Discussion class

Lecture 19 and 20

October 26, 2023

Last time

- Related rates
- Linear approximation

Today

- Differentials
- Extreme values. (Importent topic)

• Review: Problem 5, Exam 2

- Notes: ① radius is changing so $\frac{dr}{dt} \neq 0$

② Needed to use $\frac{h}{r} = \frac{16}{4} \Rightarrow r = \frac{1}{4}h$

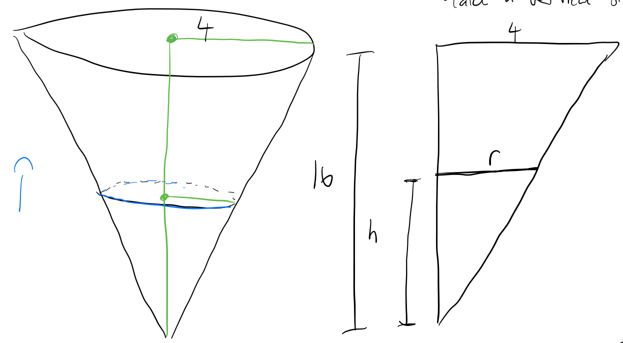
and $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 \cdot h$

Exam 2 : Problem 5

5. A container in the shape of a right circular cone with vertex pointed down has radius $4\,m$ and height $16\,m$. If water is poured into the container at a constant rate of $16\,m^3/min$ how fast is the water level rising when the water is $8\,m$ deep? Note: The volume of a cone formula is $V = \frac{\pi r^2 h}{3}$.

- Given : $V = \frac{\pi r^2 h}{3}$

Take a vertical slice



Note $\frac{r}{h} = \frac{4}{16}$

$\boxed{r = \frac{1}{4} h}$ important!

What is $\frac{dr}{dt}$?

Since $r = \frac{1}{4}h$, we know

$\boxed{\frac{dr}{dt} = \frac{1}{4}\frac{dh}{dt}}$

- Given : Water pours into the container at a constant rate $16 \frac{m^3}{min}$

- What is $\left.\frac{dh}{dt}\right|_{h=8}$.

- Method 1

Plug in $r = \frac{1}{4}h$ to get

$V = \frac{\pi}{3}\left(\frac{1}{4}h\right)^2 h$

$V = \frac{\pi}{48} h^3$

Then $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$\Rightarrow \frac{dV}{dt} = \frac{3\pi}{48} h^2 \cdot \frac{dh}{dt}$

$3 \overline{)48}$ = 16

Fix point when $h = 8$

Then $16 = \frac{\pi}{16} \cdot 64 \cdot \left.\frac{dh}{dt}\right|_{h=8}$

$\Rightarrow 16 = 4\pi \left.\frac{dh}{dt}\right|_{h=8}$

$\Rightarrow \boxed{\left.\frac{dh}{dt}\right|_{h=8} = \frac{\pi}{4} \frac{m}{min}}$.

- Method 2

$\frac{dV}{dt} = \frac{\pi}{3} 2r\left(\frac{dr}{dt}\right)h + \frac{\pi}{3}r^2\frac{dh}{dt}$ (product rule)

$\frac{dV}{dt} = \frac{2\pi}{3} r \cdot \left(\frac{1}{4}\frac{dh}{dt}\right) h + \frac{\pi}{3} r^2 \frac{dh}{dt}$

Fix point in time when $h = 8$.

So $r = \frac{1}{4}(8) = 2$

$16 = \left(\frac{2\pi}{3}(\cancel{2}) \cdot \frac{1}{\cancel{4}} \cdot 8 + \frac{\pi}{3} \cdot 2^2\right) \left.\frac{dh}{dt}\right|_{h=8}$

$16 = \left(\frac{8\pi}{3} + \frac{4\pi}{3}\right) \left.\frac{dh}{dt}\right|_{h=8}$

$16 = 4\pi \cdot \left.\frac{dh}{dt}\right|_{h=8}$

$\boxed{\left.\frac{dh}{dt}\right|_{h=8} = \frac{16}{4\pi} = \frac{4}{\pi} \frac{m}{min}}$

• Extreme Values:

Given a function $f: \mathbb{R} \to \mathbb{R}$:

- Global / Absolute maximum and minimum?

