COMPLETELY POSITIVE MAPS INDUCED BY COMPOSITION OPERATORS

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Abstract. We consider the completely positive map on the Toeplitz operator system given by conjugation by a composition operator; that is, we analyze operators of the form

$$C_\varphi^* T_f C_\varphi$$

We prove that every such operator is weakly asymptotically Toeplitz, and compute its asymptotic symbol in terms of the Aleksandrov-Clark measures for φ. When φ is an inner function, this operator is Toeplitz, and we show under certain hypotheses that the iterates of T_f under a suitably normalized form of this map converge to a scalar multiple of the identity. When φ is a finite Blaschke product, this scalar is obtained by integrating f against a conformal measure for φ, supported on the Julia set of φ. In particular the composition operator C_φ can detect the Julia set by means of the completely positive map.

1. Introduction

The purpose of this paper is to study the completely positive map induced by a composition operator on the Hardy space H^2. That is, we are interested in the mapping

$$A \to C_\varphi^* A C_\varphi \tag{1.1}$$

where A is a bounded operator on H^2; our focus is on the case where A is a Toeplitz operator T_f.

This work was originally motivated by the desire to understand C*-algebraic relations that obtain between Toeplitz and composition operators; such relations have already been studied by the author [10, 11] (for composition operators with automorphic symbols) and Kriete, MacCluer and Moorhouse [13, 12] in the case of non-automorphic linear fractional symbols, and by Hamada and Watatani when the symbol is a finite Blaschke product with an interior fixed point [9]. The expressions $C_\varphi^* T_f C_\varphi$ arise naturally in this last setting. It is easy to show, using the

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Brown-Halmos criterion, that such an operator is Toeplitz; the problem of calculating its symbol lies somewhat deeper. It turns out that this question is closely related to the properties of the dynamical system obtained from the action of φ on the unit circle, and in fact this completely positive map gives a close and very direct link between C_φ and the dynamical system, which does not seem to be apparent when one considers only "spatial" properties of the operator (e.g. spectral or cyclicity properties, both much studied in the composition operator literature). In particular the operator C_φ, by means of the map (1.1) can "see" the Julia set of φ and can recover conformal measures on the Julia set.

The paper is organized as follows: after the preliminary material of Section 2, in Section 3 we analyze the the operators $C_\varphi^* T_f C_\varphi$. We prove that these operators are *asymptotically Toeplitz* (Definition 2.4) and compute the asymptotic symbol in terms of f and the Aleksandrov measures for φ. In particular if φ is inner then $C_\varphi^* T_f C_\varphi$ is a Toeplitz operator. In sections 4 and 5 we specialize to the case when φ is inner (resp. a finite Blaschke product). We show that, under suitable hypotheses, if C_φ is replaced by a certain weighted composition operator $W_{\varphi,h} = T_h C_\varphi$ then the iterates of T_f under the corresponding completely positive map converge in norm to a scalar multiple of the identity. In turn the map sending f to this scalar determines a measure on the circle with important dynamical properties (e.g. when φ is a finite Blaschke product, this measure is supported on the Julia set of φ, invariant under φ and *conformal*).

2. Preliminaries

Throughout the paper, φ denotes a holomorphic mapping of the open unit disk $\mathbb{D} \subset \mathbb{C}$ into itself. By the Littlewood subordination principle, the map

$$C_\varphi : f \to f \circ \varphi$$

defines a bounded operator on $H^2(\mathbb{D})$, the space of all holomorphic functions on $\mathbb{D}$ with square-summable power series. Passing to boundary values H^2 may be identified with a closed subspace of L^2 on the unit circle $\mathbb{T}$, and we let $P : L^2 \to H^2$ denote the orthogonal projection. It is then the case that every $f \in L^\infty(\mathbb{T})$ defines a bounded operator

$$T_f : g \to P(fg)$$

on H^2, called the *Toeplitz operator with symbol f*. The Toeplitz operator with symbol $f(z) = z$ is called the *unilateral shift* and denoted by S.

As is nearly always the case in the study of composition operators, we require the Denjoy-Wolff theorem:

Theorem 2.1 (Denjoy-Wolff Theorem)**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. If φ is not conjugate to a rotation then there exists a unique point $w \in \overline{\mathbb{D}}$ such that $\varphi^n \to w$ uniformly on compact subsets of $\mathbb{D}$. Moreover if φ fixes a point $z \in \mathbb{D}$ then $z = w$, and if $|w| = 1$ then φ has a finite angular derivative at w (in the sense of Carathéodory) and $0 < \varphi'(w) \leq 1$.*

2.1. **Aleksandrov measures.** One of the main technical tools in our study will be the Aleksandrov measures associated to an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$; we begin by defining them and recalling some of their properties. We refer to [5, Chapter 9] and its references for details. For notational simplicity, all integrals in this paper are taken over the unit circle unless otherwise indicated.

Definition 2.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The* Aleksandrov measures *for φ are the measures μ_α defined for each $\alpha \in \mathbb{T}$ by*

$$\frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int \frac{1 - |z|^2}{|1 - \overline{z}\zeta|^2}\, d\mu_\alpha(\zeta)$$

Each μ_α has a Lebesgue decomposition

$$\mu_\alpha = h_\alpha dm + \sigma_\alpha$$

where dm denotes normalized Lebesgue measure on the circle. The absolutely continuous part is given by

$$h_\alpha(\zeta) = \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2}$$

Consider the set

$$E_\varphi = \{\zeta \in \mathbb{T} : |\varphi(\zeta)| = 1\}$$

The function h_α is zero almost everywhere on E_φ, and the singular part σ_α is carried by $\varphi^{-1}(\alpha) \subset E_\varphi$. When φ is understood we will write E for E_φ.

For a bounded Borel function f on the circle, define

$$A_\varphi(f)(\alpha) = \int f(\zeta)\, d\mu_\alpha(\zeta).$$

The function $A_\varphi(f)$ is again a bounded Borel function, and A_φ extends to a well-defined, bounded operator on $L^\infty(\mathbb{T})$ (and is in fact bounded on all of the L^p spaces for $p \geq 1$). Moreover A_φ is also bounded on $C(\mathbb{T})$.

We also define operators A^c_φ and A^s_φ, for f a bounded Borel function and $\alpha \in \mathbb{T}$, by

$$A^c_\varphi(f)(\alpha) = \int f(\zeta) h_\alpha(\zeta)\, dm(\zeta)$$

and

$$A^s_\varphi(f)(\alpha) = \int f(\zeta) d\sigma_\alpha(\zeta)$$

The operators A^c_φ and A^s_φ will be called the *continuous* and *singular* parts of A_φ respectively. Clearly $A_\varphi = A^s_\varphi + A^c_\varphi$. Alternatively, we could define

$$A^c_\varphi(f) = A_\varphi((1-\chi_{E_\varphi})f)$$
$$A^s_\varphi(f) = A_\varphi(\chi_{E_\varphi} f)$$

and from this latter definition it is clear (from the corresponding results for A_φ) that A^c_φ and A^s_φ are well-defined and bounded on $L^p(\mathbb{T})$ for all $1 \le p \le \infty$ (note however that these operators are not separately bounded on $C(\mathbb{T})$ in general).

We will also need the *Aleksandrov disintegration theorem*: if $f \in L^1(\mathbb{T}, dm)$, then $f \in L^1(\mathbb{T}, d\mu_\alpha)$ for almost every $\alpha \in \mathbb{T}$, and

$$\int \left(\int f(\zeta)\, d\mu_\alpha(\zeta) \right) dm(\alpha) = \int f(\zeta)\, dm(\zeta)$$

The significance of the Aleksandrov operator in the present context is shown by the following proposition:

Proposition 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and suppose $\varphi(0) = 0$. Then $C^*_\varphi = A_\varphi$.*

Proof. By [5, Corollary 9.1.7], if $\lambda \in \mathbb{D}$ and $\varphi(0) = 0$ then for every $\alpha \in \mathbb{T}$

$$A_\varphi(k_\lambda)(\alpha) = \int \frac{1}{1-\overline{\lambda}\zeta}\, d\mu_\alpha(\zeta) = \frac{1}{1-\overline{\varphi(\lambda)}\alpha} = C^*_\varphi(k_\lambda)(\alpha)$$

Thus C^*_φ and A_φ agree on the span of the kernel functions k_λ, which is dense in H^2. Since A_φ is bounded on $L^2(\mathbb{T})$ the result follows. $\square$

2.2. **Asymptotic Toeplitz operators.** The notion of an asymptotic Toeplitz operator was introduced by Barría and Halmos [2], in connection with the problem of finding the essential commutant of a Toeplitz operator. Very recently Nazarov and Shapiro [15] have considered versions of this property (with various topologies) for single composition operators. In this paper we are interested in operators of the form

$C_\varphi^* T_f C_\varphi$; we will prove that every such operator is strongly asymptotically Toeplitz and calculate the asymptotic symbol.

