Discussion Nov 30, 2023

Last time

- Anti-derivatives / Indefinite Integrals (L27)

Today

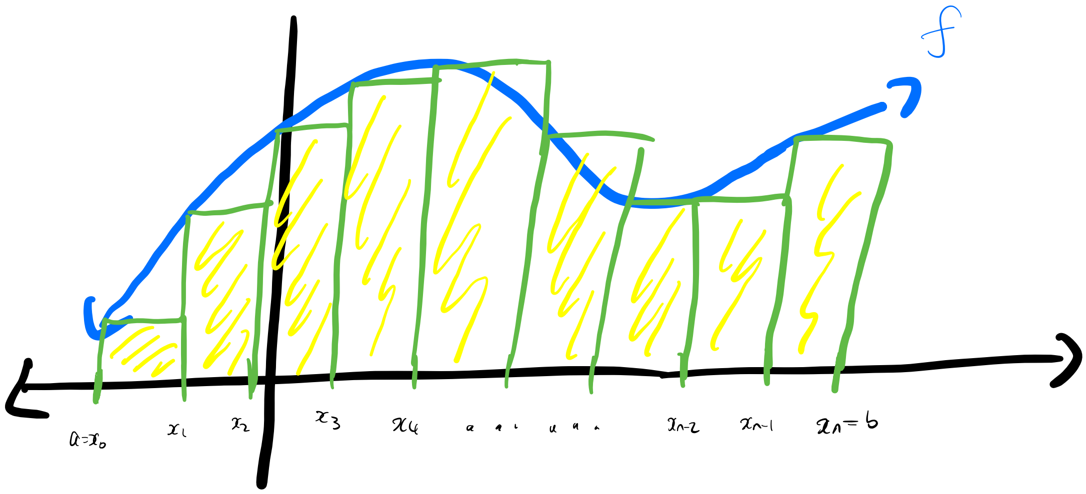
- Areas and Riemann Sums (L28)
- The Definite Integral. (L29)

Admin

- Course Evaluations (Due Dec 8)
  - Please complete them. Be honest about feedback, and provide constructive feedback
  - The best way for me to get better.
- Last discussion class

Areas and Riemann Sums:

The Area Problem: Given a curve, how do we find the area under the curve?



- **Riemann Sum:** Given $f:[a,b]\to\mathbb{R}$, $a=x_0<x_1<x_2<\ldots<x_n=b$

the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ where $x_i^*\in[x_{i-1},x_i]$ is called a Riemann sum.

- **Right, Middle, left-endpoint approximations:** (Examples of Riemann Sums)

$[x_{i-1}, x_i]$

- $R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$, $x_i^* = x_i$ — x_{i-1}, x_i (right end point approximation)
- $L_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$, $x_i^* = x_{i-1}$ — x_{i-1}, x_i (left end point approximation)
- $M_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$, $x_i^* = x_{i-1}+\frac{\Delta x}{2}$ — x_{i-1}, x_i (middle point approximation)

- **Definite Integral:**

Given a function $f:[a,b]\to\mathbb{R}$ we define

$$\int_a^b f(x)\,dx := \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $a=x_0<x_1<x_2<\ldots<x_n=b$ a partision of $[a,b]$ and $x_i^*\in[x_{i-1},x_i]$

If the limit exists, then we say f is Riemann Integrable

- There are functions that are not Riemann integrable.

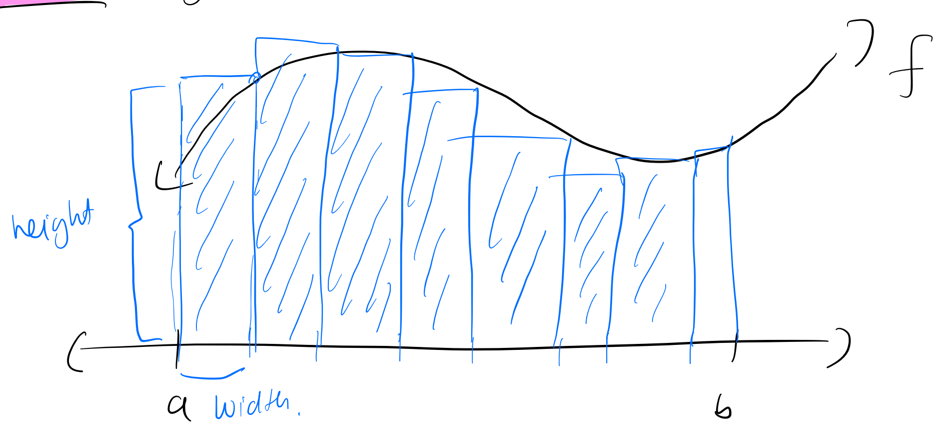
But this need not be a problem since all the functions in this course are continuous, or continuous except at finitely many points:

