An optimal approximation problem for noncommutative polynomials

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Outline

Optimal Polynomial Approximants (opa's)

Classical Setting

For $f \in H^2(\mathbb{D})$, we say a polynomial p_n of degree at most n is an optimal approximant of order n if p_n minimizes

$$\|pf - 1\|_{H^2}$$

for $p \in \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of all polynomials of degree at most n.

Non commutative Setting

For an nc function, f in d noncommuting arguments, we find an nc polynomial p_n, of degree at most n, to minimize

$$\|pf - 1\|_{\mathcal{F}_d}$$

over $\mathcal{P}_n$, the set of all polynomials of total degree at most n.

Classical $H^2(\mathbb{D})$ setting

Due to Beurling: for $f \in H^2(\mathbb{D})$

$$\|p_n f - 1\|_{H^2} \to 0 \text{ as } n \to \infty$$

iff

f is an outer function.

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iff

$$f(z) \neq 0 \text{ in the disk } |z| < 1.$$

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So, for polynomial $f \in H^2(\mathbb{D})$:
$\|p_n f - 1\|_{H^2} \to 0$ as $n \to \infty$ iff $f(z) \neq 0$ in the disk $|z| < 1$.

Classical $H^2(\mathbb{D})$ setting

So, for polynomial $f \in H^2(\mathbb{D})$:
$\|p_n f - 1\|_{H^2} \to 0$ as $n \to \infty$ iff $f(z) \neq 0$ in the disk $|z| < 1$. A sketch:

Let us first consider $f(z) = 1 - z$. (Or, $1 - \alpha z$ for $|\alpha| \leq 1$)
Consider $q_n := 1 + z + z^2 + \cdots + z^n$. Then $1 \in \overline{\{p(z)f(z)\}}^{\|\cdot\|}$ since

- $(1 + z + z^2 + \cdots + z^n)(1 - z) \to 1$ pointwise in $\mathbb{D}$, and
- $sup_n \|1 - z^{n+1}\| < \infty$.

Thus $1 \in \overline{\{p(z)f(z)\}}^{\text{wk}}$ and hence $1 \in \overline{\{p(z)f(z)\}}^{\|\cdot\|}$.

Rest follows by induction on the factors of the polynomial(with no zeros on $\mathbb{D}$).

In the Non-commutative(nc) setting

Main Theorem

For an nc polynomial f,

$$\|p_n f - 1\|^2_{\mathcal{F}_d} \to 0$$

$$\text{if and only if}$$

$$\det(f) \neq 0 \text{ in row-ball.}$$

Remark: Main theorem gives a new proof of the cyclicity theorem in [1][1] with an estimate on the rate of decay.

[1] [Jury, Martin, Shamovich, 2021]

Quantitative Cyclicity

Classically[2], we have :
f is an outer polynomial then $\|p_n f - 1\|^2_{H^2} = O(\frac{1}{n})$.

We now prove that:

Noncommutative case:

f is an nc polynomial nonsingular on the row ball then $\|p_n f - 1\|^2_{\mathcal{F}_d} = O(\frac{1}{n^p})$ for some $p > 0$ depending on f.

[2] [[2]BÉNÉTEAU, CONDORI, LIAW, SECO, AND SOLA, 2015]

Fock Space

Definition

Let $\mathbb{x} = \{x_1, \ldots, x_d\}$ be freely noncommuting (abbreviated as nc) indeterminates.

The free monoid $\langle \mathbb{x} \rangle$ is called the word set in the letters $x_1, \ldots, x_d$ with $\varnothing$ representing the empty word.

If $w \in \langle \mathbb{x} \rangle$ and $w = x_{i_1} \ldots x_{i_m}$, then the length of w, $|w| = m$ while $|\varnothing| = 0$.

Let $\mathbb{C}\langle \mathbb{x} \rangle$ denote the **free algebra**. Define an inner product by declaring $\{w\}_{w \in \mathbb{C}\langle \mathbb{x} \rangle}$ orthonormal.

The completion of $\mathbb{C}\langle \mathbb{x} \rangle$ with respect to the inner product gives us $\mathcal{F}_d$, the **Fock space** on d letters.

Fock Space

Definition

Concretely,

$$\mathcal{F}_d = \left\{ \sum_{w \in \langle \mathbb{x} \rangle} a_w w \; : \; \sum_{w \in \langle \mathbb{x} \rangle} |a_w|^2 < \infty \right\}.$$

Shift Operators

Definition

The **left** d**-shift** is the tuple of operators $L = (L_1, \ldots, L_d)$ where each

$$L_i : \mathcal{F}_d \to \mathcal{F}_d$$

is given by

$$L_i : f \mapsto x_i f.$$

We similarly define the **right** d**-shift**.