Definition 2.4. [2] *Let S denote the unilateral shift on H^2. An operator $A \in B(H^2)$ will be called* (strongly) asymptotically Toeplitz *if the limit*

$$\lim_{n\to\infty} S^{*n} A S^n$$

exists in the strong operator topology. If this limit is 0*, A will be called* asymptotically compact.

Proposition 2.5. *An operator $A \in B(H^2)$ is strongly asymptotically Toeplitz if and only if*

$$A = T + Q$$

where T is a Toeplitz operator and Q is asymptotically compact. Moreover every asymptotically Toeplitz operator can be written uniquely in this form, and T is given by $\lim_{n\to\infty} S^{*n} A S^n$.

Proof. Recall (ref) that T is a Toeplitz operator if and only if $S^*TS = T$. Thus if $A = T + Q$ then

$$\lim_{n\to\infty} S^{*n} A S^n = \lim_{n\to\infty} S^{*n} T S^n + \lim_{n\to\infty} S^{*n} Q S^n = T$$

so A is asymptotically Toeplitz. Conversely, if A is asymptotically Toeplitz let $B = \lim S^{*n} A S^n$. By definition, $S^*BS = B$ so B is Toeplitz and $A - B$ is asymptotically compact. To prove uniqueness, observe that if T is an asymptotically compact Toeplitz operator then $T = \lim S^{*n} T S^n = 0$. $\square$

If A is asymptotically Toeplitz then the symbol of T will be called the *asymptotic symbol* of A.

3. The action of C_φ on Toeplitz operators

In this section we prove that $C_\varphi^* T_f C_\varphi$ is asymptotically Toeplitz and compute its asymptotic symbol. We also give sufficient conditions under which $C_\varphi^* T_f C_\varphi$ is a Toeplitz operator; in particular this will be the case whenever φ is an inner function.

We begin with the following two lemmas, which will be used to reduce the proof of the main theorem to the case $\varphi(0) = 0$.

Lemma 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. For fixed $w \in \mathbb{D}$ let $\psi(z) = \frac{w-z}{1-\overline{w}z}$. If μ_α and ν_α are the Aleksandrov measures for φ and $\psi \circ \varphi$ respectively, then*

$$\mu_\alpha = |\psi'(\alpha)|^2 \nu_{\psi(\alpha)}$$

Proof. The result follows immediately from the definition of the Aleksandrov measures and the identity

$$\frac{1-|\psi(\varphi(z))|^2}{|\psi(\alpha)-\psi(\varphi(z))|^2} = \frac{1}{|\psi'(\alpha)|^2}\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2}$$

□

Lemma 3.2. *If $f\in L^\infty(\mathbb{T})$ and $\psi(z)=\frac{w-z}{1-\overline{w}z}$, then*

$$C_\psi^* T_f C_\psi = T_{|\psi'|^2 f\circ\psi^{-1}}$$

Proof. Fix $h\in H^2$ and consider the quadratic form $A\to\langle Ah,h\rangle$. Then making the change of variable $\eta=\psi(\zeta)$ in the first integral below,

$$\begin{aligned}\langle C_\psi^* T_f C_\psi h, h\rangle &= \int f(\zeta)|h(\psi(\zeta))|^2\,dm(\zeta)\\ &= \int f(\psi^{-1}(\eta))|\psi'(\eta)|^2|h(\eta)|^2\,dm(\eta)\\ &= \langle T_{|\psi'|^2 f\circ\psi^{-1}}h,h\rangle\end{aligned}$$

which proves the lemma. □

Theorem 3.3. *Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic and let $f\in L^\infty(\mathbb{T})$. Then $C_\varphi^*T_fC_\varphi$ is strongly asymptotically Toeplitz, with asymptotic symbol $A_\varphi^s(f)$. Moreover, if $f=0$ almost everywhere on E^c, then $C_\varphi^*T_fC_\varphi$ is a Toeplitz operator.*

Proof. We let $S=T_z$ denote the unilateral shift and observe the relation

$$C_\varphi S = T_\varphi C_\varphi \tag{3.1}$$

This follows immediately from the definitions of the operators. To prove that $C_\varphi^*T_fC_\varphi$ is asymptotically Toeplitz, it suffices to assume that $f\geq 0$, since every L^∞ function is a linear combination of four bounded nonnegative functions. In this case, the sequence of positive operators

$$S^{*n}C_\varphi^*T_fC_\varphi S^n \tag{3.2}$$

is monotone decreasing, since

$$\begin{aligned} C_\varphi^*T_fC_\varphi - S^*C_\varphi^*T_fC_\varphi S &= C_\varphi^*T_{f(1-|\varphi|^2)}C_\varphi && (3.3)\\ &\geq 0 && (3.4)\end{aligned}$$

by the relation (3.1) and the fact that $T_\varphi^*T_fT_\varphi = T_{f|\varphi|^2}$. It follows that the sequence (3.2) is convergent in the strong operator topology, so $C_\varphi^*T_fC_\varphi$ is asymptotically Toeplitz.

To compute the symbol, we look at the sequence more closely. We have for each n

$$S^{*n}C_\varphi^* T_f C_\varphi S^n = C_\varphi^* T_\varphi^{*n} T_f T_\varphi^n C_\varphi.$$

Now since φ is analytic, we have $T_\varphi^{*n} T_f T_\varphi^n = T_{|\varphi|^{2n} f}$. Since $|\varphi|^{2n} \to \chi_E$ pointwise a.e. on $\mathbb{T}$ as $n \to \infty$, we now claim that

$$C_\varphi^* T_{|\varphi|^{2n} f} C_\varphi \to C_\varphi^* T_{\chi_E f} C_\varphi$$

in the weak operator topology. To see this, let $g, h \in H^2$. Then

$$\begin{aligned}\langle C_\varphi^* T_{|\varphi|^{2n} f} C_\varphi g, h\rangle &= \langle T_{|\varphi|^{2n} f} C_\varphi g, C_\varphi h\rangle \\ &= \int |\varphi|^{2n} f \cdot (g\circ\varphi)\overline{(h\circ\varphi)}\, dm\end{aligned}$$

which converges (by the dominated convergence theorem) to

$$\int \chi_E f \cdot (g\circ\varphi)\overline{(h\circ\varphi)}\, dm = \langle C_\varphi^* T_{\chi_E f} C_\varphi g, h\rangle$$

This proves that $C_\varphi^* T_{\chi_E f} C_\varphi$ is a Toeplitz operator. The “moreover” statement of the theorem follows immediately, since to say $f = 0$ a.e. on E^c just means $f = \chi_E f$ a.e.

