1. Let $\mathbf{F}(x,y) = 2ye^{xy}\vec{i} + (2xe^{xy} + 3y^2)\vec{j}$ be a vector field.
(a) [2 points] Verify that $\mathbf{F}$ is conservative by finding the function f such that $\nabla f = \mathbf{F}$.
Solution:

$$\int 2ye^{xy}\, dx = 2e^{xy} + G(y),$$
$$\int 2xe^{xy} + 3y^2\, dx = 2e^{xy} + y^3 + H(x).$$

There are two terms: $2e^{xy}$ and y^3. The first term explicitly appears in both; the second term explicitly appears in the second equation (y^3) and implicitly in the first equation ($G(y)$. Therefore, $\boxed{f(x,y) = 2e^{xy} + y^3 + C}$ is the potential function.

(b) [1 point] Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve $y = 5 - x^2$ starting at $(-2,1)$ and ending at $(2,1)$.
Solution:
The path of the curve does not matter. Only the initial and final points matter since the function is conservative by the *Fundamental Theorem of Line Integrals.*

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(2,1) - f(-2,1), \\ &= (2e^2 + 1) - (2e^{-2} + 1), \\ &= \boxed{2e^2 - 2e^{-2}}.\end{aligned}$$

2. Let $\mathbf{F}$ be the vector field given by

$$\langle -y^2 + x\sin(x), 2xy - 3y^2\cos(y^3)\rangle.$$

Use *Green's theorem* to evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the (positively-oriented) curve which traces out the triangle with corners $(0,0)$, $(1,0)$, and $(0,3)$.

Solution:

The region bounded is a triangle bounded by $x = 0$, $y = 0$, and $y = 3 - 3x$. Use Green's Theorem:

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_0^1 \int_0^{3-3x} (2y - (-2y))\, dydx \\
&= \int_0^1 \int_0^{3-3x} 4y\, dydx \\
&= \int_0^1 2(3-3x)^2\, dx \\
&= 2\int_0^1 (9 - 18x + 9x^2)\, dx \\
&= 2\left(9x - 9x^2 + 3x^3\right)\Big|_0^1 \\
&= 2(9 - 9 + 3) = \boxed{6}.
\end{aligned}$$