Blaschke products and the curve of geodesic centers

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Bucknell University/National Science Foundation

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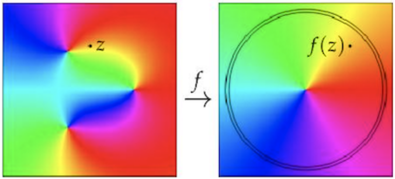
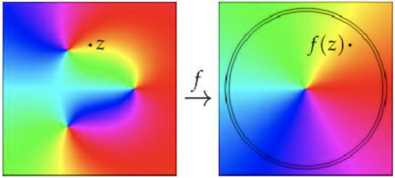
[2]Fanciest pictures by Elias Wegert, from The Beauty of Blaschke Products, Daepp, G., Semmler, and Wegert

How things work



G., Ueli Daepp, Elias Wegert, Gunter Semmler, Complex Beauties 2019

How things work



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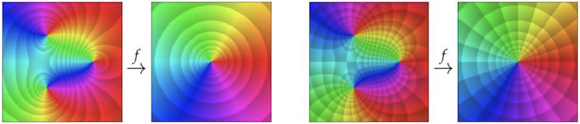
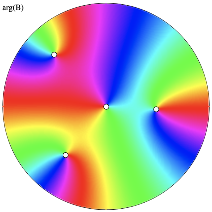


Figure: Visualizing complex functions

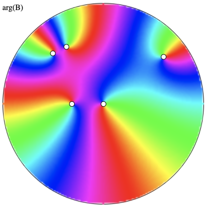
Ingredients: Blaschke Products

$$B(z) = \alpha \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } a_j \in \mathbb{D}, |\alpha| = 1.$$

Basic fact: A Blaschke product of degree n maps the unit circle onto itself n times; the argument is increasing and $B(z) = \lambda$ has exactly n distinct solutions for each $\lambda \in \mathbb{T}$.



(a) Degree 4



(b) Degree 5

Blaschke products

One more look at Blaschke products

$$B(1/\overline{z}) = 1/\overline{B(z)}.$$

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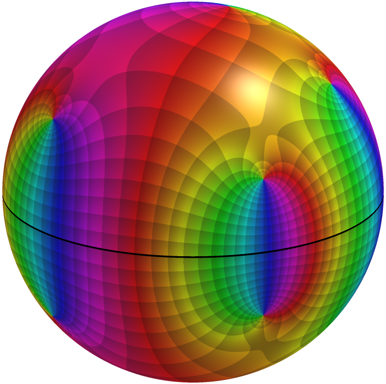
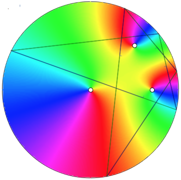


Connecting points where $B(z) = \lambda$, $\lambda \in \mathbb{T}$

$$B(z) = z\left(\frac{z-a}{1-\overline{a}z}\right)\left(\frac{z-b}{1-\overline{b}z}\right).$$



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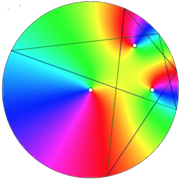

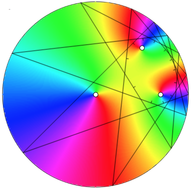

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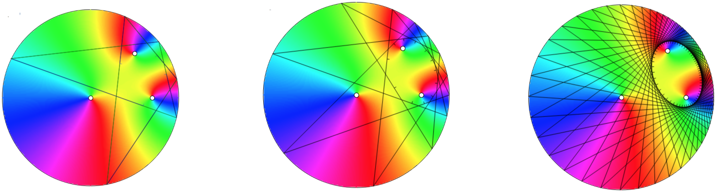

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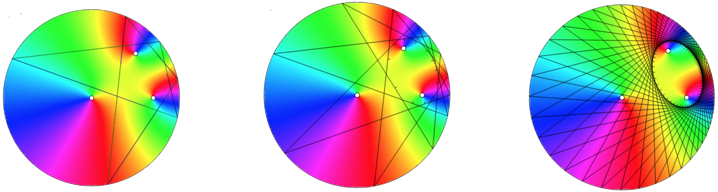


Two triangles, four triangles, many triangles

It's true!

