Involutions under Bruhat order and labeled Motzkin paths

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Joint work with Zachary Hamaker.

Inversions

Definition

An **inversion** of a permutation π is a (i,j) such that $i < j$ and $\pi_i > \pi_j$.
Let $\mathsf{Inv}(\pi)$ be the set of inversions of π and $\ell(\pi) = |\mathsf{Inv}(\pi)|$.

Ex:

$\mathsf{Inv}(2413) = \{(1,3),(2,3),(2,4)\}$.

$\mathsf{Inv}(4231) = \{(1,2),(1,3),(1,4),(2,4),(3,4)\}$.

Bruhat order

Definition

The **(strong) Bruhat order** on $\mathfrak{S}_n$ is a poset $(\mathfrak{S}_n, <)$ such that $\pi \lessdot (i\ j)\pi$ if $\ell((i\ j)\pi) = \ell(\pi) + 1$.

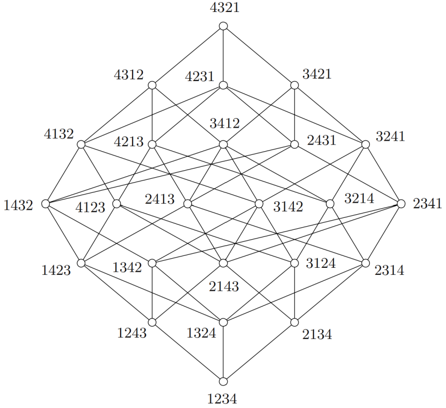

Figure: Bruhat order on $\mathfrak{S}_4$ [A. Björner, F. Brenti 2005].

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As ℓ is the rank function of this poset, we have

$$R_{\mathfrak{S}_n}(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\ell(\pi)} = \prod_{i=1}^{n} [i]_q \qquad \text{where} \qquad [i]_q = q^0 + q^1 + \ldots + q^{i-1}.$$

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What happens if we restrict $(\mathfrak{S}_n, <)$ to subsets of $\mathfrak{S}_n$?

Bruhat order for involutions

Definition

A permutation σ is an **involution** if $\sigma^2 = Id_n$. Let $\mathcal{I}_n$ denote the set of involutions of $\mathfrak{S}_n$.

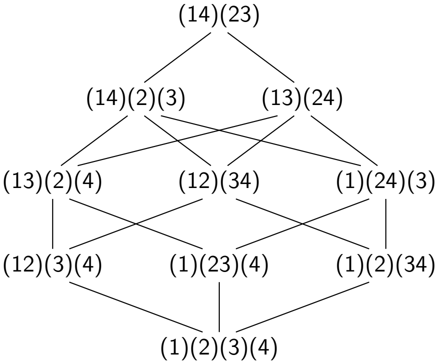


Figure: Bruhat order for $\mathcal{I}_4$.

Visible Inversions

Definition (Z. Hamaker, E. Marberg, P. Paulowski 2017)

An inversion $(i, j) \in \mathrm{Inv}(\pi)$ is **visible** if $\pi_j \leqslant i$. Let $\widehat{Inv}(\pi)$ be the set of such inversions of π and $\hat{\ell}(\pi) = |\widehat{Inv}(\pi)|$.

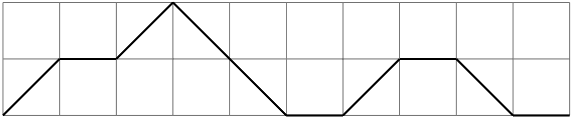
For $\mathcal{I}_n$, $\hat{\ell}$ is its corresponding rank function.
With visible inversions, it is easy to verify the following recursion

$$R_{\mathcal{I}_n}(q) = R_{\mathcal{I}_{n-1}}(q) + q[n-1]_q R_{\mathcal{I}_{n-2}}(q).$$

Motzkin paths

Definition

A **Motzkin path** μ of length n is a lattice path M from $(0,0)$ to $(n,0)$ with up steps $U = (1,1)$, down steps $D = (-1,1)$, and horizontal steps $H = (0,1)$ that stays in the first quadrant. Let $\mathcal{M}_n$ be the set of Motzkin paths of length n.



