1. (a) [3 points] What is the directional derivative at $(2, 1, 4)$ in the direction of $\vec{u} = \langle 1, 2, 2\rangle$?

Solution:

The directional derivative is the dot product of the unit vector in the direction of u and the gradient of f at $(2, 1, 4)$. The gradient of f is

$$\nabla f(x, y, z) = \langle y^2, 2xy, 3\rangle.$$

At $(2, 1, 4)$, this is

$$\nabla f(2, 1, 4) = \langle 1^2, 2(2)1, 3\rangle = \langle 1, 4, 3\rangle.$$

The unit vector in the direction of u is

$$\frac{u}{|u|} = \frac{1}{\sqrt{1^2 + 2^2 + 2^2}}\langle 1, 2, 2\rangle = \frac{1}{3}\langle 1, 2, 2\rangle.$$

Thus, their dot product is

$$\frac{1}{3}(1 * 1 + 4 * 2 + 3 * 2) = \frac{1}{3}(15) = \boxed{5}.$$

(b) [1 point] At $(2, 1, 4)$, what direction is the maximum rate of increase?

What is said maximum rate of increase?

Solution: Since the directional derivative of f in the direction of u is $|\hat{u}||\nabla f| \cos(\theta) = |\nabla f| \cos(\theta)$, the directional derivative is maximized when $\cos(\theta) = 1$, where θ is the angle between u and ∇f when treating them as vectors. Thus, the value is maximized when $\theta = 0$, so the *direction* of maximum rate of increase is $\boxed{\langle 1, 4, 3\rangle \text{ or any positive multiple of this vector}}$ (including its unit vector). The maximum *rate* of increase will be $|\nabla f| = \sqrt{1 + 16 + 9} = \boxed{\sqrt{26}}$. This is obtained from the setting $\theta = 0$ for the equation in the first sentence of this solution.

CHECK THE BACK.

2. [2 points] Find $\frac{\partial f}{\partial \theta}$ for $f(x,y) = 5xy + e^{2x}$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Express the answer in terms of r and θ. **Solution:**
By chain rule, we have

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

So, finding all relevant parts, this should be

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= (5y + 2e^{2x})(-r\sin(\theta)) + (5x + 0)(r\cos(\theta)), \\ &= (5r\sin(\theta) + 2e^{2r\cos(\theta)})(-r\sin(\theta)) + (5r\cos(\theta))(r\cos(\theta)), \\ &= \boxed{-5r^2\sin^2(\theta) - 2re^{2r\cos(\theta)}\sin(\theta) + 5r^2\cos^2(\theta)}. \end{aligned}$$

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3. [2 points] Find the equation for the tangent plane for the surface $z^2 = x^2 + 4y^2 + 4$ at the point $(4, 2, 6)$

Solution:

First, we find the gradient of the equation. As this is an implicit gradient, this means we must move all variable to one side ($F:\ 0 = x^2 + 4y^2 + 4 - z^2$) before taking partials, so we have

$$\nabla F = \langle 2x, 8y, -2z\rangle \overset{@(4,2,6)}{\rightarrow} \langle 8, 16, -12\rangle$$

Then, the equation of the tangent plane will be

$$\begin{aligned} 0 &= \nabla F_{@(4,2,6)} \cdot \langle x-4, y-2, z-6\rangle \\ &= \boxed{8(x-4) + 16(y-2) - 12(z-6) = 0} \end{aligned}$$

This can optionally be simplified more to get

$$\boxed{8x + 16y - 12z + 8 = 0}$$

and even more to get

$$\boxed{2x + 4y - 3z + 2 = 0}$$