- Local maximum and minimum?

— Local maximum: f attains a local maximum at c if there exists an open interval I contained in the domain of f s.t $f(c) \geq f(x)$ for every $x \in I$.

— Local minimum: f attains a local minimum at c if there exists an open interval I contained in the domain of f s.t $f(c) \leq (x)$ for every $x \in I$.

- Extreme value theorem:

  - If f is continuous on a closed interval $[a,b]$, then f obtain its maximum and minium on $[a,b]$.

- Critical number / value:

  — A number c in the domain of f is called a critical value of f provided $f'(c) = 0$ or $f'(c)$ DNE.

- Fermat's thm

  - If f has a local extrema at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

  - Important: If f has a local extrema at $x = c$, then c is a critical number.

  - That is c being a critical number is a necessary condition for f to have an extreme value at c.

  - We want to find extreme values. Fermat's theorem tells us we only have to look through the critical values to find local extrema. What about global extrema?

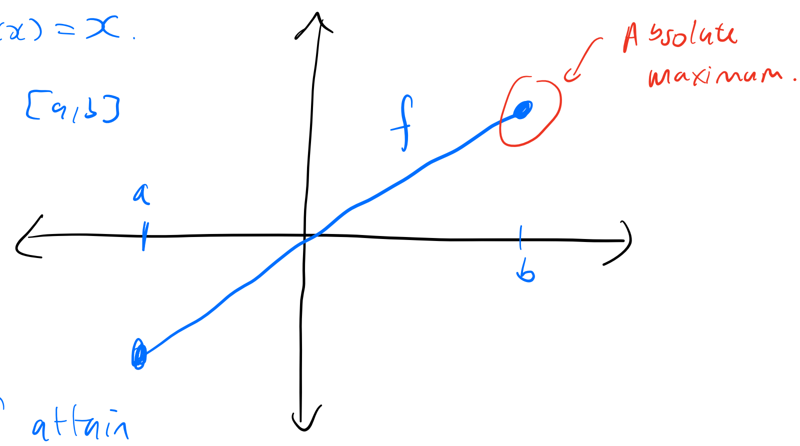
- Given $f:[a,b]\to\mathbb{R}$, can f have a local extrema at one of the endpoints a or b?
  - No, a cannot be a local extrema, because we cannot find an interval I s.t $a\in I$, I open interval, and $I\subseteq[a,b]$. Same idea for b.

<u>Example</u>: $f(x)=x$.

$f'(x)=1$ on $[a,b]$

So $f'(b)=1$.

Hence b not a critical point. But f attain absolute maximum at b.



- <u>Conclusion</u>: Extreme values of a function can occur at 1) Critical points
  2) End points.

- <u>Test in Understanding</u>: (Justify or give a counter example)
  - Can a function have no extreme values?

Examples: Find critical numbers of

1) $f(\theta) = \cos(\theta) + \sin^2(\theta)$, on $[0, 2\pi]$
2) $f(x) = x^{-2}\ln(x)$, on $(0, \infty)$
3) $g(\theta) = 4\theta - \tan\theta$, on $[0, 2\pi]$

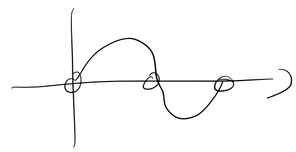
1) $f'(\theta) = -\sin\theta + 2\sin\theta \cdot \cos(\theta)$

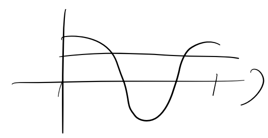
Then $-\sin\theta + 2\sin\theta \cdot \cos\theta = 0$

$\Rightarrow (-\frac{1}{2} + \cos\theta)(\sin\theta) = 0$

$\Rightarrow \boxed{\theta = 0, \pi, 2\pi}$

And $\cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta = \frac{\pi}{3}, \quad \theta = \frac{5\pi}{6}}$





2) $f'(x) = -2x^{-3}\ln(x) + x^{-2}\cdot\frac{1}{x} = \frac{-2\ln(x)}{x^3} + \frac{1}{x^3} = \frac{1 - 2\ln(x)}{x^3}$

- $x = 0$? Not included in $(0, \infty)$ so not a critical point.
- $1 - 2\ln(x) \Rightarrow \ln(x) = \frac{1}{2} \Rightarrow e^{\ln(x)} = e^{\frac{1}{2}} \Rightarrow \boxed{x = e^{\frac{1}{2}}}$

3) $g'(\theta) = 4 - \sec^2(\theta)$

Solve $4 - \sec^2(\theta) = 0$

$\Rightarrow \sec^2(\theta) = 4$

$\Rightarrow \sec(\theta) = \pm 2$

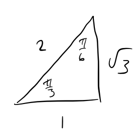
Note: g not defined at $\frac{\pi}{2}, \frac{3\pi}{2}$.