So far we have established that $C_\varphi^* T_f C_\varphi$ is strongly asymptotically Toeplitz, and its asymptotic symbol is the symbol g of the Toeplitz operator $C_\varphi^* T_{\chi_E f} C_\varphi$. To find g, first note that it suffices to assume that is f real-valued, and hence so is g since $C_\varphi^* T_{\chi_E f} C_\varphi$ is a self adjoint Toeplitz operator. We will also assume for now that $\varphi(0) = 0$. For $h \in L^2(\mathbb{T})$ let Ph denote the Szegő projection of h into H^2. Then

$$\begin{aligned}T_g 1 = Pg = C_\varphi^* T_{\chi_E f} C_\varphi 1 &= C_\varphi^* T_{\chi_E f} 1 \\ &= C_\varphi^* P(\chi_E f) \\ &= A_\varphi P(\chi_E f)\end{aligned}$$

Thus for almost every $\alpha \in \mathbb{T}$,

$$(Pg)(\alpha) = \int P(\chi_E f)\, d\mu_\alpha$$

Now since g is real-valued, we have $g = Pg + \overline{Pg} - \int Pg\, dm$. (Of course Pg need not lie in L^∞, but it does belong to L^1 which is all that is needed below.) Furthermore, by the Aleksandrov disintegration theorem [5, Theorem 9.4.11],

$$\begin{aligned}\int Pg\,dm &= \int\left(\int P(\chi_E f)(\zeta)\,d\mu_\alpha(\zeta)\right)\,dm(\alpha)\\ &= \int P(\chi_E f)(\zeta)\,dm(\zeta)\\ &= \int P(\chi_E f)dm\cdot\int d\mu_\alpha\end{aligned}$$

since μ_α is a probability measure for all α (by our assumption that $\varphi(0)=0$). Therefore,

$$\begin{aligned}g &= Pg+\overline{Pg}-\int Pg\,dm\\ &= \int P(\chi_E f)+\overline{P(\chi_E f)}-\left(\int_{\mathbb{T}} P(\chi_E f)\,dm\right)\,d\mu_\alpha\\ &= \int(\chi_E f)(\zeta)\,d\mu_\alpha(\zeta)\\ &= \int_E f(\zeta)\,d\mu_\alpha(\zeta)\end{aligned}$$

Now, since the absolutely continuous part of μ_α puts zero mass on E and the singular part σ_α is concentrated on $\varphi^{-1}(\alpha)\subset E$, we obtain finally for a.e. $\alpha\in\mathbb{T}$

$$g(\alpha)=\int f(\zeta)\,d\sigma_\alpha(\zeta)=A^s_\varphi(f)(\alpha)$$

This proves the theorem when $\varphi(0)=0$. To obtain it for general φ, we use Lemmas 3.1 and 3.2. Let now $\varphi:\mathbb{D}\to\mathbb{D}$ be an arbitrary holomorphic map and let

$$\psi(z)=\frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}$$

Then $\theta=\psi\circ\varphi$ satisfies $\theta(0)=0$, and $C_\varphi=C_\theta C_\psi$. Suppose $f=\chi_E f$, and observe that $|\varphi(\zeta)|=1$ if and only if $|\theta(\zeta)|=1$. Thus $C^*_\varphi T_f C_\varphi$ and $C^*_\theta T_f C_\theta$ are both Toeplitz operators, and

$$\begin{aligned}C^*_\varphi T_f C_\varphi &= C^*_\psi C^*_\theta T_f C_\theta C_\psi\\ &= C^*_\psi T_{A^s_\theta(f)} C_\psi\\ &= T_{|\psi'|^2(A^s_\theta(f)\circ\psi)}\end{aligned}$$

Now, let σ_α and τ_α be the singular parts of the Aleksandrov measures for φ and θ respectively. Then for almost every $\alpha\in\mathbb{T}$ we have by

Lemma 3.1

$$\begin{aligned}
|\psi'(\alpha)|^2 A^s_\theta(f)(\psi(\alpha)) &= \int |\psi'(\alpha)|^2 f(\zeta)\, d\tau_{\psi(\alpha)}(\zeta) \\
&= \int f(\zeta)\, d\sigma_\alpha(\zeta) \\
&= A^s_\varphi(f)(\alpha)
\end{aligned}$$

Thus

$$C^*_\varphi T_f C_\varphi = T_{A^s_\varphi(f)}$$

as desired. □

The following Corollary is immediate:

Corollary 3.4. *If φ is an inner function then for every $f \in L^\infty(\mathbb{T})$,*

$$C^*_\varphi T_f C_\varphi = T_{A_\varphi(f)}.$$

3.1. **Remarks.** 1. The converse to the second statement of Theorem 3.3 is not true; that is, it is possible that $C^*_\varphi T_f C_\varphi$ is Toeplitz but f is not almost everywhere 0 on E^c: for example, let $\varphi(z) = z^2$ and $f(z) = z$. Then for all analytic polynomials p, q,

$$\langle C^*_\varphi T_z C_\varphi p, q\rangle = \int \zeta p(\zeta^2)\overline{q(\zeta^2)}\, dm(\zeta) = 0$$

since the integrand is an odd trigonometric polynomial. It follows that $C^*_\varphi T_z C_\varphi = 0$ (hence trivially a Toeplitz operator).

There are special cases in which the converse does hold; for example if φ is continuous and univalent on the circle (e.g. if φ is a linear fractional transformation). The converse also holds if it is additionally assumed that the symbol f is nonnegative.

2. In general the strong convergence in Theorem 3.3 cannot be improved to norm convergence. For a simple counterexample, let $f \equiv 1$; then $A^s_\varphi(f)(\alpha) = \|\sigma_\alpha\|$. Letting $P_n = S^n S^{*n}$, by an elementary Hilbert space argument (see e.g. [18, Prop. 5.1]) the sequence of norms $\|C_\varphi P_n\|$ converges to the essential norm of C_φ, which is equal to $(\sup_{\alpha\in\mathbb{T}} \|\sigma_\alpha\|)^{1/2}$ [4]. Combining this with the inequality

$$\|C_\varphi\|^2_e = \lim \|P_n C^*_\varphi C_\varphi P_n\| \leq \liminf \|S^{*n} C^*_\varphi C_\varphi S^n\|$$

we see that if $S^{*n} C^*_\varphi C_\varphi S^n \to T_{\|\sigma_\alpha\|}$ in norm then

$$\sup_\alpha \|\sigma_\alpha\| \leq \|T_{\|\sigma_\alpha\|}\| = \operatorname{ess.sup}_\alpha \|\sigma_\alpha\|,$$

which is obviously false if, say, C_φ is non-compact but $m(E) = 0$; the map $\varphi(z) = (z+1)/2$ is an example.

Indeed, it is known that an operator A is *uniformly asymptotically Toeplitz* (that is, $S^{*n}AS^n$ is norm convergent) if and only if A is the sum of a Toeplitz operator and a compact operator. It would be interesting to know when this is the case for $C_\varphi^*C_\varphi$; from what we have done so far this will be the case when φ is inner (the "pure Toeplitz" case) and when C_φ is compact (the "pure compact" case). Interestingly, the intermediate case is also possible, that is, we may have $C_\varphi^*C_\varphi = T_f + K$ with both f and K nonzero. While this example is interesting, it is tangential to the remaining results of the paper so is relegated to the appendix. It is known [11] that $C_\varphi^*C_\varphi$ is a Toeplitz operator if and only if φ is inner.

4. Convergence of weighted iterates for inner symbols

In the remainder of the paper, we will consider only inner symbols φ. Broadly, our goal is to relate the behavior of the map

$$T_f \to C_\varphi^* T_f C_\varphi$$

to the dynamics of φ viewed as a transformation of the unit circle $\mathbb{T}$. When φ fixes a point in the interior of the disk, the analysis is relatively simple; this is the subject of the present section. The next section handles the case of boundary fixed points under the additional assumption that φ is a finite Blaschke product. The proofs in the next section require techniques from the theory of complex dynamical systems, and so the methods of proof used there do not carry over to the case of general inner functions. This will be discussed further at the end of the next section.

We begin by considering certain weighted composition operators. In particular, when φ is inner, it is possible to modify C_φ by a Toeplitz operator T_a such that

$$V_\varphi = C_\varphi T_a = T_{a\circ\varphi} C_\varphi$$

is an isometry, and such that a version of Theorem 3.3 still holds. In order to state this theorem, we introduce the *normalized Aleksandrov operator*, defined by

$$\widetilde{A}_\varphi(f)(\alpha) = \frac{1}{\|\sigma_\alpha\|} \int f(\zeta)\, d\sigma_\alpha(\zeta)$$

When φ is a finite Blaschke product, the next two theorems will be superseded by the theorems of Section 5.