Cyclic function

Analogous to the notion of cyclicity in Hardy space:

Definition

A function $f \in \mathcal{F}_d$ is **cyclic** for the left d-shift if the set

$$\{pf \ : \ p \in \mathbb{C}\langle \mathbb{x} \rangle\}$$

is dense in $\mathcal{F}_d$.

Frobenius-Fock norm

Definition

For any $k, \ell \geq 1$, we define the Frobenius norm on $M_{k\times\ell}(\mathbb{C})$ by

$$\|A\|_{F,k\times\ell} := \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{\ell}|A_{i,j}|^2},$$

where $A = (A_{i,j})_{i,j=1}^{k,\ell} \in M_{k\times\ell}(\mathbb{C})$.
Thus Frobenius-Fock norm on $M_{k\times\ell}(\mathcal{F}_d)$ is defined by

$$\|f\|_F := \sqrt{\sum_{w\in\langle \mathbb{x}\rangle}\|A_w\|_{F,k\times\ell}^2}$$

where $f(\mathbb{x}) = \sum_{w\in\langle \mathbb{x}\rangle} A_w w \in M_{k\times\ell}(\mathcal{F}_d)$.

Row ball

Definition

We define $\|A\|_{\text{row}} := \|A_1A_1^* + \cdots + A_dA_d^*\|_{op}^{1/2}$ for $A = (A_1, \cdots, A_d)$ $n \times n$ matrices.

Then $\mathfrak{B}_n^d$ is the set of all A with $\|A\|_{\text{row}} < 1$.

The set $\mathfrak{B}^d := \bigsqcup_{n=1}^{\infty} \mathfrak{B}_n^d$, which is closed under direct sums and conjugation by unitaries, is called the **row ball**.

If $\|A\|_{\text{row}} \leq 1$ then it is **row contraction** and if $\|A\|_{\text{row}} < 1$ then it is a **strict row contraction**.

Analogously, we have **column contraction** and **strict column contraction**.

Linear Pencil

Definition

An $m \times \ell$ nc linear pencil (in d indeterminates) is an expression of the form

$$L_A(\mathbb{x}) = A_0 + A\mathbb{x}$$

where $A\mathbb{x} = A_1x_1 + \cdots + A_dx_d$. If $A_0 = I$ then we call $L_A(\mathbb{x})$ a monic linear pencil.

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Definition

A matrix tuple $A = (A_1, \ldots, A_d) \in M_m(\mathbb{C})^d$ is **irreducible** if

$$M_m(\mathbb{C}) = \{p(A) : p \in \mathbb{C}\langle \mathbb{x} \rangle\}.$$

The monic linear pencil L_A is **irreducible** if A is irreducible.

Stable Associativity

Definition

Given $A \in M_{k\times k}(\mathbb{C}\langle \mathbb{x}\rangle)$ and $B \in M_{\ell\times\ell}(\mathbb{C}\langle \mathbb{x}\rangle)$.
We say A and B are **stably associated** if

$$\exists\, N \in \mathbb{Z}^+ \text{ and } P, Q \in \mathrm{GL}_N(\mathbb{C}\langle \mathbb{x}\rangle) \text{ such that}$$

$$P\begin{pmatrix} A & \\ & I \end{pmatrix} Q = \begin{pmatrix} B & \\ & I \end{pmatrix}.$$

Notation:

$$A \sim_{\mathrm{sta}} B.$$

Free Zero Locus

Definition

If $F \in M_{k\times k}(\mathbb{C}\langle \mathbb{x} \rangle)$, then

$$\mathcal{Z}_n(F) = \left\{X \in M_n(\mathbb{C})^d \ : \ \det(F(X)) = 0\right\},$$

and

$$\mathcal{Z}(F) := \bigsqcup_{n\geq 1} \mathcal{Z}_n(F).$$

The set $\mathcal{Z}(F)$ is the **free zero locus** of F.

Observe:

If $F \sim_{\text{sta}} G$ then $\mathcal{Z}(F) = \mathcal{Z}(G)$.

Main Result

Theorem

For $F \in M_{k\times k} \otimes \mathbb{C}\langle \mathbf{x} \rangle$,

$$\|P_n F - 1\|^2 \to 0$$

if and only if

$$det(F) \neq 0 \text{ on } \mathfrak{B}^d.$$

In particular,

Say F *is nonsingular in the row ball and* F *is a product of exactly* ℓ *atomic factors, then the opa's,* P_n*, for* F *satisfy*

$$\|P_n(x)F(x) - I\|_2^2 \lesssim \frac{1}{n^p}$$

where $p = \frac{1}{3^{\ell-1}}$.