Thm: If $f: [a,b] \to \mathbb{R}$ continuous, or if f has at most finitely many points of discontinuity, then
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x \text{ exists.}$$

3 ways to visualize Integrals

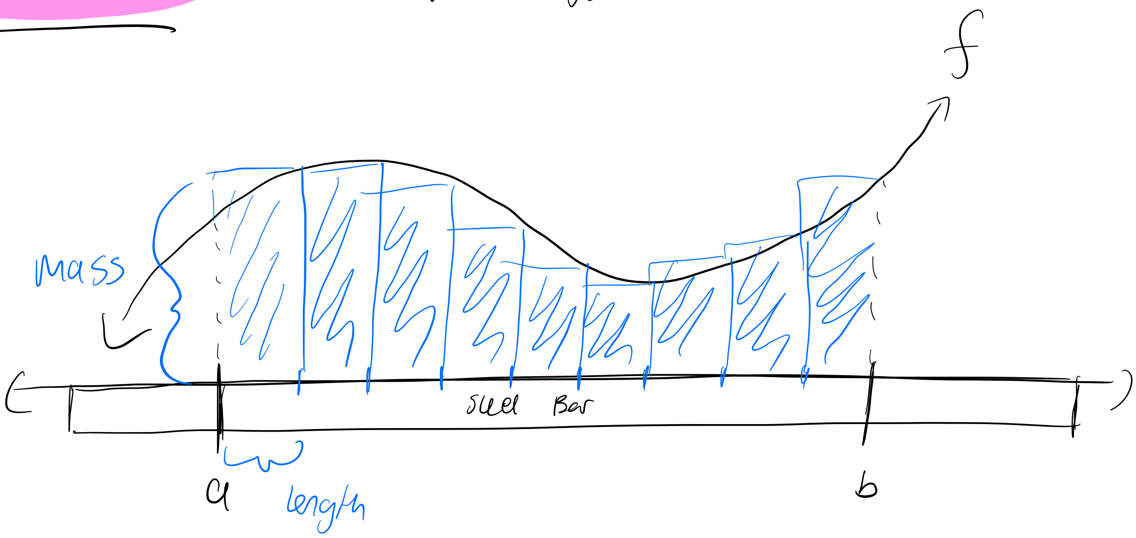
Area, Mass, Electric charge

1. Area: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, $a = x_0 < x_1 < \dots < x_n = b$.



2. Mass :

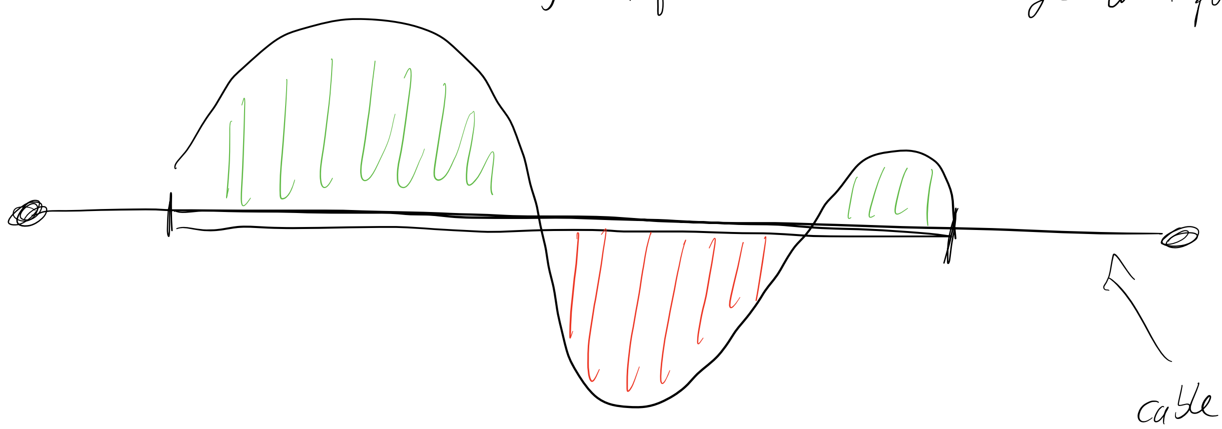
density $= \frac{\text{mass}}{\text{volume}}$ $\Rightarrow$ mass $=$ density $\cdot$ Volume



