1. [4 points] Determine whether the limit exists. If the limit exists, find its value.

(a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + xy - y^2}$.

Solution:

Along the x-axis:

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

Along the y-axis:

$$\lim_{y\to 0} \frac{-y^2}{-y^2} = 1.$$

Along the $x = y$ curve:

$$\lim_{x\to 0} \frac{x^2 - x^2}{x^2 + x^2 - x^2} = \frac{0}{x^2} = 0.$$

The limit for the $x = y$ curve is different from the limit for the x-axis (and y-axis). So, $\boxed{\text{the limit does not exist}}$.

(b) $\lim_{(x,y)\to(0,0)} \dfrac{y\sin(\sqrt{x^2+y^2})}{x^2+y^2}$ We can switch to polar coordinates:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y\sin(\sqrt{x^2+y^2})}{x^2+y^2} &= \lim_{r\to 0^+} \frac{r\sin(\theta)\sin(r)}{r^2} \\
&= \lim_{r\to 0^+} \frac{\sin(\theta)\sin(r)}{r} \\
&= \sin(\theta) \lim_{r\to 0^+} \frac{\sin(r)}{r} \\
&= \sin(\theta).
\end{aligned}$$

Note that $\cos(\theta)$ depends on θ, so this value is not the same for all paths. For example, going along the x-axis gives a value of 0, going along the positive y-axis gives a value of 1, and going along the positive $y = x$ curve gives a value of $\frac{\sqrt{2}}{2}$. Thus, $\boxed{\text{the limit does not exist}}$.

CHECK THE BACK.

2. [2 points] Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x, y) = 5x\sin(2y) + x^2\cos(y)$.

$$\boxed{\frac{\partial f}{\partial x} = 5\sin(2y) + 2x\cos(y).}$$

$$\boxed{\frac{\partial f}{\partial y} = 10x\cos(2y) - x^2\sin(y).}$$

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3. [2 points] Use implicit differentiation to find $\frac{\partial z}{\partial x}$ of the relation

$$xyz = \ln(yz) + x^y.$$

. Since we are finding $\frac{\partial z}{\partial x}$, take the partial of both sides with respect to x and treat y as a constant.

$$\begin{aligned}
\frac{\partial}{\partial x}(xyz) &= \frac{\partial}{\partial x}\left(\ln(yz) + x^y\right) \\
yz + xy\frac{\partial z}{\partial x} &= \frac{1}{yz}\frac{\partial z}{\partial x} + yx^{y-1} \\
xy\frac{\partial z}{\partial x} - \frac{1}{yz}\frac{\partial z}{\partial x} &= yx^{y-1} - yz \\
\left(xy - \frac{1}{yz}\right)\frac{\partial z}{\partial x} &= yx^{y-1} - yz \\
\frac{\partial z}{\partial x} &= \boxed{\frac{yx^{y-1} - yz}{xy - \frac{1}{yz}}}.
\end{aligned}$$