Theorem (Daepp, G., Mortini, 2002)

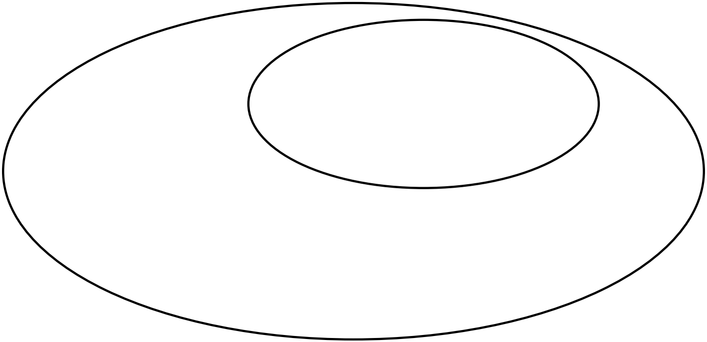
Let B be a Blaschke product with zeros $0, a$ and b. For $\lambda \in \mathbb{T}$, let z_1, z_2 and z_3 be the distinct solutions to $B(z) = \lambda$. Then the lines joining z_j and z_k, for $j \neq k$, are tangent to the ellipse given by

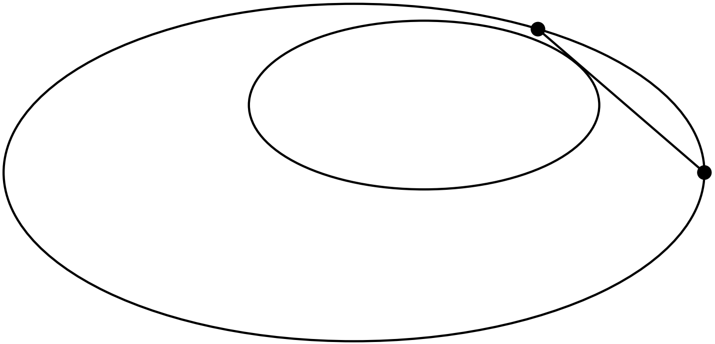
$$|w - a| + |w - b| = |1 - \bar{a}b|.$$

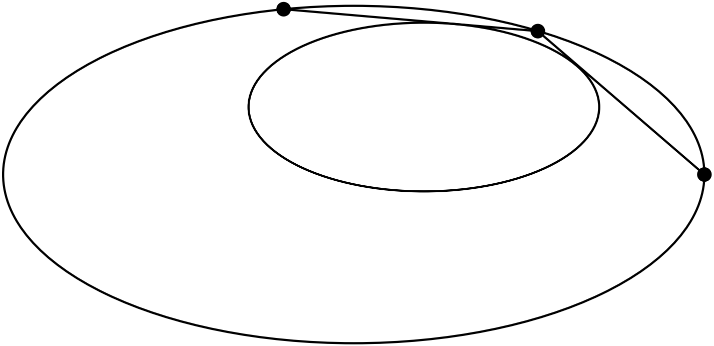
Conversely, every point on the ellipse is the point of tangency of a line segment that intersects $\mathbb{T}$ at points for which $B(z_1) = B(z_2)$.

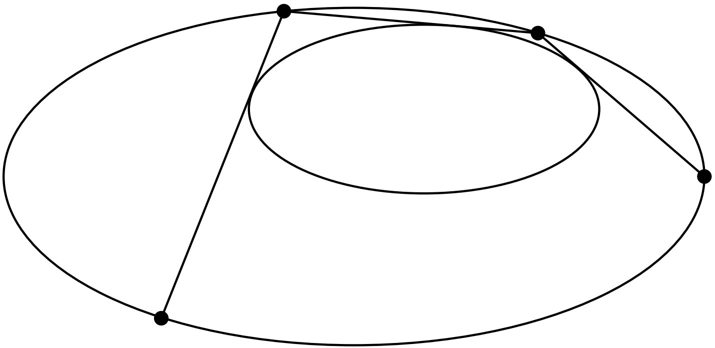
Poncelet's theorem, 1813

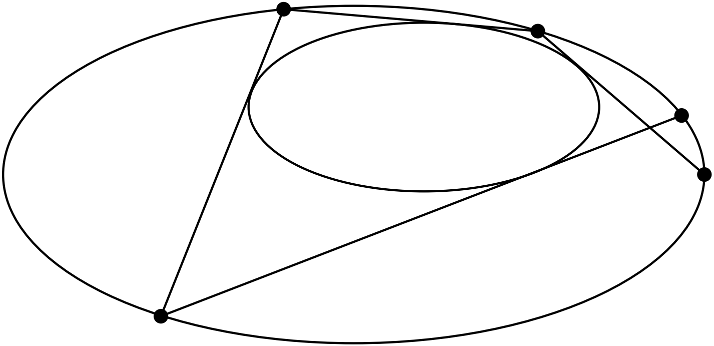
Let E_1 and E_2 be ellipses with E_1 entirely contained in E_2. Starting at a point on E_2 draw a tangent to E_1:

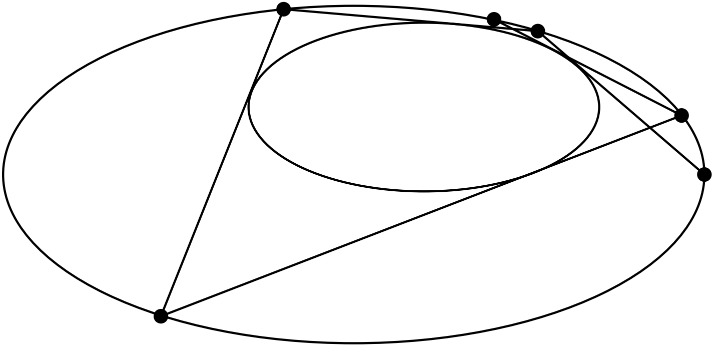


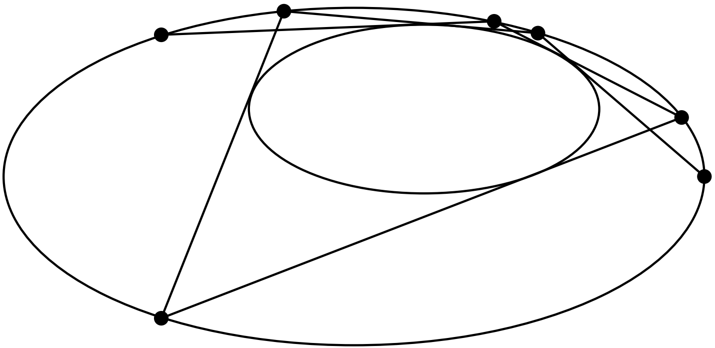


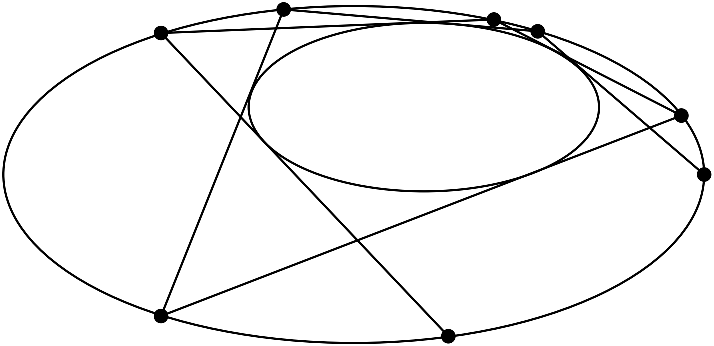


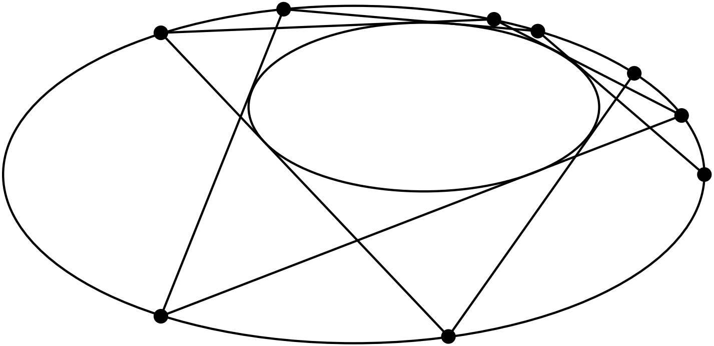




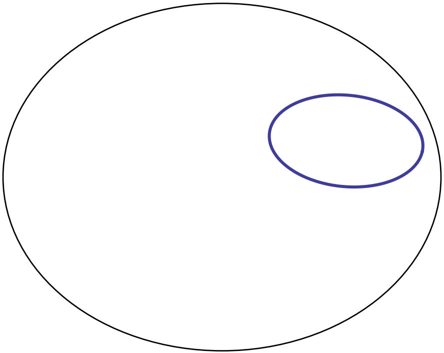


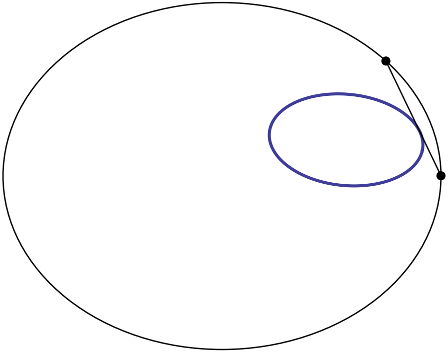


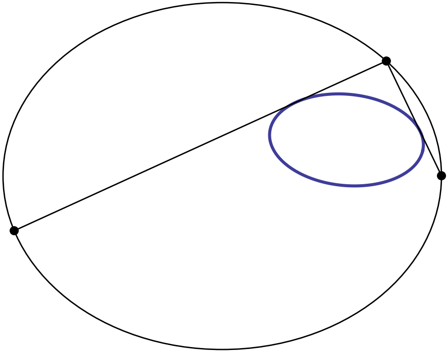


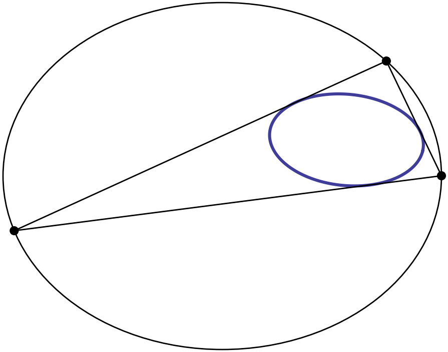


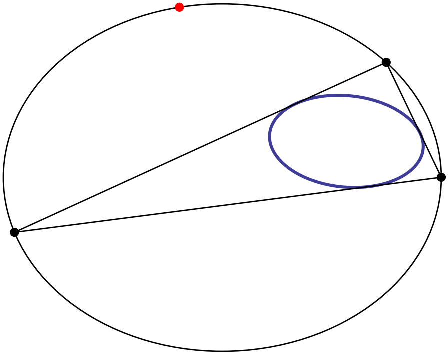
