x) - f(a)$$

$$\Rightarrow \boxed{f(x) \approx \underbrace{f'(a)(x-a) + f(a)}}$$ call this $L(x)$.

value we want to approximate

assuming these are easier to calculate than $f(x)$.

We put "close" in quotation because how close we need to get would depend on how accurate we want our esstimation.

Example: Use linearization of

$$f(x) = (x^2 - 3x - 2)\cos(x) \quad \text{at} \quad x = 0$$

to approximate $f(-0.1)$

- $f'(x) = (2x-3)\cos(x) + (x^2-3x-2)\cdot(-\sin(x))$
- $f(0) = -2$ $\qquad f'(0) = -3$

So $\quad f(x) \approx \boxed{f'(0)(x-0) + (-2)}$

$\quad = -3x - 2$

a lot easier to calculate than $f(-0.1)$

plug in 0.1

So $\quad f(0.1) \approx -3(-0.1) - 2 = -1.7$