Theorem 4.1. *Let φ be an inner function and let*

$$a(z) = \frac{1 - \overline{\varphi(0)}z}{(1 - |\varphi(0)|^2)^{1/2}}$$

Then the operator

$$V_\varphi := C_\varphi T_a$$

is an isometry, and for every $f \in L^\infty(\mathbb{T})$

$$V_\varphi^* T_f V_\varphi = T_{\widetilde{A}_\varphi(f)}$$

Proof. By Theorem 3.3, $C_\varphi^* C_\varphi$ is a Toeplitz operator with symbol

$$A_\varphi^s(1)(\alpha) = \int d\sigma_\alpha = \|\sigma_\alpha\|$$

On the other hand, the symbol of $C_\varphi^* C_\varphi$ can be computed directly ([3, Proposition 3] or [11, Lemma 2.5]) as

$$b(\alpha) = \frac{1 - |\varphi(0)|^2}{|1 - \overline{\varphi(0)}\alpha|^2} = \left| \frac{(1 - |\varphi(0)|^2)^{1/2}}{1 - \overline{\varphi(0)}\alpha} \right|^2$$

With g defined as above we have $I = T_a^* T_b T_a = V_\varphi^* V_\varphi$. Furthermore, if $f \in L^\infty$ then

$$\begin{aligned} V_\varphi^* T_f V_\varphi &= T_a^* T_{A_\varphi^s(f)} T_a \\ &= T_{|a|^2 A_\varphi^s(f)} \end{aligned}$$

By the calculation above, we have $|a(\alpha)|^2 = \|\sigma_\alpha\|^{-1}$ for all $\alpha \in \mathbb{T}$, and the theorem follows. $\square$

Theorem 4.2. *Suppose φ is an inner function with Denjoy-Wolff point $w \in \mathbb{D}$. Then for every $f \in C(\mathbb{T})$,*

$$\lim_{n\to\infty} V_\varphi^{*n} T_f V_\varphi^n = \left(\int f \, dm_w \right) \cdot I$$

in norm, where m_w is harmonic measure at w.

Proof. As before first assume $\varphi(0) = 0$; in this case 0 is the Denjoy-Wolff point and harmonic measure m_0 is Lebesgue measure m. Since 0 is fixed C_φ is an isometry and in particular $V_\varphi = C_\varphi$. Fix $f \in C(\mathbb{T})$; then by Theorem 4.1 we have for all $n \geq 0$

$$V_\varphi^{*n} T_f V_\varphi^n = T_{A_\varphi^n f}$$

Since the norm of a Toeplitz operator is equal to the uniform norm of its symbol, proving the theorem amounts to proving

$$A_\varphi^n f \to \int f \, dm$$

uniformly. To prove this, it suffices to prove it for f equal to the Poisson kernel P_z for each $z \in \mathbb{D}$, since the linear span of these functions is dense in $C(\mathbb{T})$. We now have by the definition of A_φ

$$A_\varphi(P_z)(\alpha) = \int P_z \, d\sigma_\alpha = P_{\varphi(z)}(\alpha)$$

Iterating, we obtain

$$A_\varphi^n(P_z) = P_{\varphi^n(z)}$$

By the Denjoy-Wolff theorem, for each $z \in \mathbb{D}$ the sequence $\varphi^n(z)$ converges to $\varphi(0) = 0$, and it is then clear that

$$A_\varphi^n(P_z)(\alpha) = P_{\varphi^n(z)}(\alpha) = \frac{1 - |\varphi^n(z)|^2}{|\alpha - \varphi^n(z)|^2} \to 1 = \int P_z \, dm$$

uniformly in α.

In the general case $w \neq 0$, let

$$\psi(z) = \frac{w - z}{1 - \overline{w}z}$$

and define a unitary operator U_ψ on H^2 by

$$(U_\psi h)(z) = \frac{(1 - |w|^2)^{1/2}}{1 - \overline{w}(z)} h(\psi^{-1}(z))$$

It is a straightforward calculation to verify the identity

$$U_\psi^* T_f U_\psi = T_{f \circ \psi}$$

Moreover, if V_φ is the isometry of Theorem 4.1, then

$$U_\psi^* V_\varphi U_\psi = C_{\psi^{-1} \circ \varphi \circ \psi} = C_\theta$$

and the symbol of this latter composition operator is an inner function fixing 0. Therefore

$$\begin{aligned} \lim_{n \to \infty} U_\psi^* V_\varphi^{*n} T_f V_\varphi^n U_\psi &= \lim_{n \to \infty} C_\theta^{*n} T_{f \circ \psi} C_\theta^n \\ &= P[f](\psi(0)) \cdot I \\ &= \left(\int f \, dm_w \right) \cdot I \end{aligned}$$

and conjugating by U_ψ completes the proof. $\square$

4.1. **Remarks.** 1. From the point of view of dynamical systems, Theorem 4.2 can be interpreted as a very strong mixing property for φ. If φ is an inner function fixing 0, then φ preserves Lebesgue measure on the circle. Aaronson [1] and independently Neuwirth [16] proved that such a φ is "strong mixing" (and hence ergodic) on the circle; this is essentially equivalent to the statement that for every $f \in L^1(\mathbb{T})$, the iterates $A_\varphi^n f$ converge to the constant $\int f\, dm$ in L^1 (indeed this is how Aaronson proves the result, though without discussing the Aleksandrov operator explicitly.) Since continuous functions are dense in L^1, this is obviously implied by Theorem 4.2.

2. It is not difficult to modify the above proof to show that if we omit the convergence factor T_g then

$$C_\varphi^{*n} T_f C_\varphi^n \to \left(\int f\, dm \right) T_{P_w}$$

where P_w is the Poisson kernel at w. Indeed, in this case for each $g, h \in H^2$ the measures

$$g \circ \varphi^n \overline{h \circ \varphi^n} dm$$

will converge weak-* to

$$\left(\int g\bar{h}\, dm_w \right) dm.$$

However the convergence factor will be necessary in the next section to obtain convergence in the case $|w| = 1$. Indeed, consider

$$\varphi(z) = \left(\frac{3z+1}{z+3} \right)^2$$

This φ has Denjoy-Wolff point 1 and $|\varphi'(1)| = 1$. It is easy to see that the iterates $C_\varphi^{*n} T_f C_\varphi^n$ will not converge in general; for $f \equiv 1$, the operator $C_\varphi^{*n} C_\varphi^n$ is Toeplitz with symbol

$$\frac{1 - |\varphi^n(0)|^2}{|1 - \overline{\varphi^n(0)} z|^2}$$

Since $\varphi^n(0) \to 1$ along the real axis, this sequence is divergent even in the weak operator topology. Thus to obtain convergence some normalization as in Theorem 4.1 is necessary. In the next section we show that the renormalized iterates are in fact norm convergent. As we have already mentioned, however, the proofs in the next section depend heavily on the theory of rational dynamics, and do not seem to extend to cover general inner functions. Nonetheless there is good reason to think that we should still have convergence of the normalized iterates when $|w| = 1$.

Question 4.3. *Suppose φ is an inner function which does not fix any point in $\mathbb{D}$ and $f \in C(\mathbb{T})$. Is the sequence of operators*

$$V_\varphi^{*n} T_f V_\varphi^n$$

norm convergent?

3. While elementary, the proof of the previous theorem in some sense masks its underlying dynamical content. Indeed, one may be initially tempted to conjecture that when the Denjoy-Wolff point w of φ lies on the boundary, the iterates $V_\varphi^{*n} T_f V_\varphi^n$ will converge to the scalar $f(w)$. However this is not the case; the proofs in the next section will give a clearer idea of what may be expected in the boundary case. From this point of view, the harmonic measure m_w arises as the unique $|a|^2$-conformal probability measure supported on the Julia set of φ (which coincides with the circle when $|w| < 1$).