Maybe you keep going – never returning to the starting point.
Maybe, though, it does return to the initial point.

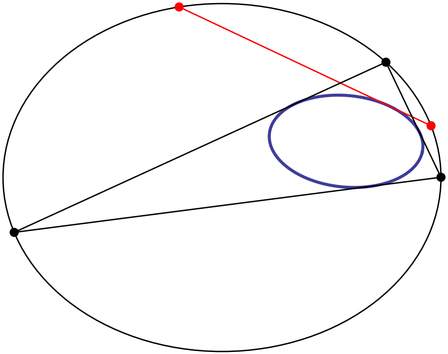


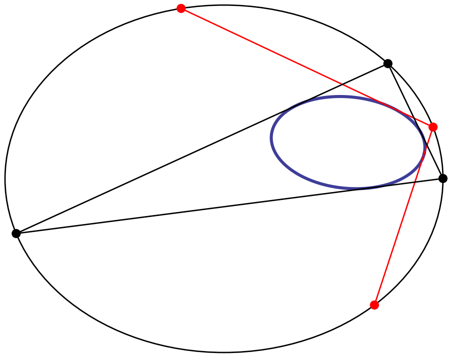


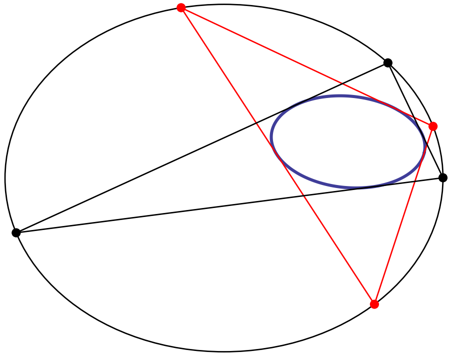


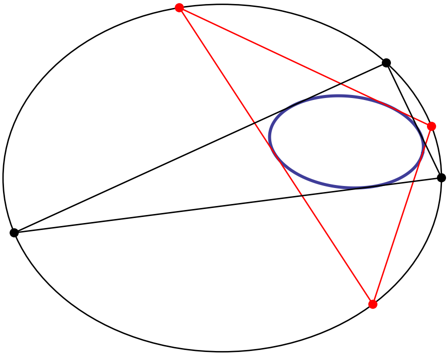












Poncelet's theorem: Following these rules, if the path closes in n steps, then *no matter where you begin* it will close in n steps.

