MODERN ANALYSIS, FALL 2014
HOMEWORK 7

Problems marked with * are to be turned in for grading.

1. Let \( f, g : X \rightarrow Y \) be continuous functions and let \( E \) be a dense subset of \( X \).
   a) Prove that \( f(E) \) is dense in \( f(X) \).
   b) Prove that if \( f(p) = g(p) \) for all \( p \in E \), then \( f = g \). (That is, a continuous function is determined by its values on a dense set.)

*2. Let \( X \) be a metric space and \( C \subset X \) a nonempty closed set. For each \( x \in X \) define
   \[ f_C(x) := \inf \{ d(x, y) : y \in C \} . \]
   \( f_C(x) \) is called the distance from \( x \) to \( C \).
   a) Prove that \( f_C(x) = 0 \) if and only if \( x \in C \).
   b) Prove that the function \( f_C : X \rightarrow \mathbb{R} \) is continuous.
   c) Let \( X = \mathbb{R}^d \) and let \( C \subset \mathbb{R}^d \) be closed. Prove that for each \( x \in \mathbb{R}^d \), there exists a point \( x^* \in C \) such that \( f_C(x) = \|x - x^*\| \) (in other words, the infimum in the definition of \( f_C \) is attained). (Hint: for each \( n \) there exists \( x_n \in C \) with \( d(x, x_n) < f_C(x) + \frac{1}{n} \).) Show by example that the point \( x^* \) need not be unique.

3. Consider the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by
   \[ f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ in lowest terms}, n \geq 1 . \end{cases} \]
   Prove that \( f \) is continuous precisely on the irrationals.

*4. Suppose that \( X, Y \) are metric spaces, \( f : X \rightarrow Y \) is a function, and \( X \) is compact. Let \( X \times Y \) have the metric described in Homework 2, problem 1. The graph of \( f \) is the subset \( G \subset X \times Y \) defined by
   \[ G = \{(x, f(x)) : x \in X \} . \]
   Prove that \( f \) is continuous if and only if its graph \( G \) is compact.
5. Let $f : X \to Y$ be continuous. Prove that for any $E \subset X,$

$$f(E) \subset f(E).$$

(here the bar denotes the closure of the set.) Show by example that the inclusion can be proper.

6. Let $E \subset \mathbb{R}$ be bounded and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Prove that $f$ is bounded. Show, by example, that the conclusion can be false if “uniformly” is omitted.

7. Suppose that $f : X \to Y$ is uniformly continuous. Prove that if $(x_n)$ is a Cauchy sequence in $X,$ then $(f(x_n))$ is a Cauchy sequence in $Y.$ Show by example that the conclusion can be false if $f$ is only assumed continuous.