Discussion Class

November 2, 2023

Last time

- Differentials (L19)
- Extreme Values (L20)

Today

- Mean Value Theorem (L21)
- 1st Derivative Test (L22)
- Concavity and 2nd Derivative test (L23)

Rolle's Thm

Thm: If $f: [a,b] \to \mathbb{R}$ continuous on $[a,b]$, and differentiable on (a,b), and $f(a) = f(b)$, then there exists $c \in (a,b)$ s.t $f'(c) = 0$

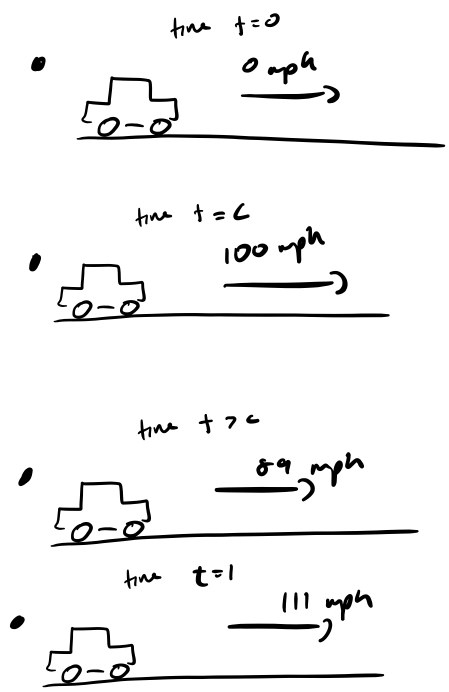
Mean Value Thm

Thm: If $f: [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Intuitively: If f represents velocity, it means over a time interval $[a,b]$, the average velocity and instentemeous velocity coincide at at least one point in time

- Think about it like this:
- If average 100 mph from time 0 to time 1 hour, then there must have been at least one c in time $(0,1)$ such that we drove exactly 100 mph at time c.



Example: Use the mean value theorem to show that for any real numbers $a < b$ we have

$$-1 \leq \frac{\cos(b) - \cos(a)}{b - a} \leq 1.$$

- To use the MVT we need 1) a function and 2) an interval.
- Consider the interval $[a,b]$ and function $f(x) = \cos(x)$.

  – Why can we apply MVT?

  ⋮

  – Then there exists c in (a,b) s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

  That is $-\sin(c) = \dfrac{\cos(b) - \cos(a)}{b - a}$.

Then $\left| \dfrac{\cos(b) - \cos(a)}{b - a} \right| = |-\sin(c)| \leq 1$

$\Rightarrow \quad -1 \leq \dfrac{\cos(b) - \cos(a)}{b - a} \leq 1$, as required.

Exercise 1: $f(x) = x^{\frac{2}{3}} - 2$, continuous on $[-1,1]$.

Note that $f(-1) = \sqrt[3]{(-1)^2} - 2$
$= \sqrt[3]{1^2} - 2$
$= f(1)$.

But $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \neq 0$ for any $x \in [-1,1] \setminus \{0\}$.
This does not contradict Rolle's thm because f is not differentiable on $(-1,1)$ since $f'(0)$ DNE.

Exercise 2: If $f(2) = -2$, $f'(x) \geq 1$ for x in $[2,5]$
How small can $f(5)$ be?
Suppose f satisfy hypothesis of MVT.
Then there exists c in $(2,5)$ s.t

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{f(5) - f(2)}{3}$$

Then $f'(c) \geq 1 \Rightarrow \frac{f(5) - f(2)}{3} \geq 1$
$\Rightarrow f(5) \geq 3 + f(2)$
$\Rightarrow f(5) \geq 3 - 2 = 1$.

So $f(5)$ at least 1.

Exercise 3: Suppose f is an odd function which is differentiable on $(-\infty, \infty)$. Show for $a > 0$ there is some x in $(-a,a)$ s.t $f'(x) = \frac{f(a)}{a}$.

Apply MVT since f cont. on $[0,a]$ to get c_1 in $(0,a)$ st $f'(c) = \frac{f(a)}{a}$.

Exercise 4: Does there exists a function f such that

- $f(0) = -1$
- $f(2) = 4$
- $f'(x) \leq 2$ for every x in $[0,2]$?

How to approach?