2015, Monthly proof Halbeisen and Hungerbühler. Proof relies on duality (Brianchon's and Pascal's theorems)

A Poncelet ellipse is one that can be inscribed in a (convex) polygon that is itself inscribed in the larger ellipse.

Higher Degree

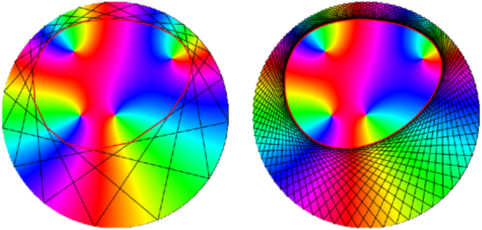


Figure: Poncelet curves

A Poncelet curve is a smooth curve that can be inscribed in a (convex) polygon that is itself inscribed in the unit circle.

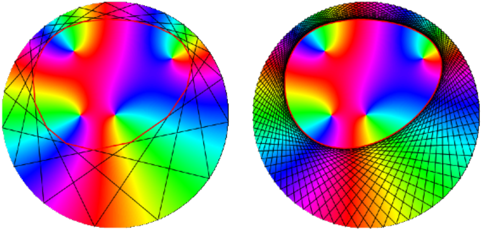


Figure: Poncelet curves

A Poncelet curve is a smooth curve that can be inscribed in a (convex) polygon that is itself inscribed in the unit circle. Some geometric properties remain, but these are no longer ellipses.

References: Gau-Wu, Mirman

Euclidean model: points, lines, distance is $|a - b|$

Poincaré model: points are points in $\mathbb{D}$, lines are (open) arcs orthogonal to $\mathbb{T}$ or (open) diameters, hyperbolic distance is

$$d(z,w) = \log \frac{1+\rho(z,w)}{1-\rho(z,w)}, \qquad \rho(z,w) = \left| \frac{z-w}{1-\overline{w}z} \right|.$$

Polynomials	**Blaschke**
p(z)=w, n solutions in $\mathbb{C}$	B(z) $=$ w, n solutions in $\mathbb{D}$
self-maps of Riemann sphere, valency n	selfmaps of $\mathbb{D}$, valency n
factor as $c(z - z_1)\dots(z - z_n)$	factor as $\lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$
conformal selfmaps of $\mathbb{C}$ are $az + b$	conformal selfmaps of $\mathbb{D}$
class of products of n conformal selfmaps of $\mathbb{C}$	class of products of n conformal selfmaps of $\mathbb{D}$
$n - 1$ critical pts. in $\mathbb{C}$ determine p up to comp. with a selfmap of $\mathbb{C}$	$n - 1$ critical pts. in $\mathbb{D}$ determine B up to comp. with a selfmap of $\mathbb{D}$

Maybe we should be working in the Poincaré disk

Gauss-Lucas and Walsh

Theorem (Gauss-Lucas, Euclid)

If p is a (non-constant) polynomial, then the critical points of p belong to the convex hull of the zeros of p.

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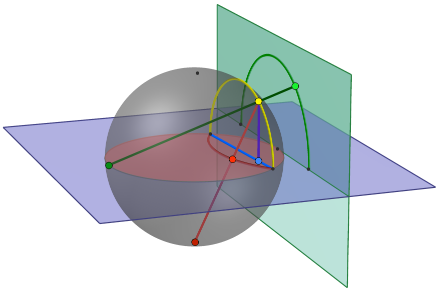
If p is a (non-constant) polynomial, then the critical points of p belong to the convex hull of the zeros of p.

Theorem (Walsh, Poincaré)

Let B be a Blaschke product. Then the critical points of B inside $\mathbb{D}$ in the non-Euclidean convex hull of the zeros of B with respect to the Poincaré metric.



1. Hemisphere: Lines in yellow
2. Poincaré disk model: Lines in red (stereographic projection from south pole)
3. Klein model: Lines in blue (project the hemisphere orthogonally onto the equator).

Map between Poincaré and Klein model can be found explicity.