f : input is point on the rod and output is density at that point.

3.) Net Electric Charge

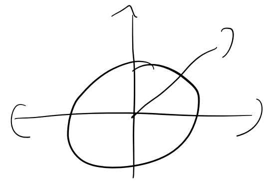
f - represents electric charge at a point

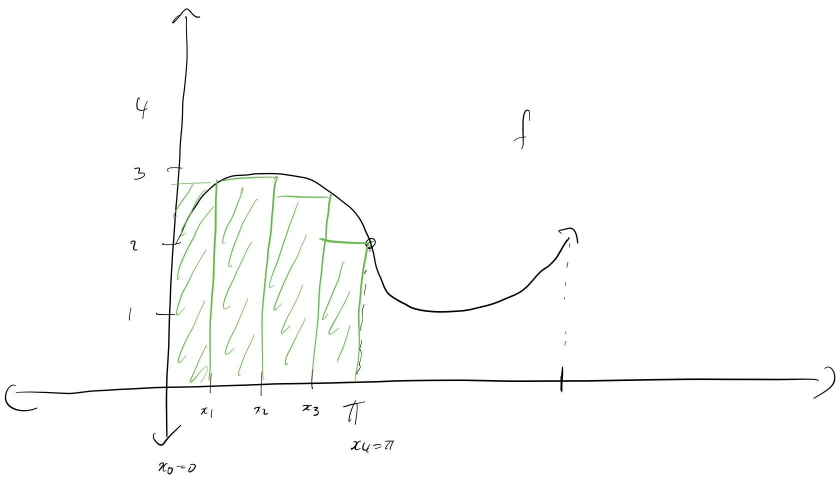


$\int_a^b f(x)\,dx$ = net charge over the rod from $[a,b]$

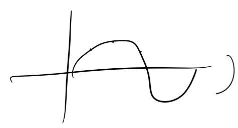
Example: Evaluate the upper and lower sum for $f(x) = 2+\sin x$

$n=4$, $0, \pi$





$$R_4 = \sum_{i=1}^{4} f(x_i)\cdot \Delta x \quad , \quad \Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$x_1 = 0 + \frac{\pi}{4} = \frac{\pi}{4}$, $f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

$x_2 = 0 + 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = 2+1$

$x_3 = 0 + 3\cdot\frac{\pi}{4} = \frac{3\pi}{4}$, $f(\frac{3\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