- Either find an example.
- OR explain why it cannot happen.

let's say such an f exist. Then f is differentiable on $[0,2]$ and so continuous on $[0,2]$. So Mean Value Theorem applies. Meaning there exists c in $(0,2)$ s.t

$$f'(c) = \frac{f(2) - f(0)}{2}$$

But then $\frac{f(2) - f(0)}{2} = \frac{4-(-1)}{2} = \frac{5}{2}$

But $f'(c) \leq 2$, which canot happen since $2 < \frac{5}{2}$.

So no so f exists.

Example 2 : Given • $f(x) = \frac{x^2}{x^2+3}$ (see quiz a)

• $g(x) = e^{2x} + e^{-x}$

For f and g, find the following:

a) Intervals on which they are increasing / decreasing

b) Local minima and local maxima

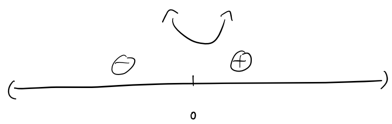
c) Intervals of concavity and inflection points.

1)

$f(x) = \frac{x^2}{x^2+3}$, $f'(x) = \frac{(2x)(x^2+3) - (x^2)(2x)}{(x^2+3)^2}$

$= \frac{2x^3 + 6x - 2x^3}{(x^2+3)^2}$

$= \frac{6x}{(x^2+3)^2}$



$\Rightarrow$ $f' > 0$ on $(0, \infty)$

$f' < 0$ on $(-\infty, 0)$

a) So f is increasing on $(0,3) \cup (3,\infty)$

and f is decreasing on $(-\infty,-3) \cup (-3,0)$

b). f has a local minimum at $x = 0$

. f does not have a local maximum.

c) $f''(x) = \frac{d}{dx}\left(\frac{6x}{(x^2+3)^2}\right) = \frac{d}{dx}\left(6x \cdot (x^2+3)^{-2}\right)$

$= 6 \cdot (x^2+3)^{-2} + 6x\left(-2(x^2+3)^{-3} \cdot 2x\right)$

$= 6(x^2+3)^{-2} - 24x^2(x^2+3)^{-3}$

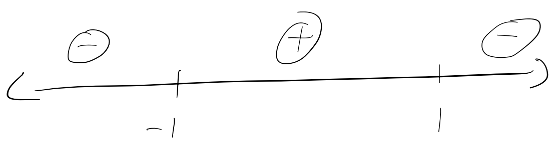
$= \frac{6(x^2+3) - 24x^2}{(x^2+3)^3}$

$= \frac{6x^2 + 18 - 24x^2}{(x^2+3)^3}$

$= \frac{-18x^2 + 18}{(x^2+3)^3}$

$= \frac{(-18)(x^2-1)}{(x^2+3)^3}$

$\Rightarrow x = \pm 1$

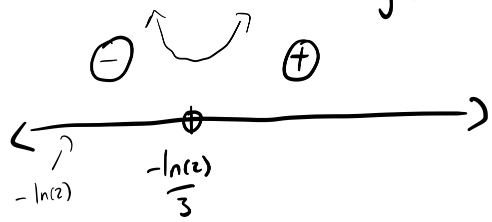


• $x = -1, 1$ are inflection points

For g :

a) We know g is increasing on an interval I if $g'(x) > 0$ for all $x \in I$.

$g(0) > 0$



• well $f'(x) = 2e^{2x} - e^{-x}$

Now $2e^{2x} - e^{-x} = 0$

$\Rightarrow (2e^{3x} - 1)(e^{-x}) = 0$

Now $2e^{3x} - 1 = 0$

$\Rightarrow e^{3x} = \frac{1}{2}$

$\Rightarrow 3x\ln(e) = \ln(\frac{1}{2})$

$\Rightarrow x = \frac{\ln(\frac{1}{2})}{3}$

$= \frac{-\ln(2)}{3} < 0$

Try $g'(-\ln(2))$

$= 2e^{2(-\ln(2))} - e^{\ln(2)}$

$= 2e^{\ln(2^{-2})} - e^{\ln(2)}$

$= 2 \cdot 2^{-2} - 2$

$= \frac{2}{4} - 2 < 0$

• So g is strictly increasing on $(-\frac{\ln(2)}{3}, \infty)$.