Looking at this from another viewpoint

1. We can connect points of like color using geodesics;
2. We can consider the centers of the geodesics;
3. We can ask for the inner and outer boundary of the union of the geodesics.

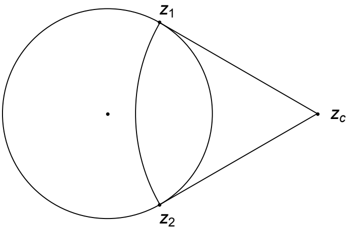
Assume: Closed, smooth, strictly convex curve.

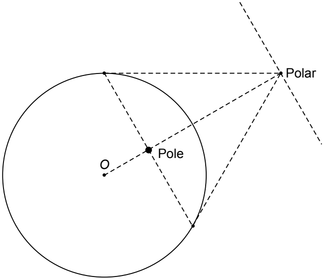
We focus on (2, 3) here. Process:

1. Draw a tangent line at a point on the curve.
2. Consider the two points of intersection.
3. Draw the geodesic.
4. Locate its center.

This set of points forms the curve of geodesic centers.

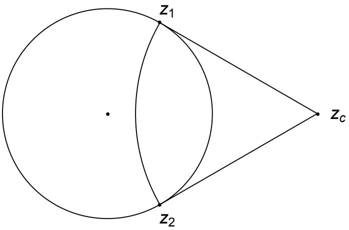
Curve of geodesics:

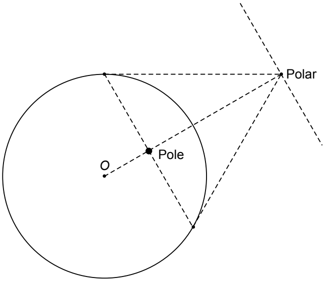




Not hard to check that z_c is the reflection of the midpoint $(z_1 + z_2)/2$ with respect to $\mathbb{T}$.

Curve of geodesics:





Not hard to check that z_c is the reflection of the midpoint $(z_1 + z_2)/2$ with respect to $\mathbb{T}$.

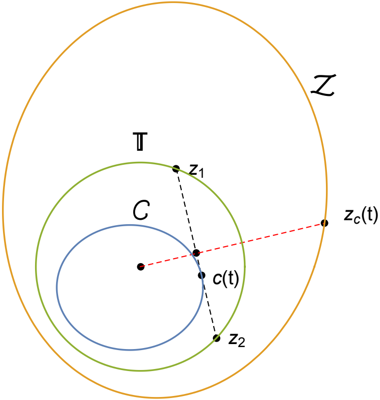
Other language you may know: Let P be a point and $P^\star$ the inversion with respect to $\mathbb{T}$. The line through $P^\star$ perpendicular to the line $\overline{PP^\star}$ is the *polar* and the point P is the *pole*.

Duality

Insightful beauty can manifest itself in a flash of insight, or in a slowly growing appreciation over time. There are many mathematical ideas that I didn't appreciate until I had seen them arise over and over again in disparate places. One recurring theme throughout mathematics is *duality*, natural pairings that exist between mathematical ideas...Recognizing duality is like using a mirror to see how two creatures that look and behave differently are really the same. I didn't appreciate duality until I saw it in many contexts; now I think it is beautiful.

Mathematics for Human Flourishing, Frances Su

The curve of geometric centers



Note that this gives a process that can be continued!

Theorem (Classical)

The curve of geodesic centers of an ellipse E with respect to a circle is

1. *an ellipse, if the origin of the circle lies in the interior of E;*
2. *a parabola, if the origin lies on E;*
3. *a hyperbola, if the origin lies outside E.*

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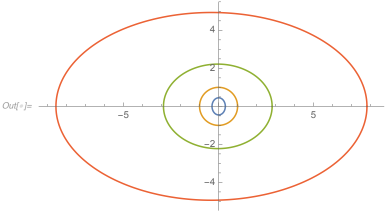
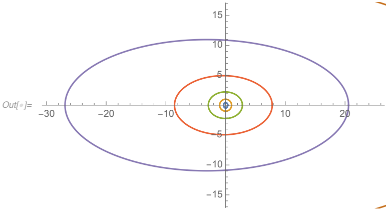
Theorem (Classical)

Let $\mathcal{C}$ be a smooth, closed, strictly convex curve in $\mathbb{D}$ containing 0 in its interior with parametrization $(x(t), y(t))$ for t in an open interval $\mathcal{I}$. Then the curve of geodesic centers of $\mathcal{C}$ is smooth, strictly convex, closed and has parametrization

$$z_c(t) = \left(\frac{y'(t)}{y'(t)x(t) - x'(t)y(t)}, \frac{-x'(t)}{y'(t)x(t) - x'(t)y(t)} \right).$$

Putting this together...