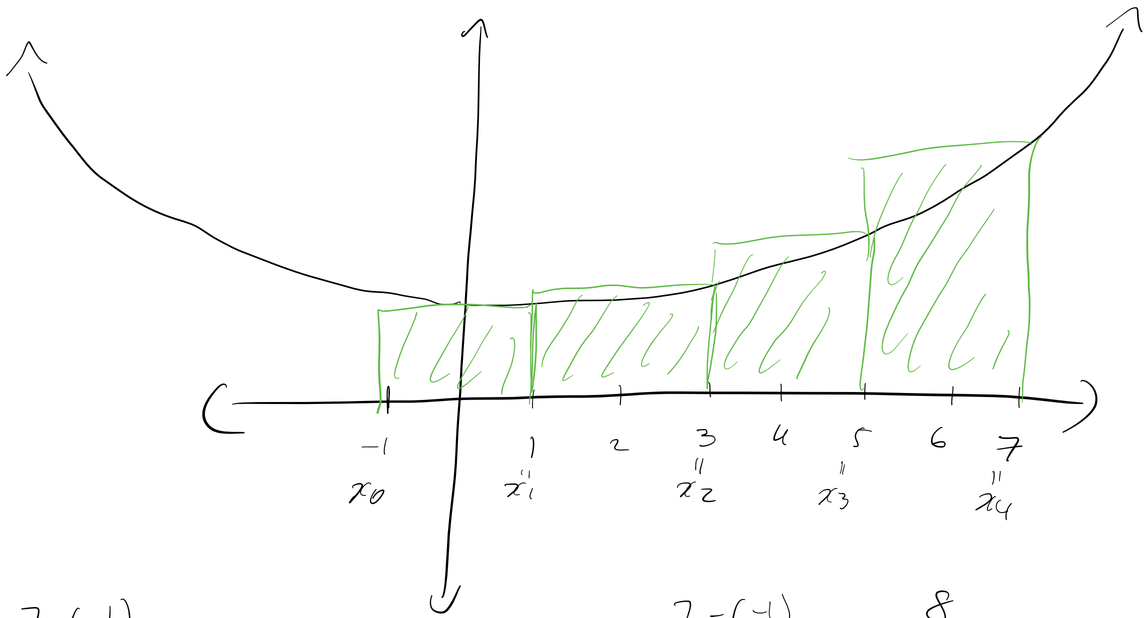
$x_4 = 0 + 4\cdot\frac{\pi}{4} = \pi$, $f(\pi) = 2+0$

Then $R_4 = \sum_{i=1}^{4} f(i\cdot\frac{\pi}{4})\cdot\frac{\pi}{4} = (9+\sqrt{2})\cdot\frac{\pi}{4}$

$$L_4 = \sum_{i=0}^{3} f(x_i)\cdot\Delta x = (9+\sqrt{2})\cdot\frac{\pi}{4}$$

Example (Final Exam Fall 2022)

Given $f(x) = x^2+1$ on $[-1,7]$. Find the right endpoint Riemann sum approximation of $\int_{-1}^{7} f(x)\,dx$ with $n=4$ rectangles.



$$\Delta x = \frac{7-(-1)}{4} = 2 \quad , \quad \Delta x = \frac{7-(-1)}{n} = \frac{8}{n}$$

And $x_i = x_0 + i \cdot \Delta x = -1 + i \cdot \frac{8}{n}$

general

Then
$$\boxed{\begin{aligned}\int_{-1}^{7} f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(-1+\frac{8i}{n}\right)^2+1\right)\cdot\frac{8}{n}\end{aligned}}$$

$n=4$

So $\boxed{R_4 = \sum_{i=1}^{4}\left((-1+2i)^2+1\right)\cdot 2}$ (Approximation)

Properties of the Definite Integral

Let $f:[a,b] \to \mathbb{R}$ be continuous.

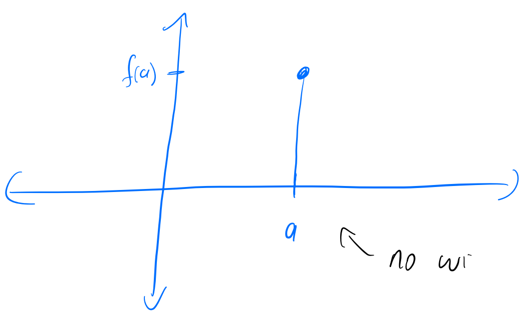
How is a definite Integral defined?

Through sums and limits. Therefore if a property is preserved by sums and limits it preserved by definite integrals.

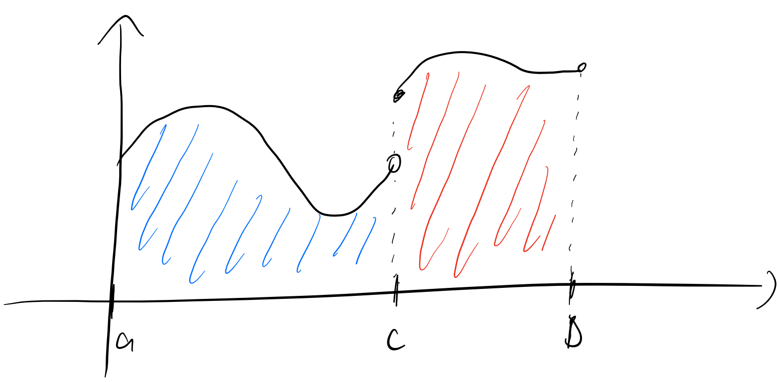
1) Constant multiple: $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ (special case $\int_a^b c\,dx = c(b-a)$)

