Interpolating sequences for pairs of kernels

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Definition

A sequence $\{z_n\}$ in $\mathbb{D}$ is interpolating for H^∞ if for every sequence $\{w_n\} \in \ell^\infty$, there exists $f \in H^\infty$ such that

$$f(z_n) = w_n, \quad \forall n.$$

Write $\{z_n\}$ satisfies (IS).

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(WS) is weakly separated if there exists $\delta > 0$ such that,

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(CM) satisfies the Carleson measure condition if there exists $M > 0$ such that

$$\sum_j (1 - |z_j|^2)|f(z_j)|^2 \leq M \int_{\partial\mathbb{D}} |f|^2 dm, \quad \forall f \in \mathbb{C}[z],$$

i.e. $\mu = \sum_j (1 - |z_j|^2)\delta_{z_j}$ is a Carleson measure on $\mathbb{D}$.

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i.e. $\mu = \sum_j (1 - |z_j|^2)\delta_{z_j}$ is a Carleson measure on $\mathbb{D}$.

Theorem (Carleson, 1958)

For a sequence $\{z_n\}$ in $\mathbb{D}$, (IS) $\Leftrightarrow$ (WS) + (CM).

H^∞ as a multiplier algebra

Let
$$H^2 = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathrm{Hol}(\mathbb{D}) : ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

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This is a reproducing kernel Hilbert space on $\mathbb{D}$: For all $f \in H^2$ and $w \in \mathbb{D}$,
$$f(w) = \langle f, k_w \rangle_{H^2},$$

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The multiplier algebra of H^2 is
$$\mathrm{Mult}(H^2) = \{\phi : \mathbb{D} \to \mathbb{C} : \phi \cdot f \in H^2 \text{ for all } f \in H^2\},$$
equipped with $||\phi||_{\mathrm{Mult}(H^2)} := ||M_\phi||_{B(H^2)}$, where $M_\phi(f) = \phi \cdot f$.

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Observation

$\mathrm{Mult}(H^2) = H^\infty$ with equality of norms.

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Lemma (Shapiro–Shields)

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$$f \mapsto \left\{ f(z_n)\sqrt{1-|z_n|^2} \right\}_n = \left\{ \frac{f(z_n)}{||k_{z_n}||} \right\}_n$$

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Bishop, Marshall-Sundberg (1994): Used this idea to characterize interpolating sequences for the multiplier algebra of the Dirichlet space

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Key property

H^2 and $\mathcal{D}$ are complete Pick spaces.

Nevanlinna-Pick Interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, z_2, \ldots, z_n \in \mathbb{D}$ and $w_1, w_2, \ldots, w_n \in \mathbb{C}$. There exists $\phi \in \text{Mult}(H^2) = H^\infty$ with

$$\phi(z_i) = w_i \text{ for } 1 \leq i \leq n \quad \text{and} \quad ||\phi||_{\text{Mult}(H^2)} \leq 1$$

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if and only if the matrix

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{i,j=1}^n = \left[(1 - w_i\overline{w_j})k(z_i, z_j)\right]_{i,j=1}^n$$

is positive semi-definite.

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is positive semi-definite. Recall that $k(z, w) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Complete Pick spaces

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Definition

- $\mathcal{H}_k$ is called a Pick space if this condition is also sufficient.
- $\mathcal{H}_k$ is called a complete Pick space if the analogue of this condition for matrix-valued functions is sufficient.

Examples

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with norm $||f||^2_{\mathcal{D}} = ||f'||^2_{L^2(\mathbb{D})} + ||f||^2_{H^2}$ is a complete Pick space.

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- The Drury-Arveson space H^2_d is the RKHS on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$, with reproducing kernel

$$k(z,w) = \frac{1}{1 - \langle z, w\rangle} = \frac{1}{1 - \sum_{i=1}^d z_i \overline{w}_i}.$$

H^2_d is a complete Pick space and is also universal among all complete Pick spaces (McCullough–Quiggin, Agler–McCarthy).

A distance function for RKHS's

Let $\mathcal{H}_k$ be a RKHS on a set X with kernel k. Also, let $\hat{k}_x := \frac{k_x}{||k_x||}$ denote the *normalized* kernel function at x.

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Example

If $\mathcal{H}_k = H^2$, *then*

$$d_k(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

is the pseudohyperbolic metric on $\mathbb{D}$.

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Old and new developments

Lemma

In every RKHS $\mathcal{H}_k$, (IS) $\Rightarrow$ (WS) + (CM).

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- (Bøe, 2005) holds in every space on the unit ball $\mathbb{B}_d$ with kernel

$$k(z,w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}, \quad \text{where } \alpha \in (0,1);$$

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- (Aleman–Hartz–McCarthy–Richter, 2017) holds if $\mathcal{H}_k$ is a complete Pick space!