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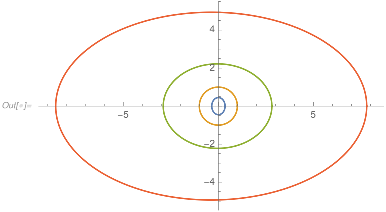
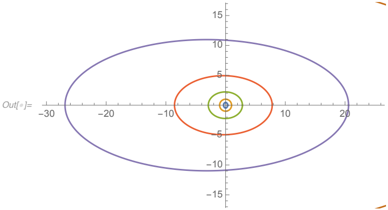
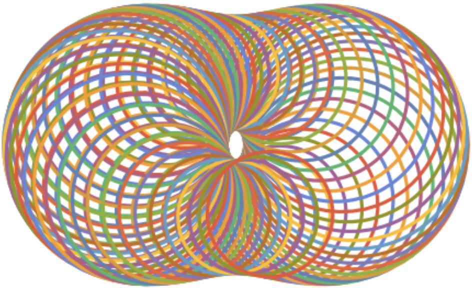




Theorem

There exists an infinite Poncelet dual chain of ellipses symmetric about the x-axis if and only if they are centered at 0.

Envelopes



The envelope of the geodesics

Let $\mathcal{Z}$ (parametrized $(x(t), y(t))$) be the curve of geodesic centers, $q(t) = x(t)y'(t) - y(t)x'(t)$ for $t \in [0, 2\pi]$ and

$$\beta_{\pm}(t) := \frac{q(t) \pm \sqrt{q^2(t) - |z'(t)|^2}}{|z'(t)|}.$$

Theorem

If $\mathcal{G}$ is the union of all geodesic circles with centers on $\mathcal{Z}$, then the boundary of $\mathcal{G}$ consists of all points of the form

$$c_{ext}(t) = \left(\beta_+(t)\frac{y'(t)}{|z'(t)|}, -\beta_+(t)\frac{x'(t)}{|z'(t)|}\right);$$

$$c_{int}(t) = \left(\beta_-(t)\frac{y'(t)}{|z'(t)|}, -\beta_-(t)\frac{x'(t)}{|z'(t)|}\right).$$

So, $c_{int}(t) \cdot \overline{c_{ext}(t)} = 1$ for all t, points $c_{int}(t)$ are inside $\mathbb{T}$ and points $c_{ext}(t)$ are outside $\mathbb{T}$.

Special case of an ellipse

Theorem

The boundary $\mathcal{C}$ of $\mathcal{R}$ consists of all points in the images of the parametrizations

$$c_{int}(t) = \left(\frac{e_x(t)}{1 + \sqrt{1 - e_x(t)^2 - e_y(t)^2}}, \frac{e_y(t)}{1 + \sqrt{1 - e_x(t)^2 - e_y(t)^2}} \right);$$

$$c_{ext}(t) = \left(\frac{e_x(t)}{1 - \sqrt{1 - e_x(t)^2 - e_y(t)^2}}, \frac{e_y(t)}{1 - \sqrt{1 - e_x(t)^2 - e_y(t)^2}} \right).$$

Remark: The Klein-Poincaré map is

$$k^{-1}(x, y) = \left(\frac{x}{1 + \sqrt{1 - x^2 - y^2}}, \frac{y}{1 + \sqrt{1 - x^2 - y^2}} \right).$$

Singer, 2006

Theorem

Let B be a Blaschke product and let γ be the envelope of the non-Euclidean geodesics (with respect to the Poincaré metric) joining pairs of points that satisfy $B(z_1) = B(z_2)$. Then γ is part of an algebraic curve with real foci that are the critical points of B in $\mathbb{D}$ together with their inverses with respect to $\mathbb{D}$.