2) Sums and differences: $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3) Endpoints equal: $\int_a^a f(x)\,dx = 0$



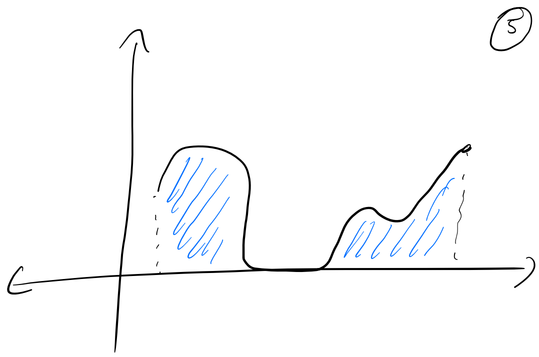
4) Splitting: For $a \leq c \leq b$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



More properties

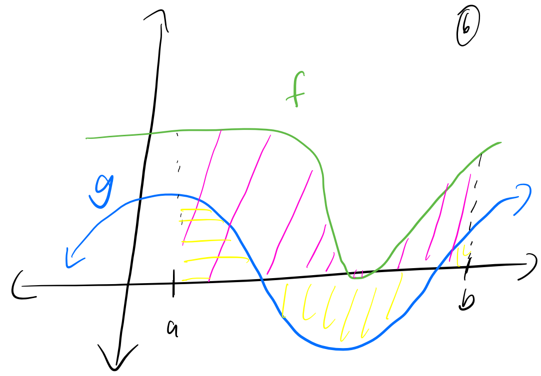
5.) If $f(x) \geq 0$ for all x in $[a,b]$,

then $\int_a^b f(x)\,dx \geq 0$.



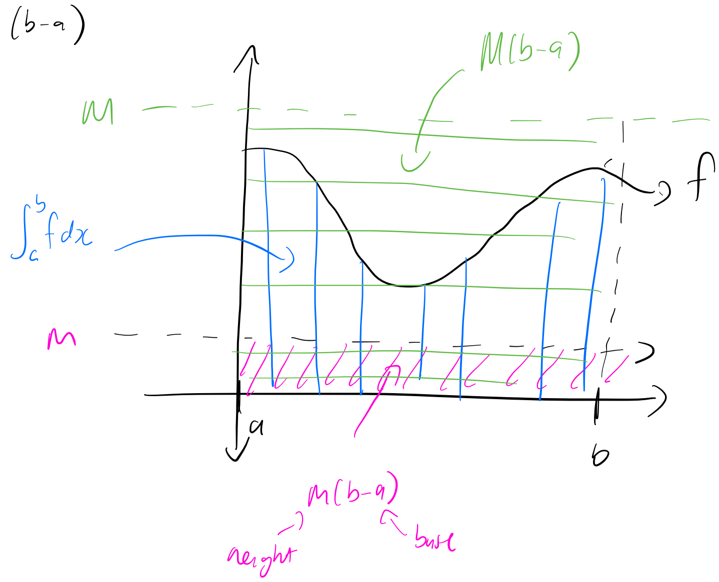
6.) If $g(x) \leq f(x)$ for every x in $[a,b]$,

then $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$.



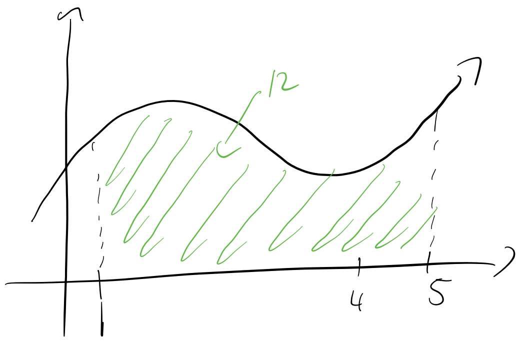
7.) If $m \leq f(x) \leq M$ for all x in $[a,b]$,