$h_i(\mu) =$ 1 1 2 2 1 0 1 1 1 0

Definition

The **height** of the i-th step $h_i(\mu)$ is the largest y-coordinate of its endpoints.

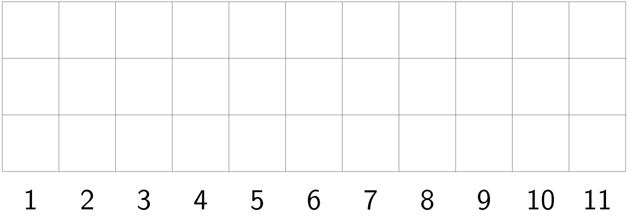
Biane's Bijection

Definition

A **labeled Motzkin path** (μ, λ) is a Motzkin path $\mu = \mu_1 \mu_2 \dots \mu_n$ where every downstep μ_i is associated with an integer $\lambda_i \in [h_i(\mu)]$.

When restricted to involutions, Biane's bijection (1993) becomes a bijection between $\mathcal{I}_n$ and $\mathcal{M}_n^L$.

$$\pi = (1\ 10)(24)(3)(59)(6\ 11)(7)(8)$$



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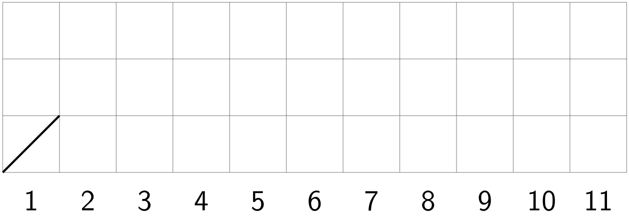

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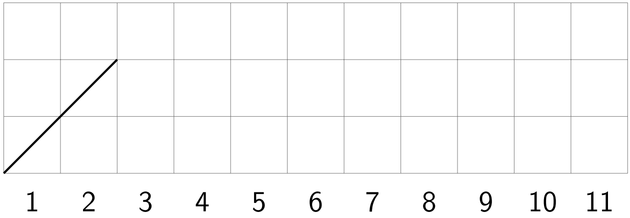

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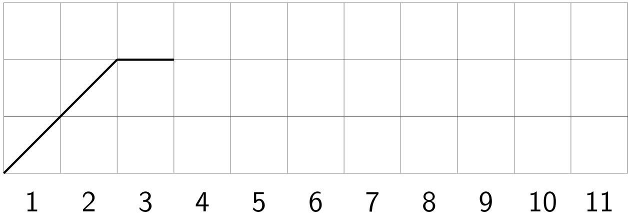

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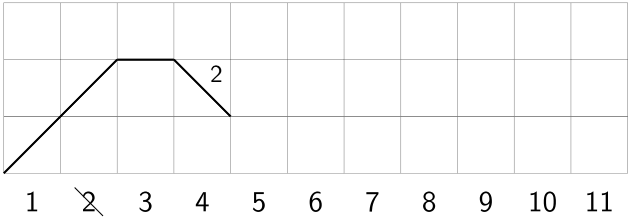

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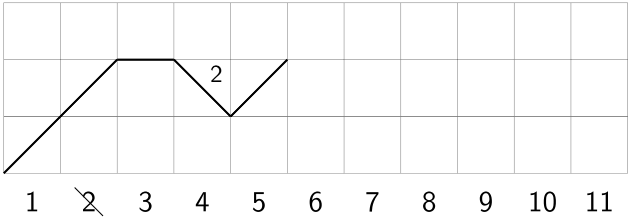

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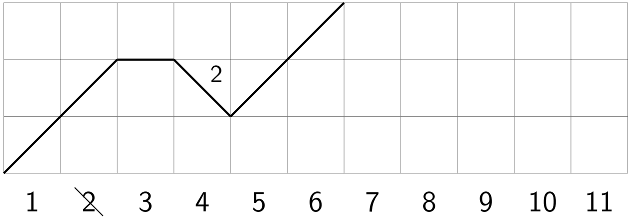