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Proof.

One direction ($\Rightarrow$): easy.

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One direction ($\Rightarrow$): easy.

For the converse ($\Leftarrow$), we need the following lemma. □

The Lemma

Lemma

Assume that M is column contraction, i.e., $\|M\|_{col} \leq 1$,

then $I - M\mathbb{x}$ *is cyclic in* $M_{k\times k}(\mathbb{C}) \otimes \mathcal{F}_d$.

In fact, $\exists$ 1*-variable polynomials,* π_n *of degree* n *so that:*

$$\|\pi_n(M\mathbb{x})\|_\infty \lesssim n$$

and

$$\|\pi_n(M\mathbb{x})(1 - M\mathbb{x}) - 1\|_2^2 \lesssim \frac{1}{n}.$$

The Lemma

Proof.

A (qualitative) sketch.

For $F(\mathbb{x}) = I - M\mathbb{x} \in M_{k\times k}(\mathbb{C}) \otimes \mathcal{F}_d$, $\|M\|_{col} \leq 1$, following are true:

1. $(I + M\mathbb{x} + \cdots + (M\mathbb{x})^n)(I - M\mathbb{x}) \to I$, pointwise in $\mathfrak{B}^d$.
2. $\sup_n \|I - (M\mathbb{x})^n\|_F < \infty$.

And done.

Note. For quantitative form of lemma we import estimates from [2][a]

□

[a] [BÉNÉTEAU, CONDORI, LIAW, SECO, AND SOLA, 2015]

Proof of main result continues...

($\Leftarrow$) Let $F \in M_{k\times k}(\mathbb{C}\langle \mathbb{x}\rangle)$ be nonsingular in $\mathfrak{B}^d$.

We can assume, without loss of generality, that $F(0) = I$.

BIG IDEA: Using **Linearization trick** [3][3] we have that $F \sim_{\mathrm{sta}} L_A(\mathbb{x})$ where $L_A(\mathbb{x})$ is a monic linear pencil.

[3] [Helton, Klep, Volcic, 2018]

Proof of main result

We have that $F \sim_{\mathrm{sta}} L_A(\mathbb{x})$ where $L_A(\mathbb{x})$ is a monic linear pencil.
So $L_A(\mathbb{x})$ is nonsingular in $\mathfrak{B}^d$.

Proposition 4.1, [1][4] implies that outer spectral radius of A, $\rho(A) \leq 1$ where it is defined as follows:

Outer spectral radius $\rho(X)$: Let $n \in \mathbb{N}$ and $X \in M_{n\times n}(\mathbb{C})^d$ we have the associated completely positive map on $M_{n\times n}$ as

$$\Psi_X(T) = \sum_{j=1}^{d} X_j T X_j^* .$$

Then **outer spectral radius** is $\rho(X) := \lim_{k\to\infty} \|\Psi_X^k(I)\|^{1/2k}$.

[4] [Jury, Martin, Shamovich, 2021]

Proof of main result

Proposition 4.1, [1] implies that joint spectral radius of A, $\rho(A) \leq 1$.

Proof of main result

Proposition 4.1, [1] implies that joint spectral radius of A, $\rho(A) \leq 1$.
Now it follows from Burnside's theorem, any monic linear pencil is similar to a block upper-triangular linear pencil of the form:

$$\begin{bmatrix} L_1(\mathbb{x}) & * & * & \dots & * \\ 0 & L_2(\mathbb{x}) & * & \dots & * \\ 0 & 0 & L_3(\mathbb{x}) & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & L_\ell(\mathbb{x}) \end{bmatrix}$$

where for every k, linear pencils, $L_k = I$ or L_k is an irreducible.
Say for each k, $L_k(\mathbb{x}) = I - \bar{A}^{(k)}\mathbb{x}$ where $\bar{A}^{(k)} = (\bar{A}_1^{(k)} \; \bar{A}_2^{(k)} \; \cdots \; \bar{A}_d^{(k)})$. That is, $L_k(\mathbb{x}) = I - \sum_{j=1}^{d} \bar{A}_j^{(k)} x_j$.

Proof of Main Result

Thus from above we get that, for each k, $\rho(\bar{A}^{(k)}) \leq 1$.

Hence for each k, $\bar{A}^{(k)}$ is irreducible with spectral radius ≤ 1.

[5] [Salomon, Shalit, Shamovich, 2020]

Proof of Main Result

Thus from above we get that, for each k, $\rho(\bar{A}^{(k)}) \leq 1$.

Hence for each k, $\bar{A}^{(k)}$ is irreducible with spectral radius ≤ 1.