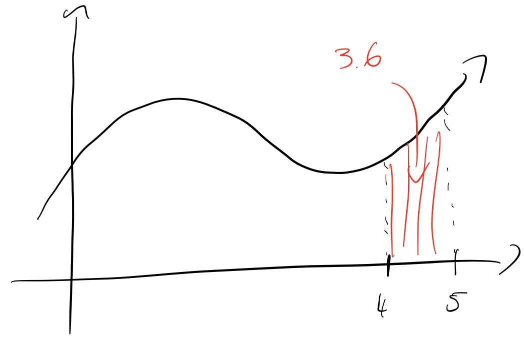
then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

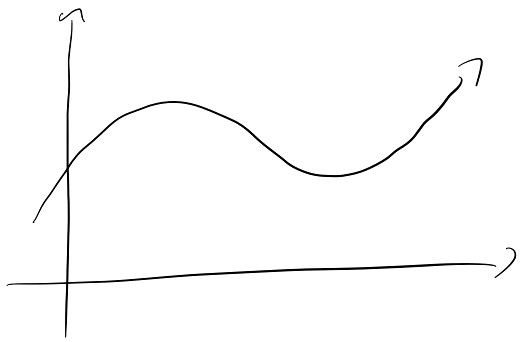


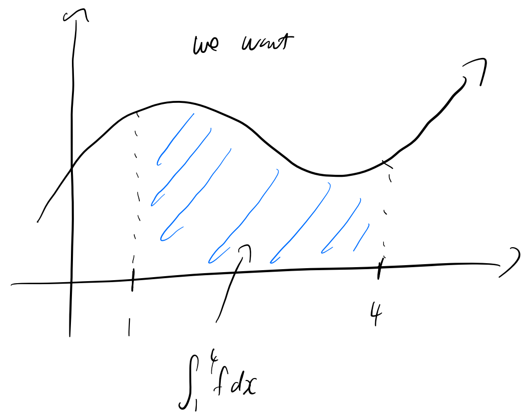
Examples: Use properties of the definite integral to find the following:

Example 1: Given $\int_1^5 f\,dx = 12$, $\int_4^5 f\,dx = 3.6$, find $\int_1^4 f\,dx$









$$\text{So} \quad \int_1^5 f\,dx - \int_4^5 f\,dx = \int_1^4 f\,dx$$

$$\Rightarrow \int_1^4 f\,dx = 12 - 3.6 = \underline{\underline{8.4}}$$

Example 2:

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1}\,dx = 0$$

Why?

Example 3:

$$\int_{6}^{-10} f(x)\,dx = 23 \quad , \quad \int_{-10}^{6} g(x) = -9$$

Find $\displaystyle\int_{-10}^{6} 2f(x) - 10\,g(x)\,dx$

$$= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

$$= -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

$$= -2(23) - 10(-9)$$

$$= -46 + 90$$

$$= 44$$

Examples : Suppose f, g continuous functions.

Given $\int_{-1}^{1} f(x)\,dx = 7$, $\int_{-1}^{4} 2g(x)\,dx = 4$, $\int_{-1}^{4} [f(x) + g(x)]\,dx = 9$

What is $\int_{1}^{4} f(x)\,dx$?

- $\int_{-1}^{4} 2g(x)\,dx = 4 \Rightarrow \int_{-1}^{4} g(x) = 2$.

- $\int_{-1}^{4} f(x) + g(x)\,dx = \int_{-1}^{4} f(x)\,dx + \int_{-1}^{4} g(x)\,dx$

$\Rightarrow \quad 9 = \int_{-1}^{4} f(x)\,dx + 2$

So $\int_{-1}^{4} f(x)\,dx = 7$

- Then $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$

$\Rightarrow \quad 7 = 7 + \int_{1}^{4} f(x)\,dx$

$\Rightarrow \quad \boxed{\int_{1}^{4} f(x)\,dx = 0.}$

Fundamental Theorem of Calculus:

Part 2:

If f is differentiable on $[a,b]$ and f' continuous on (a,b),

then $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Part 1:

Let f is continuous on $[a,b]$.