5. The transfer operator and conformal measures for finite Blaschke products

In this section φ will be a finite Blaschke product. We would like to establish convergence of the sequence

$$C_\varphi^{*n} T_f C_\varphi^n$$

for continuous symbols f. However, as remarked above, this sequence can be badly divergent even for very reasonable φ and f. In this section we prove that if we replace C_φ with certain weighted composition operators

$$W_{h,\varphi} := T_h C_\varphi \tag{5.1}$$

then the sequence

$$W_{\varphi,h}^{*n} T_f W_{\varphi,h}^n$$

will converge in norm to a scalar multiple of the identity operator. The map assigning the symbol f to this limiting scalar then determines a measure μ on the circle, which turns out to be a *conformal measure* which appears in the theory of complex dynamical systems.

The weight functions h appearing in (5.1) will be continuous functions on the circle which extend analytically to the open unit disk, with the property that

$$g := \log |h|^2$$

is an *admissible weight*, which we now define:

Definition 5.1. *A function $g : \mathbb{T} \to \mathbb{R}$ will be called an* admissible weight *if*

(1) *g is Hölder continuous, that is, for some $C > 0$ and $0 < \alpha < 1$ we have*
$$|g(z') - g(z)| \leq C|z' - z|^{\alpha}$$
for all $z, z' \in \mathbb{T}$.

(2) *For all $z \in \mathbb{T}$,*
$$\sum_{\varphi(z')=z} \exp(g(z')) = 1$$

We may now define a unital completely positive map $\mathcal{L} : C(\mathbb{T}) \to C(\mathbb{T})$ by

$$\mathcal{L}(f)(z) = \sum_{\varphi(z')=z} \exp(g(z'))f(z') \tag{5.2}$$

The operator $\mathcal{L}$ will be called the *transfer operator* for the pair (φ, g). This operator is also sometimes called a *Perron-Frobenius-Ruelle* operator. The algebraic significance of $\mathcal{L}$ is that it is a left inverse for composition by φ on $C(\mathbb{T})$, and in fact for any $f_1, f_2 \in C(\mathbb{T})$ we have
$$\mathcal{L}(f_1 \cdot (f_2 \circ \varphi)) = \mathcal{L}(f_1) \cdot f_2.$$

In this section we prove that given any such g, there is an analytic Toeplitz operator T_h (with continuous symbol) such that for all $f \in C(\mathbb{T})$,

$$\Phi_g(T_f) = W_{\varphi,h}^* T_f W_{\varphi,h} = T_{\mathcal{L}(f)} \tag{5.3}$$

We also prove that there exists a unique invariant $\exp(g)$-conformal probability measure μ supported on the Julia set $J(\varphi)$ such that

$$\lim_{n\to\infty} \Phi_g^n(T_f) = \left(\int f\, d\mu\right) \cdot I \tag{5.4}$$

(By definition a measure μ on J is called F*-conformal* if for every Borel subset $E \subset J$ on which φ is injective,
$$\mu(E) = \int_{\varphi(E)} F\, d\mu,$$
that is, F is a measure-theoretic Jacobian for φ^{-1} with respect to μ.) Conformal measures are important dynamical invariants for φ; they were introduced by D. Sullivan [19] (in the special case $F = |\varphi'|^{\delta}$) as a technique for estimating the Hausdorff dimension of Julia sets. For example Sullivan proved that the Hausdorff dimension of $J(\varphi)$ is

equal to the infimum over all δ such that there exists a $|\varphi'|^{\delta}$-conformal measure supported on the Julia set $J(\varphi)$.

5.1. **Weighted CP maps implementing $\mathcal{L}$.** We now show that the transfer operators $\mathcal{L}$ considered in the previous section can be implemented on the symbols of Toeplitz operators by conjugating by an isometric weighted composition operator of the form $W_{h,\varphi} = T_h C_\varphi$. The convergence results of the last section then imply norm convergence of the iterates of the map

$$T_f \to W_{\varphi,h}^* T_f W_{\varphi,h}$$

to a scalar multiple of I; and this scalar is given by the integral of f against the conformal measure associated to $\mathcal{L}$.

Theorem 5.2. *Let φ be a finite Blaschke product and let g be a Hölder continuous function on $\mathbb{T}$ satisfying*

$$\sum_{\varphi(z')=z} \exp(g(z')) = 1 \tag{5.5}$$

for all $z \in \mathbb{T}$. Then there exists a function h in the disk algebra such that the weighted composition operator

$$W_{\varphi,h} = T_h C_\varphi$$

is an isometry and

$$W_{\varphi,h}^* T_f W_{\varphi,h} = T_{\mathcal{L}(f)}$$

for all $f \in C(\mathbb{T})$.

Proof. For any $h \in A(\mathbb{D})$, it follows from Corollary 3.4 that

$$W_{\varphi,h}^* T_f W_{\varphi,h} = T_{A_\varphi(|h|^2 f)} \tag{5.6}$$

Since φ is a finite Blaschke product, the Aleksandrov measures μ_α for φ are given by

$$\mu_\alpha = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta$$

(see [5, Example 9.2.4].)It follows that

$$A_\varphi(|h|^2 f)(z) = \sum_{\varphi(\zeta)=z} \frac{|h(\zeta)|^2}{|\varphi'(\zeta)|} f(\zeta)$$

It therefore suffices to produce a function $h \in A(\mathbb{D})$ such that

$$\log|h| = \frac{1}{2}\left(\log|\varphi'| + g\right) \tag{5.7}$$

Since $\log|\varphi'| + g$ is obviously integrable on $\mathbb{T}$, we can define h to be the outer function

$$h(z) = \exp\left(\frac{1}{2}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(g(e^{i\theta}) + \log|\varphi'(e^{i\theta})|\right)\, dm(\theta)\right) \tag{5.8}$$

By definition, h is analytic in $\mathbb{D}$. To see that it has continuous boundary values, we observe that since φ' is nonvanishing on the circle, the function $\log|\varphi'|$ is real analytic and hence $g + \log|\varphi'|$ is Hölder continuous on $\mathbb{T}$. Since harmonic conjugation preserves the Hölder classes [8, Theorem 5.8], it follows from that the function inside the exponential in (5.8) is continuous on the circle, and thus so is h. □

Remark: The Hölder assumption on g is necessary in the above theorem in order to show that the outer function h is continuous; if g is only assumed continuous this need not be the case. This is because harmonic conjugation preserves the Hölder classes $\Lambda_\alpha(\mathbb{T})$ (for $\alpha < 1$) but not $C(\mathbb{T})$.

5.2. **Convergence of the iterates of the transfer operator.** The convergence in (5.4) is proved using (5.3) and a Perron-Frobenius theorem for the transfer operator $\mathcal{L}$. To state this result, we first make some definitions. Given a transfer operator $\mathcal{L}$, a *unitary eigenvalue* of $\mathcal{L}$ is an eigenvalue of modulus 1, and the set of all such eigenvalues is called the *unitary spectrum*, denoted spec_u. The *unitary eigenspace* of $\mathcal{L}$ is the closed linear span in $C(\mathbb{T})$ of the eigenspaces of the unitary eigenvalues. The operator $\mathcal{L}$ is called *almost periodic* if for each $f \in C(\mathbb{T})$, the set of iterates $\{\mathcal{L}^n(f)\}_{n\geq 0}$ is precompact in $C(\mathbb{T})$. We then have the following, a special case of a result due to Ljubich [14, p. 354]:

Theorem 5.3. *Assume that $\mathcal{L}$ is almost periodic, $\mathrm{spec}_u = \{1\}$, and the unitary eigenspace of $\mathcal{L}$ is one-dimensional (and hence consists only of the scalars). Then there exists an $\mathcal{L}^*$-invariant measure μ_g on $\mathbb{T}$ such that for all $f \in C(\mathbb{T})$,*

$$\mathcal{L}_g^n(f) \to \left(\int f\, d\mu_g\right) \cdot 1$$

in norm.

We note that since the $\mathcal{L}^*$-invariance of μ_g means simply

$$\int \mathcal{L}(f)\, d\mu_g = \int f\, d\mu_g$$

for all $f \in C(\mathbb{T})$, the measure μ_g is seen to be $\exp(g)$-conformal by a standard approximation argument. We also observe that in the present setting the measure μ must be supported on the Julia set of φ, and because it is an $exp(g)$-conformal measure, the support is in fact equal to the Julia set, and therefore by the theorem of Denker and Urbański [7, Theorem 30] this measure is unique. This will be discussed further below.