$\sec\theta = 2 \Rightarrow \cos(\theta) = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$

$\sec\theta = -2 \Rightarrow \cos(\theta) = -\frac{1}{2}$

$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$



Example: 1) Find absolute maximum and minimum :

$f(x) = \ln(x^2+x+1)$ on $[-1,1]$

2) $f(x) = x - 2\tan^{-1}(x)$, $[0,4]$

1) $f'(x) = \dfrac{2x+1}{x^2+x+1}$

- $2x+1=0 \Rightarrow \boxed{x = -\frac{1}{2}}$
- $x^2+x+1 = 0$? (No, has no real roots)

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i$$

(not in $\mathbb{R}$)

2) $f'(x) = 1 - \dfrac{2}{1+x^2}$ on $[0,4]$

$$1 - \frac{2}{1+x^2} = 0$$

$$\Rightarrow \frac{1+x^2-2}{1+x^2} = 0$$

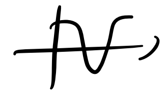
$$\Rightarrow \frac{x^2-1}{x^2+1} = 0 \Rightarrow (x^2-1) = 0 \Rightarrow \boxed{x = \pm 1} \quad , \quad -1 \notin [0,4]$$

$f(0) = 0$ $\qquad \tan(0) = 0 \Rightarrow \tan^{-1}(0) = 0$

$f(1) = 1 - 2 \cdot \frac{\pi}{4} = 1 - \frac{\pi}{2} < 0$ $\qquad$ (absolute min at 1)

$f(4) = 4 - 2\tan^{-1}(4) > 0$ $\qquad$ (absolute max at 4)

Example: $g(\theta) = 4\theta - \sec^2(\theta)$ on $[0, 2\pi]$



- Find critical points.
- Note g not defined where $\cos(\theta) = 0$, $\underline{\theta = \frac{\pi}{2}, \frac{3\pi}{2}}$.

- Now $g'(\theta) = 4 - 2\sec\theta \cdot \sec\theta \cdot \tan\theta$

$$g'(\theta) = 4 - 2\sec^2\theta \cdot \tan\theta$$

- Now $4 - 2\sec^2\theta \cdot \tan\theta = 0$

$$\Rightarrow 2 - \sec^2\theta \cdot \tan(\theta) = 0$$

$$\Rightarrow 2 - (1 + \tan^2\theta)\tan(\theta) = 0$$

$$\Rightarrow 2 - \tan\theta - \tan^3\theta = 0$$

$$\Rightarrow \tan^3\theta + \tan\theta - 2 = 0$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

- Set $x = \tan\theta$.

$\frac{\pi}{4}$

- Solve $x^3 + x - 2 = 0$
- By inspection find $x = 1$ as a root, since $1^3 + 1 - 2 = 0$.
- Then

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x - 1 & x^3 + x - 2 \\ & x^3 - x^2 \\ \hline & x^2 + x \\ & x^2 - x \\ \hline & 2x - 2 \\ & 2x - 2 \end{array}$$

Continued:

So $(x^3+x-2) = 0$

$\Rightarrow (x-1)(x^2+x+2) = 0$

$x-1=0 \Rightarrow \tan\theta - 1 = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$x^2+x+2=0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1^2-4(1)(2)}}{2}$

$\Rightarrow x = \dfrac{-1}{2} \pm \dfrac{\sqrt{-7}}{2} = \frac{-1}{2} \pm \frac{\sqrt{7}}{2} \cdot i$

So $\tan(\theta) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2} \cdot i$, and since tan outputs only real numbers, no $\theta \in \mathbb{R}$ satisfy this.

Hence the only two critical points are $\frac{\pi}{4}, \frac{5\pi}{4}$.