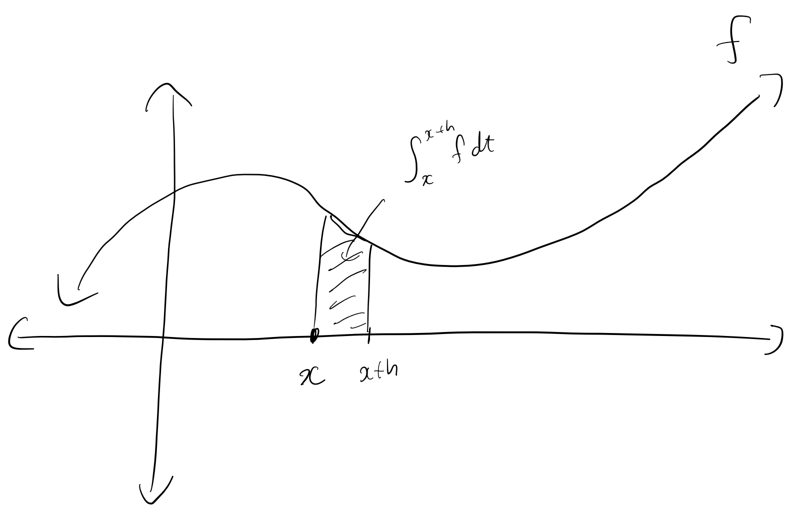
Define $F(x) = \int_a^x f(t)\,dt$ on $[a,b]$,

Then F is continuous and $F'(x) = f(x)$ on $[a,b]$.

Think about it:

$$f(x) = F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_x^{x+h} f(t)\,dt\right)$$



So as $h \to 0$, the average value $\frac{1}{h}\int_x^{x+h} f(t)\,dt \to f(x)$.

Consequence of Part I:

- Take f continuous and define $F(x) = \int_a^x f(t)\,dt$, x in $[a,b]$.

Then $F'(x) = f(x)$. That is $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

We want a formula show how to get

$$\boxed{\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = -f(v)\cdot\frac{dv}{dx} + f(u)\frac{du}{dx}}$$

We know $\frac{d}{dx}\underbrace{\overbrace{\int_a^x f(t)\,dt}^{F(x)}}_{\text{Function of } x} = f(x)$

Now $F(u) = \int_a^u f(t)\,dt$, $u(x)$

Then $\frac{d}{dx}F(u) = \frac{d}{du}F(u)\cdot\frac{du}{dx} = \frac{d}{du}\underbrace{\int_a^u f(t)\,dt}_{F(u)}\cdot\frac{du}{dx} = f(u)\cdot\frac{du}{dx}$

And $\frac{d}{dx}\int_v^a f(t)\,dt = -\frac{d}{dx}\int_a^v f(t)\,dt$, $v(x)$

$= -\frac{d}{dv}\left(\int_a^v f(t)\,dt\right)\cdot\frac{dv}{dx}$

$= - f(v)\cdot\frac{dv}{dx}$

And $\int_v^u f(t)\,dt = \int_v^a f(t)\,dt + \int_a^u f(t)\,dt$

$\Rightarrow \frac{d}{dx}\int_v^u f(t)\,dt = \frac{d}{dx}\int_v^a f(t)\,dt + \frac{d}{dx}\int_a^u f(t)\,dt$

$= -f(v)\cdot\frac{dv}{dx} + f(u)\cdot\frac{du}{dt}$, as required.

Example 1: Differentiate the following integral

$$\int_{\sqrt{x}}^{3x} \underbrace{t^2 \sin(1+t^2)}_{f(t)}\, dt$$

- $u(x) = 3x$, $v(x) = \sqrt{x}$, $f(t) = t^2 \sin(1+t^2)$

Then $$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = -f(v)\cdot\frac{dv}{dx} + f(u)\cdot\frac{du}{dx}$$

$$= -v^2\sin(1+v^2)\cdot\frac{1}{2}x^{-\frac{1}{2}} + u^2\sin(1+u^2)\,3$$

$$= -x\sin(1+x)\cdot\frac{1}{2}x^{-\frac{1}{2}} + 9x^2\sin(1+9x^2)\cdot 3$$

$$= -\frac{\sqrt{x}}{2}\cdot\sin(1+x) + 27x^2\sin(1+9x^2)$$