Given the formula (5.3), which will be proved in Section 4.2, the convergence in (5.4) will follow once we have proved that our transfer operator $\mathcal{L}$ satisfies the hypotheses of Theorem 5.3. The spectral conditions will be checked presently. To prove almost periodicity, we observe that since $\mathcal{L}$ is contractive, the iterates $\{\mathcal{L}^n f\}$ are uniformly bounded and hence by the Arzela-Ascoli theorem it suffices to show that this set is equicontinuous on $\mathbb{T}$. This is done in Section 5.3 below.

We now prove that $\mathcal{L}$ satisfies the spectral hypotheses of Theorem 5.3. The argument is essentially the same as that given by Ljubich [14, Lemma 2] for the case $g \equiv \log d$.

Theorem 5.4. *For $\mathcal{L}$ as above, $spec_u(\mathcal{L}) = \{1\}$, and the corresponding eigenspace contains only the constants.*

Proof. Let $\lambda \in \mathrm{spec}_u(\mathcal{L})$ and let f be a nonzero eigenfunction. Let $z \in \mathbb{T}$ maximize $|f(z)|$. Then since $\mathcal{L}(f)(z) = \lambda f(z)$ is a convex combination of the values of f at the preimages $\zeta \in \varphi^{-1}(z)$, it follows that

$$f(\zeta) = \lambda f(z)$$

for each such ζ. Iterating, we find

$$f(\zeta) = \lambda^n f(z)$$

for all $\zeta \in \varphi^{-n}(z)$. Since the backwards orbit of z accumulates on the Julia set $J(\varphi)$, for any $w \in J(\varphi)$ we can find a sequence $\zeta_n \in \varphi^{-n}(z)$ such that $\zeta_n \to w$. Thus

$$\lambda^n f(z) \to f(w)$$

and $\lambda = 1$. Assume now that f is real. It follows that f is constant on the Julia set and if f attains its maximum at z, $f(z)$ is equal to this constant. The same argument applied to the minimum of f shows that f is constant; considering the real and imaginary parts of f separately completes the proof. $\square$

5.3. **Equicontinuity of $\{\mathcal{L}^n f\}$.** Our proof of equicontinuity follows broadly the argument of Przytycki [17, Appendix A] but also draws on the earlier argument of Ljubich [14]. Before beginning the proof we make a few preliminary observations about the dynamics of φ.

In our setting the arguments of [17] simplify considerably since, first, we consider only unital transfer operators, and second, the dynamics of Blaschke products are comparatively simple—

The first observation is that a finite Blaschke product φ has no critical points on the circle, and the forward orbit of the set of critical points accumulates only at the Denjoy-Wolff point. Indeed, let

$$\varphi(z) = \lambda \prod_{i=1}^{d} \frac{\alpha_i - z}{1 - \overline{\alpha_i} z}$$

Then by an elementary computation,

$$|\varphi'(z)| = \sum_{i=1}^{d} \frac{1 - |\alpha_i|^2}{|\alpha_i - z|^2}$$

for all $|z| = 1$. Thus every critical point of φ in the Riemann sphere $\hat{\mathbb{C}}$ lies either in the interior or the exterior of the unit disk. Moreover, by the Denjoy-Wolff theorem, if the Denjoy-Wolff point w of φ lies on the circle then the iterates of φ converge uniformly to w on every compact subset disjoint from $\mathbb{T}$. In particular the forward orbit of the critical points of φ accumulates only at w. (If $|w| < 1$ then this orbit does not accumulate at any point of $\mathbb{T}$.)

To prove the equicontinuity of the iterates $\mathcal{L}^n f$, it suffices (since $\mathbb{T}$ is compact) to prove equicontinuity on a neighborhood of every point of $\mathbb{T}$. In the proof of Theorem 5.6 we will first do this for points other that the Denjoy-Wolff point w of φ. Our second observation is that for $z_0 \neq w$ on $\mathbb{T}$, by the remarks above there is a neighborhood U of z_0 such that the closure of U is disjoint form the forward orbit of the critical points of φ. Thus on U there exists a system of inverse branches $\{\varphi_k^{-m}\}$, $k = 1, \dots d^m$ of φ^m for each $m = 1, 2, \dots$. Moreover these branches may be chosen compatibly, in the sense that for any $m < n$ and $1 \leq i \leq d^n$ there exists $1 \leq j \leq d^{n-m}$ such that

$$\varphi^m \circ \varphi_i^{-n} = \varphi_j^{m-n} \tag{5.9}$$

on U. By a standard application of Montel's theorem [14, Proposition 2] the family $\{\varphi_k^{-m}\}$ is equicontinuous on U.

The next lemma is elementary; we state it so as to have a ledger of the constants that will be used in the proof of Theorem 5.6 below.

Lemma 5.5. *Let φ be a finite Blaschke product, g an admissible weight and f a continuous function on the circle. Let $\epsilon > 0$ be given. Then there exist:*

(1) *β and γ such that* $\sup_{\mathbb{T}} e^g \leq \beta < \gamma < 1$

(2) ϵ' *such that* $|1-e^x| < \epsilon/(4\|f\|_\infty)$ *whenever* $|x|<\epsilon'$
(3) $n_0 \in \mathbb{N}$ *such that*
 a) $C\sum_{j=n_0}^{\infty}(\gamma^\alpha)^j < \frac{\epsilon'}{2}$, *where* C *and* α *are as in the Hölder inequality for* g, *and*
 b) $\sum_{j=n_0}^{\infty}\left(\frac{\beta}{\gamma}\right)^j < \frac{\epsilon}{8\|f\|_\infty}$
(4) $\delta_1 > 0$ *such that* $d(z,\zeta)<\delta_1$ *implies* $|f(z)-f(\zeta)| < \frac{\epsilon}{2}$
(5) $\delta_2 > 0$ *such that* $d(z,\zeta)<\delta_2$ *implies* $|\varphi_k^{-m}(z)-\varphi_k^{-m}(\zeta)|<\delta_1$ *for all* $k=1,\dots d^m$, $m=1,2,\dots$
(6) $\delta_3>0$ *such that* $d(z,\zeta)<\delta_3$ *implies*

$$\sum_{m<n_0}\sum_{k=1}^{d^m}\left|\left(g(\varphi_k^{-m}(z))-g(\varphi_k^{-m}(\zeta))\right)\right| < \frac{\epsilon'}{2}$$

Proof. (1) follows from the unitarity condition in the definition of an admissible weight, together with the fact that g is continuous.

(2) is of course the continuity of e^x; (3a) follows from the fact that $\gamma,\alpha<1$ and (3b) from $\beta<\gamma$.

Finally, (4) is the uniform continuity of f, (5) is the equicontinuity of the branches $\{\varphi_k^{-m}\}$ mentioned above, and (6) the equicontinuity of the finite family of functions $g\circ\varphi_k^{-m}$ for $m<n_0$, $k=1,\dots d^m$. □

To streamline the notation, define for each n

$$E_n(z)=\exp\left(\sum_{j=0}^{n-1} g(\varphi^j(z))\right)$$

It follows that for all n and all $z\in\mathbb{T}$

$$\mathcal{L}^n(f)(z)=\sum_{\varphi^n(\zeta)=z}E_n(\zeta)f(\zeta)$$

and in particular

$$\sum_{\varphi^n(\zeta)=z}E_n(\zeta)\equiv 1.$$

Theorem 5.6. *The sequence of functions* $\{\mathcal{L}^n(f)\}$ *is equicontinuous on* $\mathbb{T}$.

