MAC 2311

March 16, 2021 **Quiz 8 - Keeran & York**

Remember to show all of your work.

Problem 1. Prove that $f(x) = x^4 - x^3 - x^2 - x - 1$ has at least one horizontal tangent line on $[-1, 2]$ (no need to find the x value, just prove it has one).

Sample Solution: this is just a straight forward application of Rolle's Theorem. If all three conditions hold, then the statement is proved:

(i) $f(x)$ continuous on $[-1, 2]$: $f(x)$ is a polynomial, and thus is continuous everywhere. So, of course it's continuous on $[-1, 2]$.

(ii) $f(x)$ differentiable on $(-1, 2)$: again, because $f(x)$ is a polynomial, it is differentiable everywhere. You can also check by taking the derivative: $f'(x) = 4x^3 - 3x^2 - 2x - 1$, and this is a polynomial, and hence is defined everywhere. Thus, $f(x)$ is differentiable on $(-1, 2)$.

(iii) $f(a) = f(b)$: here we just have to check $f(-1)$ and $f(2)$:

$$
\begin{aligned}
f(-1) &= (-1)^4 - (-1)^3 - (-1)^2 - (-1) - 1 \\
&= 1 + 1 - 1 + 1 - 1 \\
&= 1 \\
f(2) &= (2)^4 - (2)^3 - (2)^2 - (2) - 1 \\
&= 16 - 8 - 4 - 2 - 1 \\
&= 1
\end{aligned}
$$

so $f(-1) = f(2)$

Now, we've proven that the three conditions for Rolle's Theorem hold. Therefore, $f(x)$ must have a horizontal tangent line somewhere on the interval $[-1, 2]$ via Rolle's Theorem.

Problem 2. Let $f(x) = x^2 + \frac{1}{x^2}$, and consider the interval $\left[-3, -\frac{1}{2}\right]$

- Find the absolute maximum (x and y value) if one exists.
- Find the absolute minimum (x and y value) if one exists.

Sample Solution: Note that $f(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$, and therefore is continuous on $[-3, -1/2]$. Thus, we can apply the Extreme Value Theorem. Now, we have to find any critical points of $f(x)$, and then find the values of $f(x)$ for these critical points, as well as the endpoints. The largest $f(x)$ value is the absolute max, the lowest is the absolute min. Start by finding the critical points:

$$
\begin{aligned}
f(x) &= x^2 + \frac{1}{x^2} \\
f'(x) &= 2x - \frac{2}{x^3} \\
0 &= 2x - \frac{2}{x^3} \\
\frac{2}{x^3} &= 2x \\
1 &= x^4 \\
x &= \pm 1
\end{aligned}
$$

(NOTE: 0 cannot be a critical point because it is not in the domain of $f(x)$!)

So, we have two critical points, but we're only concerned about -1, because the problem gives us the interval $[-3, -1/2]$. So, we have to check $f(-3)$, $f(-1)$, and $f(-1/2)$:

$$
\begin{aligned}
f(-3) &= (-3)^2 + \frac{1}{(-3)^2} = 9 + \frac{1}{9} = \frac{82}{9} \approx 9.111 \\
f(-1) &= (-1)^2 + \frac{1}{(-1)^2} = 1 + 1 = 2 \\
f(-\frac{1}{2}) &= (-\frac{1}{2})^2 + \frac{1}{(-\frac{1}{2})^2} = \frac{1}{4} + 4 = 4.25
\end{aligned}
$$

so, $f(x)$ has an absolute max at $(-3, \frac{82}{9})$ and an absolute min at $(-1, 2)$.