From [4][5], it follows that for each k, $\bar{A}^{(k)}$ is similar to a column contraction.

Let us say that $\bar{A}^{(k)} \sim M^{(k)}$ where for $1 \leq k \leq l$, $M^{(k)}$ is a column contraction.

[5] [Salomon, Shalit, Shamovich, 2020]

Proof of Main Result

Finally we get that our F is stably associated to the following block upper-triangular form:

$$F \sim_{\mathrm{sta}} \begin{bmatrix} I - M^{(1)}(\mathbb{x}) & * & * & \dots & * \\ 0 & I - M^{(2)}(\mathbb{x}) & * & \dots & * \\ 0 & 0 & I - M^{(3)}(\mathbb{x}) & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I - M^{(\ell)}(\mathbb{x}) \end{bmatrix}$$

where $M^{(k)}$ is a column contraction.

Proof of Main Result

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where $M^{(k)}$ is a column contraction.

Now we are going to prove our result via induction on ℓ just for this monic linear pencil and that would suffice due to the following lemma:

Another Lemma

Lemma

If $F \sim_{\mathrm{sta}} G$ then

$$\|p_n F - 1\|^2 \to 0 \text{ iff } \|q_n G - 1\|^2 \to 0.$$

(with comparable rate of decay.)

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(with comparable rate of decay.)

We prove our result via induction on ℓ that is, we prove that there exists a matrix polynomial $g^{(\ell)}$ such that $\|g^{(\ell)}G - 1\| < \epsilon$.

Proof of Main Result

For $\ell = 2$, let us assume

$$F \sim_{\text{sta}} G = \begin{bmatrix} I - M^{(1)}(\mathbb{x}) & Y(\mathbb{x}) \\ 0 & I - M^{(2)}(\mathbb{x}) \end{bmatrix}$$

where for $k = 1,\ 2,\ M^{(k)}$ is a column contraction.

Consider the polynomial matrix, $g = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix}$ where $p,\ r,\ q$ are polynomials of any degree.

Then

$$gG = \begin{pmatrix} p(\mathbb{x})(I - M^{(1)}(\mathbb{x})) & -p(x)Y(\mathbb{x}) + r(\mathbb{x})(I - M^{(2)}(\mathbb{x})) \\ 0 & q(\mathbb{x})(I - M^{(2)}(\mathbb{x})) \end{pmatrix}$$

Proof of Main Result

Then

$$gG(\mathbb{x}) = \begin{pmatrix} p(\mathbb{x})(I - M^{(1)}(\mathbb{x})) & -p(\mathbb{x})Y(\mathbb{x}) + r(\mathbb{x})(I - M^{(2)}(\mathbb{x})) \\ 0 & q(\mathbb{x})(I - M^{(2)}(\mathbb{x})) \end{pmatrix}$$

From the Lemma, $I - M^{(1)}(\mathbb{x})$ is cyclic in $M_{k\times k}(\mathbb{C}) \otimes \mathcal{F}_d$, so $\exists$ a matrix polynomial, $p(\mathbb{x})$ such that $p(\mathbb{x})(I - M^{(1)}(\mathbb{x}))$ can be made close to I.

Similarly, $(I - M^{(2)}(\mathbb{x}))$ is cyclic in $M_{k\times k}(\mathbb{C}) \otimes \mathcal{F}_d$ we can find: $q(\mathbb{x})$ and $r(\mathbb{x})$ such that
$q(\mathbb{x})(I - M^{(2)}(\mathbb{x}))$ can be made close to I;
$r(\mathbb{x})(I - M^{(2)}(\mathbb{x}))$ can be made close to $p(\mathbb{x})Y(\mathbb{x})$.

Proof of Main Result

Thus, given an $\epsilon > 0$, $\exists \begin{bmatrix} p & r \\ 0 & q \end{bmatrix}$ such that

$$\left\| \begin{bmatrix} p & r \\ 0 & q \end{bmatrix} \begin{bmatrix} I - M^{(1)}(\mathbb{x}) & Y(\mathbb{x}) \\ 0 & I - M^{(2)}(\mathbb{x}) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|_F^2 < \epsilon.$$

Then the rest of the proof is just induction on the number of diagonal blocks in the matrix, ℓ.

We are done.

Open Questions

So we have proved that

$$\|P_n(x)F(x) - I\|_2^2 \lesssim \frac{1}{n^p}$$

where $p = \frac{1}{3^{\ell-1}}$.

Questions which have not been answered yet:

1. Is this p uniform?
2. Like the classical setting, is it $O(\frac{1}{n})$?
3. Can we prove a similar result for a rational nc function as well?

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