• And g is strictly decreasing on $(-\infty, \frac{-\ln(2)}{3})$.

b) g has local extrema where $g'(c) = 0$ or $g'(c)$ DNE.

We see $g'(\frac{-\ln(2)}{3}) = 0$

And by a) g has a local minimum at $\boxed{x = \frac{-\ln(2)}{3}.}$

c) • Inflection point c : g cont at c + g'' changes sign at c.

• Well $g''(x) = 4e^{2x} + e^{-x} > 0$ for all $x \in (-\infty, \infty)$ and so g'' does not change sign $\Rightarrow$ no inflection

• And g is concave upwards on $(-\infty, \infty)$

1^{st} Derivative Test

- f increasing: for every $x_1 < x_2$ we have $f(x_1) \leq f(x_2)$
- f decreasing: for every $x_1 < x_2$ we have $f(x_1) \geq f(x_2)$

Proposition: Suppose $f: I \to \mathbb{R}$ is a differentiable function.

- If $f'(x) \geq 0$ for all x in I, then f increasing on I.
- If $f'(x) \leq 0$ for all x in I, then f decreasing on I.

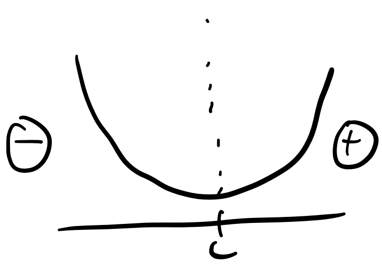
Main Questions:

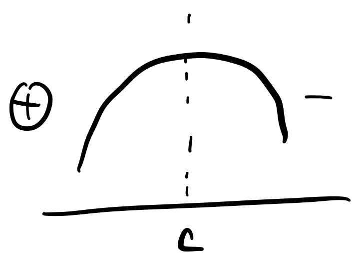
- Determine intervals where functions are increasing.
- Using derivatives to classify local extrema.

Proposition: Suppose that c is a critical number of a continuous function f.

1.) If f' changes from $\oplus$ to $\ominus$, then f has a local max at c

2.) If f' changes from $\ominus$ to $\oplus$, then f has a local min at c

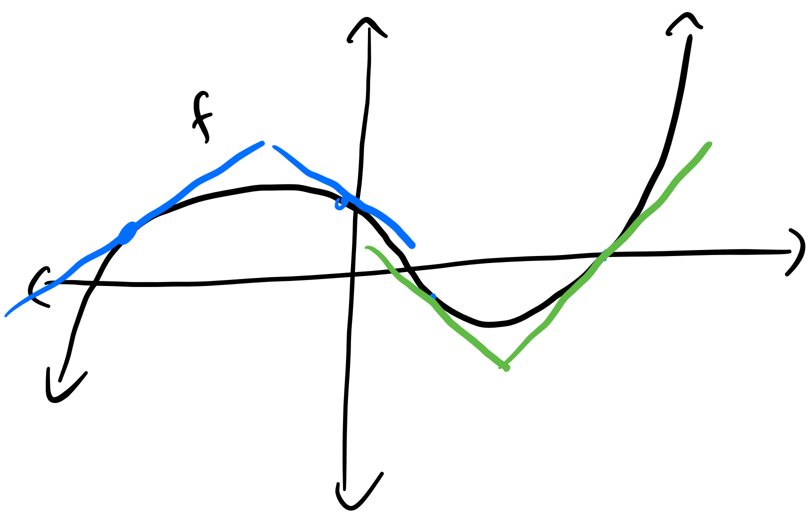




Concavity and
2^{nd} Derivative Test

def: Concave up :- Graph of f lies above tangent line ($f''(x) > 0$)

def Concave down : Graph of f lies below tangent line ($f''(x) < 0$)



def: Inflection point :
A point $P = (c, f(c))$ is called an inflection point to mean :
- f is continuous at c and
- f'' changes sign at c.

Examples

Think about
- $f(x) = x^3$
- $g(x) = e^x$
- $h(x) = \sin(x)$
- $k(x) = \frac{1}{x}$

Ask about the following for f, g, h, k:

1) Where is the function increasing and decreasing?
2) what are the local and global extrema?
(- local min, max, and global min, max)
3) What are the intervals where the function is concave up, concave down?
4) Where (if any) are inflection points?