MAC 2311

November 12, 2020 **Quiz 10**

Remember to show all of your work.

Problem 1. Evaluate the following limits:

(a) $\lim_{x\to 0} x^2 \tan(x)$

Sample Solution: This one was here to "trick" you all! For this limit, direct substitution works perfectly fine.

$$\lim_{x\to 0} x^2 \tan(x) = 0^2 \tan(0) = 0(0) = 0$$

(b) $\lim_{x\to 4} \dfrac{x^2 - 5x + 4}{x - 4}$

Sample Solution: This is a straight forward application of L'Hôpital's rule. Direct substitution gives the indeterminate form $\frac{0}{0}$, so we differentiate both numerator and denominator:

$$\lim_{x\to 4} \frac{x^2 - 5x + 4}{x - 4} = \lim_{x\to 4} \frac{2x - 5}{1} = 3$$

(c) $\lim_{x\to 0} \left(\dfrac{1}{x} - \csc(x)\right)$

Sample Solution: Direct substitution here yields the indeterminate form $\infty - \infty$. We can't quite use L'Hôpital's rule; we have to do some rearranging first

$$\lim_{x\to 0} \left(\frac{1}{x} - \csc(x)\right) = \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim_{x\to 0} \left(\frac{\sin(x) - x}{x\sin(x)}\right)$$

Now we have $\frac{0}{0}$, so we can take the derivative of numerator and denominator:

$$\lim_{x\to 0} \left(\frac{\sin(x) - x}{x\sin(x)}\right) = \lim_{x\to 0} \left(\frac{\cos(x) - 1}{\sin(x) + x\cos(x)}\right)$$

This still gives $\frac{0}{0}$, so once more:

$$\lim_{x\to 0} \left(\frac{\cos(x) - 1}{\sin(x) + x\cos(x)}\right) = \lim_{x\to 0} \left(\frac{-\sin(x)}{\cos(x) + \cos(x) - x\sin(x)}\right) = \lim_{x\to 0} \left(\frac{-\sin(x)}{2\cos(x) - x\sin(x)}\right) = 0$$

(d) $\lim_{x\to 1^+} (x-1)^{x^2-1}$

Sample Solution: Direct substitution here yields the indeterminate form 0^0. Here we'll use the exponential trick: remember that $x-1$ can be written as $e^{\ln(x-1)}$ (here I'll denote this $exp[\ln(x-1)]$. Thus:

$$\lim_{x\to 1^+} (x-1)^{x^2-1} = \lim_{x\to 1^+} exp\Big[(x^2-1)\ln(x-1)\Big] = exp\Big[\lim_{x\to 1^+}(x^2-1)\ln(x-1)\Big]$$

So now we just need to find the limit in brackets. We have to rearrange a bit to be able to use L'Hôpital's rule:

$$\lim_{x\to 1^+}(x^2-1)\ln(x-1) = \lim_{x\to 1^+}\frac{\ln(x-1)}{1/(x^2-1)}$$

Now direct substitution gives $\frac{-\infty}{\infty}$, so we can use L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to 1^+}\frac{\ln(x-1)}{1/(x^2-1)} &= \lim_{x\to 1^+}\frac{1/(x-1)}{-2x/(x^2-1)^2} = \lim_{x\to 1^+}\frac{1}{x-1} * \frac{-(x^2-1)^2}{2x} \\
&= \lim_{x\to 1^+}\frac{1}{x-1} * \frac{-(x-1)(x+1)(x^2-1)}{2x} \\
&= \lim_{x\to 1^+}\frac{-(x+1)(x^2-1)}{2x} \\
&= 0
\end{aligned}$$

Thus, we have

$$\lim_{x\to 1^+} (x-1)^{x^2-1} = exp\Big[\lim_{x\to 1^+}(x^2-1)\ln(x-1)\Big] = exp(0) = e^0 = 1$$