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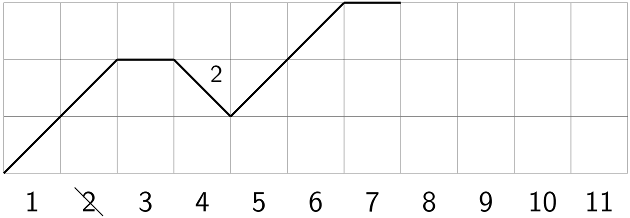

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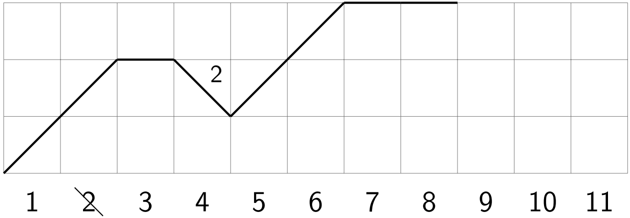

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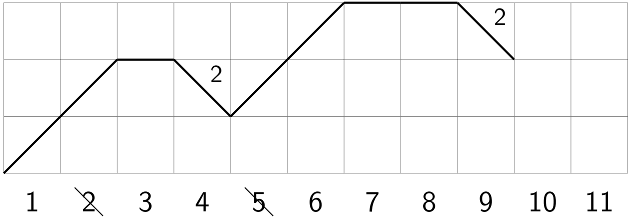

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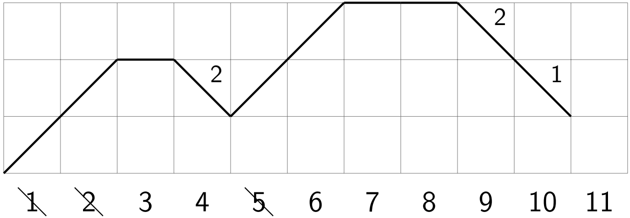

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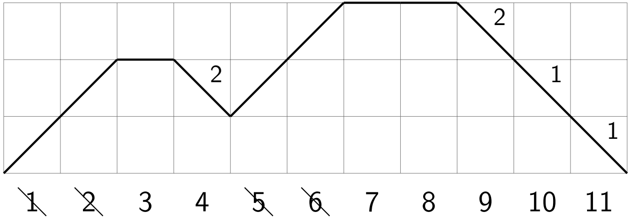

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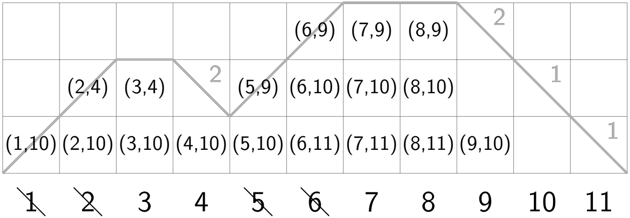

Formalization

For $(\mu, \lambda) \in \mathcal{M}_n^L$ with $\mu = \mu_1 \dots \mu_n$, define

$$H_i(\mu, \lambda) = \begin{cases} \lambda_i(\mu) - 1 & \mu_i = D \\ h_i(\mu) & \mu_i \neq D \end{cases} \quad \text{and} \quad H(\mu, \lambda) = \sum_{i=1}^{n} H_i(\mu, \lambda).$$

Proposition

Let ϕ be Biane's bijection. For all $\sigma \in \mathcal{I}_n$, $\hat{\ell}(\sigma) = H(\phi(\sigma))$.

Proof.