Proof. Fix a point on the circle $z_0\neq w$ and a neighborhood U of z_0 in $\mathbb{C}$ such that the closure of U is disjoint from the forward orbit of the critical points (in particular $w\notin U$). Now fix an open arc $I\subset\mathbb{T}\cap U$ containing z_0; we first prove that $\{\mathcal{L}^n f\}$ is equicontinuous on I. It suffices to show that given $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $z, \zeta \in I$ and $d(z, \zeta) < \delta$,

$$|\mathcal{L}^n(f)(z) - \mathcal{L}^n(f)(\zeta)| < \epsilon$$

for all $n \geq n_0$ chosen as in Lemma 5.5. Indeed, if this is possible then equicontinuity for all n follows by shrinking δ sufficiently. We claim that $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ works. After proving this we will prove equicontinuity on an arc containing w.

For $z, \zeta \in I$

$$\begin{aligned} |\mathcal{L}^n(f)(z) - \mathcal{L}^n(f)(\zeta)| &= \\ &= \left| \sum_{k=1}^{d^n} E_n(\varphi_k^{-n}(z)) f(\varphi_k^{-n}(z)) - \sum_{k=1}^{d^n} E_n(\varphi_k^{-n}(\zeta)) f(\varphi_k^{-n}(\zeta)) \right| \\ &\leq \sum_{k=1}^{d^n} E_n(\varphi_k^{-n}(z)) \left| f(\varphi_k^{-n}(z)) - f(\varphi_k^{-n}(\zeta)) \right| + \qquad \text{(I)} \\ &\quad + \sum_{k=1}^{d^n} |f(\varphi_k^{-n}(\zeta))| \left| E_n(\varphi_k^{-n}(z)) - E_n(\varphi_k^{-n}(\zeta)) \right| \qquad \text{(II)} \end{aligned}$$

We now estimate the sums (I) and (II) separately. If $d(z, \zeta) < \delta$ then by items (4) and (5) of Lemma 5.5, the sum (I) will be less than $\epsilon/2$. To estimate (II), we split it into two further sums (II$'$) and (II$''$) according to the behavior of the branches $\varphi^j(\varphi_k^{-n})$. For the constant γ of Lemma 5.5, we call a branch φ_k^{-m} "good" if

$$\operatorname{diam}(\varphi_k^{-m}(I)) \leq \gamma^m$$

and "bad" otherwise. By a simple packing argument, for each m the number of bad branches of φ^{-m} is at most γ^{-m}.[1] Now for each $n \geq n_0$ define

$$T_n = \left\{ t \in \{1, \dots d^n\} : \varphi^j \circ \varphi_t^{-n} \text{ is good for all } 0 \leq j \leq n - n_0 \right\}$$

We now split the sum (II) into two sums (II$'$) and (II$''$) according as the index $k \in T_n$ or $k \notin T_n$ respectively. Working first with (II$'$), write

$$\text{II}' = \sum_{k \in T_n} |f(\varphi_k^{-n}(\zeta))| \left| E_n(\varphi_k^{-n}(\zeta)) - E_n(\varphi_k^{-n}(z)) \right| \tag{5.10}$$

$$= \sum_{k \in T_n} |f(\varphi_k^{-n}(\zeta))| E_n(\varphi_k^{-n}(\zeta)) \left| 1 - \frac{E_n(\varphi_k^{-n}(z))}{E_n(\varphi_k^{-n}(\zeta))} \right| \tag{5.11}$$

[1] Note that the set of good branches is always nonempty, that is, γ is always strictly larger than $1/d$. To see this, combine the assumption (5.1) with item (1) of Lemma 5.5.

To show that this sum is less than $\epsilon/4$, it suffices to prove that $d(z,\zeta) < \delta$ implies

$$\left|\sum_{j=1}^{n}\left(g(\varphi^j(\varphi_k^{-n}(\zeta))) - g(\varphi^j(\varphi_k^{-n}(z)))\right)\right| < \epsilon' \tag{5.12}$$

for each $k \in T_n$, with ϵ' chosen as in Lemma 5.5. Indeed if this is so then

$$\left|1 - \frac{E_n(\varphi_k^{-n}(z))}{E_n(\varphi_k^{-n}(\zeta))}\right| < \frac{\epsilon}{4\|f\|_\infty}$$

and it follows that (5.11) is less than $\epsilon/4$.

To prove (5.12), we consider the sums from $j = 1$ to $n - n_0$ and from $n - n_0 + 1$ to n separately. For the first sum, by the definition of T_n all of the branches $\varphi^j(\varphi_k^{-n})$ are good. Therefore, for $0 \le j \le n_0$

$$d\left(\varphi^j(\varphi_k^{-n}(z)), \varphi^j(\varphi_k^{-n}(\zeta))\right) \le \gamma^{n-j}.$$

Since g is Hölder continuous, it follows that

$$\left|g(\varphi^j(\varphi_k^{-n}(z))) - g(\varphi^j(\varphi_k^{-n}(\zeta)))\right| \le C \cdot \gamma^{\alpha(n-j)}$$

By the choice of δ_3, the portion of the sum (5.12) from $n - n_0 + 1$ to n is no more than $\epsilon'/2$. Therefore the entire sum (5.12) is dominated by

$$\frac{\epsilon'}{2} + \sum_{j=1}^{n-n_0} C\gamma^{\alpha(n-j)} = \frac{\epsilon'}{2} + C\sum_{j=n_0}^{n-1}\gamma^{\alpha j} < \epsilon'$$

We have thus proved that (II$'$) is less than $\epsilon/4$.

Finally we consider (II$''$). Write temporarily $\psi_{jk} = \varphi^j(\varphi_k^{-n})$. Since $\mathcal{L}^n = \mathcal{L}^{n-j}\mathcal{L}^j$ for each j, we can rewrite (II$''$) as

$$\sum_{j=0}^{n-n_0}\left|\sum_{\{k:\psi_{jk}\text{ bad}\}} E_{n-j}(\psi_{jk}(\zeta))\mathcal{L}^j(f)(\psi_{jk}(\zeta)) - E_{n-j}(\psi_{jk}(z))\mathcal{L}^j(f)(\psi_{jk}\zeta)\right|$$

(In other words, we have sorted the branches $\psi_{jk} = \varphi^j(\varphi_k^{-n})$, for $k \notin T_n$, into those that are bad for $j = 0$, then good for $j = 0$ but bad for $j = 1$, etc.) Recalling that $\sup E_m \le \beta^m$ and that the number of bad branches of the form $\varphi^j(\varphi_k^{-n})$ is at most γ^{j-n}, we see that for each fixed $j \le n - n_0$ the term in the absolute value bars above is controlled by

$$2\|f\|_\infty\beta^{n-j}\gamma^{j-n}$$

Thus we can estimate (II$''$) by

$$\mathrm{II}'' \le 2\|f\|_\infty\sum_{j=0}^{n-n_0}\left(\frac{\beta}{\gamma}\right)^{n-j} < \epsilon/4$$

by the choice of n_0 (item 3(b) of Lemma 5.5). This completes the proof of equicontinuity on open arcs not containing w.

We now prove equicontinuity on a neighborhood of w. Let $\epsilon > 0$ be given, and let N be a positive integer such that

$$E_N(w) = \exp(Ng(w)) < \frac{\epsilon}{6\|f\|_\infty}$$

We observe that because $\varphi'(w) > 0$, for every $p \geq 1$ the number w is a root of $\varphi^p(z) = w$ of multiplicity one. Let z_i denote the roots of the equation

$$\varphi^N(z) = w, \quad i = 1, \dots d^N - 1$$

different from w (repeated according to multiplicity). Since all of the points z_i are distinct from w, by Theorem (previous) we can find $\delta' > 0$ such that

$$d(z_i, \zeta) < \delta' \quad \text{implies} \quad |\mathcal{L}^n(f)(z_i) - \mathcal{L}^n(f)(\zeta)| < \frac{\epsilon}{3}$$

for $i = 1, \dots d^N - 1$ and all $n = 1, 2, \dots$. Shrink δ' further if necessary so that

$$d(z, \zeta) < \delta' \quad \text{implies} \quad |E_N(z) - E_N(\zeta)| < \frac{\epsilon}{3d^N\|f\|_\infty}$$

for all $z, \zeta \in \mathbb{T}$. For this δ', choose $0 < \delta < \delta'$ so that