Two proofs of the A.-H.-M.-R. characterization

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- New proof uses the column-row property:

Theorem (Hartz, 2020)

Assume $\mathcal{H}_k$ is a complete Pick space and $\{\phi_n\} \subset \mathrm{Mult}(\mathcal{H}_k)$.

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Assume $\mathcal{H}_k$ is a complete Pick space and $\{\phi_n\} \subset \mathrm{Mult}(\mathcal{H}_k)$. *Then,*

$$\left|\left| \begin{bmatrix} M_{\phi_1} & M_{\phi_2} & \cdots \end{bmatrix} \right|\right| \leq \left|\left| \begin{bmatrix} M_{\phi_1} \\ M_{\phi_2} \\ \vdots \end{bmatrix} \right|\right|.$$

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Example

Let $\mathcal{H}_k = H^2$ (Hardy space) and $\mathcal{H}_\ell = L_a^2$ (Bergman space). Since $H^2 \subset L_a^2$, we have

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Actually, it is even true that

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$\mathrm{Mult}(H^2, L^2_a)$ was described by Stegenga (1980).

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Observation

If $\phi \in \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$, *then* $|\phi(z)| \leq ||\phi||_{\text{M}} \frac{||\ell_z||}{||k_z||}$, *for all* $z \in X$.

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If $\phi \in \mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$, then $|\phi(z)| \leq ||\phi||_{\mathrm{M}} \frac{||\ell_z||}{||k_z||}$, for all $z \in X$.

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A sequence $\{z_n\}$ in X is interpolating for $\mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ (write (IS)) if for all $\{w_n\} \in \ell^\infty$, there exists $\phi \in \mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ such that

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Interpolating sequences for pairs of kernels

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Lemma

If $\{z_n\} \subset X$ satisfies (IS), then it also satisfies (CM) for k and is (WS) by ℓ.

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Theorem (Aleman–Hartz–McCarthy–Richter, 2017)

The answer is yes if ℓ is a power of a complete Pick kernel.

Beyond weak separation

Definition

Let ℓ be a kernel on X and $\{z_n\} \subset X$. Given $m \geq 2$, we say that $\{z_n\}$ is *m-weakly separated by ℓ* (write (m-WS)) if there exists $\delta > 0$ such that for every m-point subset $\{\mu_1, \dots, \mu_m\} \subset \{z_n\}$ we have

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Let $X = \{1, 2, 3\}, v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, v_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$

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Then, $\{1, 2, 3\}$ will be (2-WS) but not (3-WS) by ℓ.

Main result

Let $\mathcal{H}_k, \mathcal{H}_\ell$ be two RKHS on X such that k is a complete Pick factor of ℓ. Recall: (IS)=interpolating for $\text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$.

Question (Aleman–Hartz–McCarthy–Richter)

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Fix $m \geq 2$. In general,

$$\boxed{\text{(CM) for } k} + \boxed{(m\text{-WS}) \text{ by } \ell} \not\Rightarrow \boxed{\text{(IS)}}.$$

The automatic separation property

Definition

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This is equivalent to:
for any fixed $m \geq 2$, a kernel $\hat{\ell}_z$ can be "close" to the span of m other kernels $\hat{\ell}_{w_1}, \hat{\ell}_{w_2}, \ldots, \hat{\ell}_{w_m}$ if and only if it is "close" to one of them.

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Then, ℓ does not have the automatic separation property.

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- kernels of Hardy spaces on finitely-connected planar domains;
- kernels of Bergman spaces on $\mathbb{D}$ with exponentially decaying weights;
- kernels of weighted Bargmann-Fock spaces on $\mathbb{C}^n$ (e.g. $\ell(z, w) = e^{z\overline{w}}$).

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Let k be the kernel corresp. to the Bergman space on $\mathbb{D}$ with weight $e^{-\frac{1}{1-|z|^2}}$.

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$$\ell(z, w) = \frac{k(z_1, w_1) + k(z_2, w_2)}{(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)}.$$

ℓ is "regular", but doesn't have the automatic sep. property.

A "metric" description of (m-WS)

Let ℓ be a kernel on X and assume that $\{z_n\} \subset X$ is (WS). Then, given $m \geq 3$, $\{z_n\}$ will be (m-WS) if and only if there exists $\delta > 0$ (depending on m) such that

$$d_\ell(z, w; \mu_1, \mu_2, \dots, \mu_{m-2}) > \delta,$$

for all $z \neq w$ and for any $m-2$ point subset $\{\mu_1, \dots, \mu_{m-2}\}$ of $\{z_n\}$ that does not contain either z or w, where $d_\ell(\cdot, \cdot\,; \mu_1, \mu_2, \dots, \mu_{m-2})$ is the metric associated with the subspace of $\mathcal{H}_\ell$ given by

$$\{f \in \mathcal{H}_\ell \ : \ f(\mu_1) = \cdots = f(\mu_{m-2}) = 0\}.$$