Remember to show all of your work.

Problem 1. Let $f(x) = x^3 e^x$

(a) Find all critical points of $f(x)$.

Sample Solution: critical points are when $f'(x) = 0$ or when $f'(x) = \text{undefined}$. Thus, step 1 is to find the derivative:

$$f'(x) = 3x^2e^x + x^3e^x = x^2e^x(3+x)$$

Note that this is never undefined, so we only need to find where $f'(x) = 0$:

$$0 = x^2e^x(3+x)$$

Thus $x = 0$ and $x = -3$ are critical points.

(b) Determine the intervals on which $f(x)$ is increasing or decreasing.

Sample Solution: here's where we need to plug a number from each interval into the derivative and check it's sign.

Number in $(-\infty, -3)$: $f'(-4) = (-4)^2e^{-4}(3 + (-4)) = \frac{-16}{e^4} < 0$

Number in $(-3, 0)$: $f'(-1) = (-1)^2e^{-1}(3 + (-1)) = \frac{2}{e} > 0$

Number in $(0, \infty)$: $f'(1) = (1)^2e^1(3 + 1) = 4e > 0$

Therefore, $f(x)$ is increasing on $(-3, 0) \cup (0, \infty)$ and $f(x)$ is decreasing on $(-\infty, -3)$

(c) Find all inflection points of $f(x)$.

Sample Solution: Here we have to find the second derivative and determine when $f''(x) = 0$ or $f''(x) = \text{undefined}$.

$$f''(x) = 3x^2e^x + 6xe^x + x^3e^x + 3x^2e^x = xe^x(x^2 + 6x + 6)$$

Again, note that $f''(x)$ is never undefined. Thus, we need only consider when $f''(x) = 0$:

$$0 = xe^x(x^2 + 6x + 6)$$

Using the quadratic formula to factor gives that $x = 0$, $x = -3 \pm \sqrt{3}$ are (potential) inflection points.

Problem 2. For each part below, answer "yes" or "no" and provide a short(!) explanation why.

(a) Does the Mean Value Theorem hold for $f(x) = \cos(x)\ln(x)$ on the interval $[0, 2\pi]$?

Sample Solution: here we have to check the two conditions of the Mean Value Theorem:

- $f(x) = \cos(x)\ln(x)$ is continuous on $[0, 2\pi]$
- $f(x) = \cos(x)\ln(x)$ is differentiable on $(0, 2\pi)$

Here, the first condition fails: $\cos(x)$ is continuous everywhere, but $\ln(x)$ is continuous on the **open** interval $(0, \infty)$. Thus, the Mean Value Theorem does not hold.

Sample acceptable answer: The MVT fails because $f(x)$ is not continuous on $[0, 2\pi]$.

(b) Does Rolle's Theorem hold for $f(x) = x - \frac{1}{x}$ on the interval $[1, 2]$?

Sample Solution: here we have to check the three conditions of Rolle's theorem:

- $f(x) = x - \frac{1}{x}$ is continuous on $[1, 2]$
- $f(x) = x - \frac{1}{x}$ is differentiable on $(1, 2)$
- $f(1) = f(2)$

The first condition is clear: x is continuous everywhere and $\frac{1}{x}$ is continuous on $(0, \infty)$, so $f(x)$ is continuous on $[1, 2]$.

The second condition is a little less clear: take the derivative and make sure it's defined on the interval we're considering.

$$f'(x) = 1 + \frac{1}{x^2}$$

Now $f'(x)$ is defined on $(\infty, 0) \cup (0, \infty)$. Therefore, $f'(x)$ is defined on $(1, 2)$ and thus is differentiable.

For the third condition, we just need to evaluate the function.

$f(1) = 1 - \frac{1}{1} = 0$

$f(2) = 2 - \frac{1}{2} = 1.5.$

Thus, $f(1) \neq f(2)$. Therefore Rolle's theorem does not hold.

Sample acceptable answer: Rolle's Theorem fails because $f(a) \neq f(b)$ for the interval $[1, 2]$.

Sample acceptable answer: Rolle's Theorem fails because $f(1) \neq f(2)$.