MAC 2311

March 23, 2021 **Quiz 9 - Keeran & York**

Remember to show <u>all</u> of your work.

Problem 1. Let $f(x) = \dfrac{1}{x^2 - 3x - 18}$.

(a) Find the critical point(s) of $f(x)$.

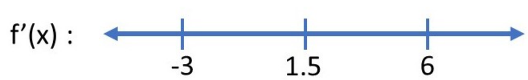
Sample Solution: Here, we have to find where $f'(x)$ is either zero or undefined (but also, remember that critical points *must* be defined for the original function $f(x)$. This will be important here.) We have

$$\begin{aligned} f(x) &= \frac{1}{x^2 - 3x - 18} \\ f'(x) &= \frac{(x^2 - 3x - 18)(0) - (1)(2x - 3)}{(x^2 - 3x - 18)^2} \\ &= \frac{-2x + 3}{(x^2 - 3x - 18)^2} \\ &= \frac{-2x + 3}{\big((x - 6)(x + 3)\big)^2} \end{aligned}$$

Now, $x = 6$ and $x = -3$ cannot be critical points, because the original function $f(x)$ is undefined there. Thus, the only critical point is $-2x + 3 = 0 \longrightarrow x = \frac{3}{2}$.

(b) Find the intervals on which $f(x)$ is increasing / decreasing.

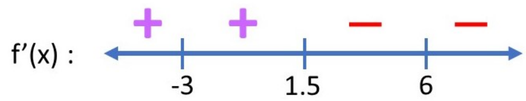
Sample Solution: We've already done the tough work here. What we have to do is construct a number line with "important points" (here, this includes $x = 6$ and $x = -3$, as well as $x = \frac{3}{2}$), and determine if $f'(x)$ is positive or negative in each interval.



Let's use the points -4, 0, 2, and 7. NOTE: the denominator of the derivative will never be negative, because it's a square. Thus, the sign of the derivative depends only on the numerator.

$$\begin{aligned} f'(-4) &\to -2(-4) + 3 = 8 + 3 = 11 > 0 \\ f'(0) &\to -2(0) + 3 = 3 > 0 \\ f'(2) &\to -2(2) + 3 = -4 + 3 = -1 < 0 \\ f'(7) &\to -2(7) + 3 = -14 + 3 = -11 < 0 \end{aligned}$$

Thus, we have the following:



So, $f(x)$ is increasing on $(-\infty, -3) \cup (-3, \frac{3}{2})$ and decreasing on $(\frac{3}{2}, 6) \cup (6, \infty)$.

Problem 2. Evaluate one of following limits (you can choose which):

- $\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right)$

Sample Solution:

$$\begin{aligned}
\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \to 0} \left(\frac{\cos x}{\sin x} - \frac{1}{x} \right) \\
&= \lim_{x \to 0} \left(\frac{x \cos x}{x \sin x} - \frac{\sin x}{x \sin x} \right) \\
&= \lim_{x \to 0} \left(\frac{x \cos x - \sin x}{x \sin x} \right)
\end{aligned}$$

Now here, if we try direct substitution we get $\frac{0}{0}$, an indeterminate form. Thus, L'Hôpital's can be used:

$$\begin{aligned}
\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \to 0} \left(\frac{x \cos x - \sin x}{x \sin x} \right) \\
&\overset{LH}{=} \lim_{x \to 0} \left(\frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} \right) \\
&= \lim_{x \to 0} \left(\frac{-x \sin x}{\sin x + x \cos x} \right)
\end{aligned}$$

Direct substitution here still gives $\frac{0}{0}$, so we apply L'Hôpital's again:

$$\begin{aligned}
\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \to 0} \left(\frac{-x \sin x}{\sin x + x \cos x} \right) \\
&\overset{LH}{=} \lim_{x \to 0} \left(\frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} \right) \\
&= \lim_{x \to 0} \left(\frac{-\sin x - x \cos x}{2 \cos x - x \sin x} \right)
\end{aligned}$$

Now using direct substitution gives

$$\begin{aligned}
\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \to 0} \left(\frac{-\sin x - x \cos x}{2 \cos x + \cos x} \right) \\
&= \left(\frac{-\sin(0) - (0)\cos(0)}{2\cos(0) + (0)\sin(0)} \right) \\
&= \frac{0}{2} = 0
\end{aligned}$$

Alternate Sample Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{1}{\tan x} - \frac{1}{x}\right)\\
&= \lim_{x\to 0}\left(\frac{x}{x\tan x} - \frac{\tan x}{x\tan x}\right)\\
&= \lim_{x\to 0}\left(\frac{x - \tan x}{x\tan x}\right)\\
&\overset{LH}{=} \lim_{x\to 0}\left(\frac{1-\sec^2 x}{\tan x + x\sec^2 x}\right)\\
&\overset{LH}{=} \lim_{x\to 0}\left(\frac{-2\sec^2 x\tan x}{\sec^2 x + \sec^2 x + 2x\sec^2 x\tan x}\right)\\
&= \frac{-2\sec^2(0)\tan(0)}{2\sec^2(0) + 2(0)\sec^2(0)\tan(0)}\\
&= \frac{0}{2+0} = \frac{0}{2} = 0
\end{aligned}$$

- $\lim\limits_{x \to 1^+} \left(x-1\right)^{x-1}$

Sample Solution:

Trying direct substitution here gives 0^0, an indeterminate form. Thus, we want to use L'Hôpital's, but we can't quite yet. We have to do a little bit of manipulating first:

$$\begin{aligned}\lim_{x\to 1^+}(x-1)^{x-1} &= \lim_{x\to 1^+}\left(e^{\ln(x-1)}\right)^{x-1}\\ &= \lim_{x\to 1^+} e^{(x-1)\ln(x-1)}\\ &= e^{\lim_{x\to 1^+}(x-1)\ln(x-1)}\\ &= e^{\lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{x-1}}}\end{aligned}$$

Now, trying direct substitution gives $-\frac{\infty}{\infty}$, so we can use L'Hôpital's. We have:

$$\begin{aligned}\lim_{x\to 1^+}(x-1)^{x-1} &= e^{\lim_{x\to 1^+}\frac{\ln(x-1)}{\frac{1}{x-1}}}\\ &\overset{LH}{=} e^{\lim_{x\to 1^+}\frac{\frac{1}{x-1}}{\frac{-1}{(x-1)^2}}}\\ &= e^{\lim_{x\to 1^+}\frac{1}{x-1}*\frac{(x-1)^2}{-1}}\\ &= e^{\lim_{x\to 1^+}\frac{(x-1)}{-1}}\\ &= e^{\lim_{x\to 1^+}(-x+1)}\end{aligned}$$

Now, direct substitution gives

$$\begin{aligned}\lim_{x\to 1^+}(x-1)^{x-1} &= e^{\lim_{x\to 1^+}(-x+1)}\\ &= e^{-(1)+1}\\ &= e^0 = 1\end{aligned}$$