Fix $\sigma \in \mathcal{I}_n$ and $i \in [n]$. Consider the amount of visible inversions of the form (i, j). If $i \leqslant \sigma(i)$, the amount is equivalent the number of 2-cycles $(\sigma(j)\ j)$ such that $\sigma(j) \leqslant i < j$. Via ϕ, this is precisely the height of the i-th step. If $i > \sigma(i)$, the amount is equal to the number of 2-cycles $(\sigma(j)\ j)$ such that $\sigma(j) < \sigma(i) < i < j$. Via ϕ, the label λ_i indicates that there are $\lambda_i - 1$ unpaired up steps as of the labeling of the i-th step. $\square$

Formalization

For $\mu \in \mathcal{M}_n$, define

$$H[\mu;q] = \sum_{\lambda:(\mu,\lambda)\in\mathcal{M}_n^L} q^{H(\mu,\lambda)}.$$

Alternatively, with

$$H_i[\mu;q] = \begin{cases} [h_i(\mu)]_q & \mu_i = D \\ q^{h_i(\mu)} & \text{else} \end{cases}, \quad \text{we have} \quad H[\mu;q] = \prod_{i=1}^{n} H_i[\mu;q].$$

Theorem (C., Z. Hamaker 2022)

$R_{\mathcal{I}_n}(q) = \sum_{\mu\in\mathcal{M}_n} H[\mu;q]$.

Corollary: FPF-involutions

We can perform the same treatment to the set of fixed point free (FPF) involutions.

Definition

An inversion $(i,j) \in \mathsf{Inv}(\pi)$ is **FPF visible** if $\pi_j < i$. Let $\widehat{Inv}^{FPF}(\pi)$ be the set of such inversions of π and $\hat{\ell}^{FPF}(\pi) = |\widehat{Inv}^{FPF}(\pi)|$.

Definition

A **Dyck path** δ of semi-length n is a Motzkin path of length $2n$ without any horizontal steps. Let $\mathcal{D}_n$ be the set of Dyck paths of semi-length n.

Corollary: FPF-involutions

Corollary

$\sum_{\delta\in\mathcal{D}_n} H[\delta,q] = q^n R_{\mathcal{I}_n^{FPF}}(q) = q^n \prod_{k=1}^{n}[2k-1]_q$.

Proof.

Observe the following.

- For $\tau \in \mathcal{I}_{2n}^{FPF}$, $\hat{\ell}(\tau) = \hat{\ell}^{FPF}(\tau) + n$.
- When restricted to $\mathcal{I}_{2n}^{FPF}$, Biane's bijection is a bijection from $\mathcal{I}_{2n}^{FPF}$ to Dyck paths of semi-length n where each down step is labeled.
- Using FPF-visible inversions, the equation $R_{\mathcal{I}_{2n}^{FPF}}(q) = \prod_{k=1}^{n}[2k-1]_q$ can be verified.

The result follows from the previous theorem. □

Corollary: FPF-involutions

Rephrasing of Corollary

Let $h'_i(\delta)$ be the height of the i-th down step of δ. Then,

$$\prod_{k=1}^{n}[2k-1]_q = \sum_{\delta \in \mathcal{D}_n} \prod_{i \in [n]} q^{h'_i(\delta)-1}[h'_i(\delta)]_q.$$

This result has proven before:

[I. Goulden, D. Jackson 1983],
[L. Billera, L. Levine, K. Mészáros 2013],
[M. Watson 2014].

Cover relations in terms of Biane's bijection

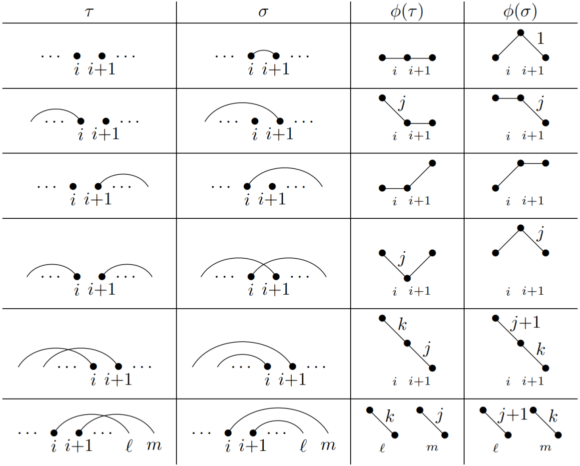

Figure: Cover relations $\tau \lessdot_W \sigma$ for $\tau, \sigma \in \mathcal{I}_n$, with involutions depicted as partial matchings.