$$d(a, \zeta) < \delta \quad \text{implies} \quad d(z_i, \zeta_i) < \delta'$$

for appropriately chosen ζ_i satisfying $\varphi^N(\zeta_i) = \zeta$. Thus, if $d(a, \zeta) < \delta$ we can estimate for all n (as at the beginning of Theorem 5.6)

$$|\mathcal{L}^{N+n}(f)(a) - \mathcal{L}^{N+n}(f)(\zeta)| \leq$$

$$\leq \sum_{k=1}^{d^N-1} E_N(z_i)\, |\mathcal{L}^n(f)(\zeta_i) - \mathcal{L}^n(f)(z_i)| + \tag{5.13}$$

$$+ \sum_{k=1}^{d^N-1} |\mathcal{L}^n(f)(\zeta_i)|\, |E_N(\zeta_i) - E_N(z_i)| + \tag{5.14}$$

$$+ |E_N(w)\mathcal{L}^n(f)(w) - E_N(\zeta)\mathcal{L}^n(f)(\zeta)| \tag{5.15}$$

By the choice of δ and δ', the two sums are each less than $\epsilon/3$. The last term is controlled by

$$(E_N(w) + E_N(\zeta))\, \|f\|_\infty$$

which is less than $\epsilon/3$ by the choice of N and δ. It follows that the family $\{\mathcal{L}^n(f)\}$ is equicontinuous on the arc of radius δ around w for $n \geq N$, and hence for all n by shrinking δ. ∎

Theorem 5.7. *Let g be an admissible weight and $\mathcal{L}$ the associated transfer operator. Then there exists a probability measure μ such that for all $f \in C(\mathbb{T})$,*

$$\lim_{n\to\infty} \mathcal{L}^n(f) = \int f\, d\mu$$

uniformly. Moreover the support of μ is equal to the Julia set of φ.

Proof. The existence of μ follows from Theorem 5.3; its hypotheses are verified by Theorems 5.4 and 5.6. It remains only to check the claim that $\operatorname{supp}(\mu) = J$.

Since

$$\mathcal{L}_g^n(f)(z) = \sum_{\varphi^n(\zeta)=z} E_n(\zeta) f(\zeta)$$

we may write

$$\mathcal{L}_g^n(f)(z) = \int f\, d\mu_n^z$$

where the measure μ_n^z is supported on the finite set $\varphi^{-n}(z)$. It follows that

$$\operatorname{supp}(\mu) \subseteq \overline{\bigcup_{j=1}^{\infty} \varphi^{-j}(z)} = K_z$$

for every $z \in \mathbb{T}$. Since the set of limit points of $\cup_j \varphi^{-j}(z)$ is precisely the Julia set J, we see that K_z is the union of J with a countable set of isolated points. Thus if μ is not supported on J it must have an atom at some point $z_0 \in K \setminus J$. Consequently, μ_n^z must have at atom at z_0 for infinitely many n, and since this must be the case for every $z \in \mathbb{T}$, we conclude that for each $z \in \mathbb{T}$ there exists an N such that $z_0 \in \varphi^{-N}(z)$, which is obviously false. Thus $\operatorname{supp}(\mu) \subset J$. Since we know that μ is an $\exp(g)$ conformal measure, it follows from [7, Theorem 30] that in fact $\operatorname{supp}(\mu) = J$.

□

Combining Theorems 5.3 and 5.2, we have proved:

Theorem 5.8. *Let φ be a finite Blaschke product, let g be a Hölder continuous function on the circle as above, and $h \in A(\mathbb{D})$ as in Theorem 5.2. Then for every $f \in C(\mathbb{T})$,*

$$\lim_{n\to\infty} W_{\varphi,h}^{*n} T_f W_{\varphi,h}^n = \left(\int f\, d\mu_g \right) \cdot I$$

in norm.

In particular, since $\operatorname{supp}(\mu_g) = J(\varphi)$, a nonnegative function $f \in C(\mathbb{T})$ vanishes on the Julia set if and only if the above limit is 0. In this sense the composition operator C_φ can "see" the Julia set of φ.

Appendix A. An example

We now give an example, mentioned in the remarks following Theorem 3.3, of a composition operator C_φ such that $C_\varphi^* C_\varphi$ is uniformly asymptotically Toeplitz (and hence of the form $T_f + K$) with both f and K nonzero. (In fact one may compute the function f described below exactly to see that it is in fact continuous, but we will not need this). In what follows we let ψ denote the conformal map from $\mathbb{D}$ to the upper half plane $\mathbb{H}$

$$\psi(z) = i\frac{1+z}{1-z}$$

Example A.1. *Let φ be the Riemann map of the unit disk onto the upper half-disk $\{z \in \mathbb{D} | Im(z) > 0\}$, conjugate to the mapping $z \to \sqrt{z}$ on the upper half plane. Then*

$$C_\varphi^* C_\varphi = T_f + K$$

for a nonzero continuous function f and nonzero compact operator K.

To begin, we observe that φ extends continuously to the unit circle, and has modulus one precisely on the upper semicircle

$$E = \{z \in \mathbb{T} : \mathrm{Im} z \geq 0\},$$

which is taken by ψ to the nonnegative real axis (including the point at infinity). Now write

$$C_\varphi^* C_\varphi = C_\varphi^* T_{\chi_E} C_\varphi + C_\varphi^* T_{\chi_{E^c}} C_\varphi$$

The first operator is Toeplitz by Theorem 3.3. Both operators are nonzero, since they each give 1/2 under the state $\langle \cdot 1, 1\rangle$. To see that $C_\varphi^* T_{\chi_E^c} C_\varphi$ is compact, we first define a measure μ on Borel subsets $F \subset \overline{\mathbb{D}}$ by

$$\mu(F) = m(\varphi^{-1}(F) \cap \mathbb{T} \cap E^c)$$

This measure satisfies

$$\int_{\overline{\mathbb{D}}} g \, d\mu = \int_{E^c} g \circ \varphi \, dm \tag{A.1}$$

for all nonnegative measurable functions g in $\overline{\mathbb{D}}$ (see [6, Lemma 2.1]). For $\zeta = e^{i\theta} \in \mathbb{T}$ and $0 < r < 1$, let $S(\zeta, r)$ denote the Carleson region

$$S(\zeta, r) = \{z \in \overline{\mathbb{D}} : |\zeta - z| \leq r\}$$

We now claim that

$$\lim_{r \to 0} \sup_{\zeta \in \mathbb{T}} \frac{\mu(S(\zeta, r))}{r} = 0. \tag{A.2}$$

In fact we will show that there is an absolute constant C (independent of ζ) so that for all r sufficiently small,

$$\mu(S(\zeta, r)) \le Cr^2$$

which clearly implies (A.2). From the definition of μ, we see that for small r, $\mu(S(\zeta, r)) = 0$ unless $S(\zeta, r)$ contains 1 or -1. Assume now $S(\zeta, r)$ contains -1, which is sent to 0 by the conformal map ψ from $\mathbb{D}$ to the upper half plane $\mathbb{H}$. The set $S(\zeta, r)$ is taken by ψ to a half-disk in $\mathbb{H}$ centered on the real line, and for small r the radius of this disk is comparable to r. The half-disk intersects the positive real and imaginary axes in two segments each of length at most $C \cdot r$. The preimage of the disk under the mapping $z \to \sqrt{z}$ then intersects the real line in a segment of length at most $C \cdot r^2$, which has length again at most $C \cdot r^2$ when conjugated back to an arc of the circle. Since $\mu(S(\zeta, r))$ is the length of the intersection of this arc with E^c, we have proved the claim for ζ near -1; the case near 1 (taken to the point at infinity under ψ) is handled similarly.

Combining (A.2) with the identity (A.1), it can be shown using standard Carleson measure arguments (along the lines of the proof of [6, Theorem 3.4]) that the operator

$$f \to \chi_{E^c} \cdot (f \circ \varphi)$$

is compact from H^2 to L^2, which clearly implies the compactness of $C_\varphi^* T_{\chi_{E^c}} C_\varphi$.

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