Remark. For $n = 3$ the curve is a non-Euclidean ellipse:

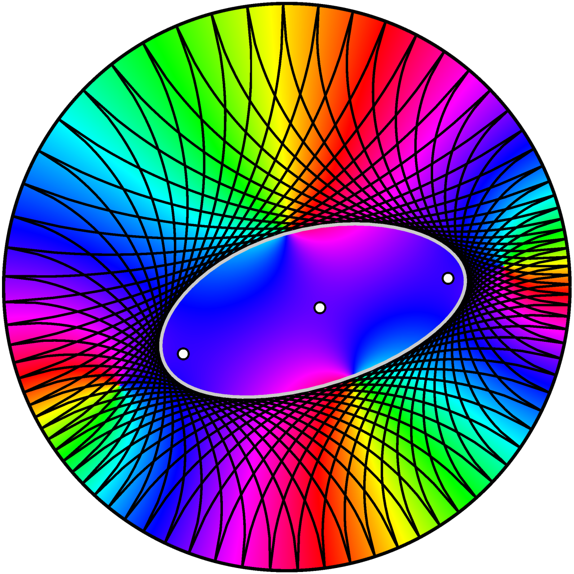
Singer, 2006

Theorem

Let B be a Blaschke product and let γ be the envelope of the non-Euclidean geodesics (with respect to the Poincaré metric) joining pairs of points that satisfy $B(z_1) = B(z_2)$. Then γ is part of an algebraic curve with real foci that are the critical points of B in $\mathbb{D}$ together with their inverses with respect to $\mathbb{D}$.

Remark. For $n = 3$ the curve is a non-Euclidean ellipse: "The ellipse in the hyperbolic plane is the locus of a point the sum of whose hyperbolic distances from two geometric foci are constant."

Hyperbolic ellipse



An application – When is 0 in the numerical range of an operator?

One of the most important open problems on numerical ranges is the discovery of necessary and/or sufficient conditions for the origin to be a point of $W(A)$. More specifically, it would be interesting to discover conditions for the origin to belong to the boundary or to the topological interior of $W(A)$. – Psarrakos and Tsatsomeros 2003

Ingredients: Operator theory

H^2 is the Hardy space; $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

An inner function is a bounded analytic function on $\mathbb{D}$ with radial limits of modulus one almost everywhere.

S is the shift operator $S : H^2 \to H^2$ defined by $[S(f)](z) = zf(z)$;

The adjoint is $[S^\star(f)](z) = (f(z) - f(0))/z$.

Theorem (Beurling's theorem)

The nontrivial invariant subspaces under S are

$$UH^2 = \{Uh : h \in H^2\},$$

where U is an inner function.

Subspaces invariant under the adjoint, $S^\star$ are $K_U := H^2 \ominus UH^2$.

Our operators and our space

Our operators: **compressions of the shift** $S_B : K_B \to K_B$ defined by

$$S_B(f) = P_B S(f),$$

where P_B is the orthogonal projection from H^2 onto K_B.

Our spaces: $K_B := H^2 \ominus BH^2$ where $B(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$.

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Applying Gram-Schmidt we get the Takenaka-Malmquist basis.

Our curves: Boundary of the numerical range of S_B

A an $n \times n$ matrix.

The *numerical range* of A is $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$. We'll look at the boundary of $W(S_B)$, where B is a finite Blaschke product.

Why should we look at the numerical range?

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$0 \in W(A)$ means $\langle Ax, x\rangle = 0$.

Compare the zero matrix and the $n \times n$ Jordan block: (Here's the 2×2)

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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$$W(A_1) = \{0\}, W(A_2) = \{z : |z| \leq 1/2\}.$$

Numerical Range

$$W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}.$$

Theorem (Elliptical range theorem)

Let A be a 2×2 matrix with eigenvalues a and b. Then the numerical range of A is an elliptical disk with foci at a and b and minor axis given by $(tr(A^\star A) - |a|^2 - |b|^2)^{1/2}$.

Theorem (The Toeplitz-Hausdorff Theorem; 1918)

The numerical range of an $n \times n$ matrix is convex.

For compressed shifts

Gau and Wu show that the boundary is a strictly convex curve with tangents at all points. In fact, it's smooth and everything we have done so far applies.

How we get our curves

Take a finite Blaschke product B.

Let $\widehat{B}(z) = zB(z)$.

Form the convex polygons, P_λ, with vertices at the points of $\mathbb{T}$ at which $\widehat{B}(z) = \lambda$.

The envelope of these polygons is the boundary of $W(S_B)$.

Remember these?

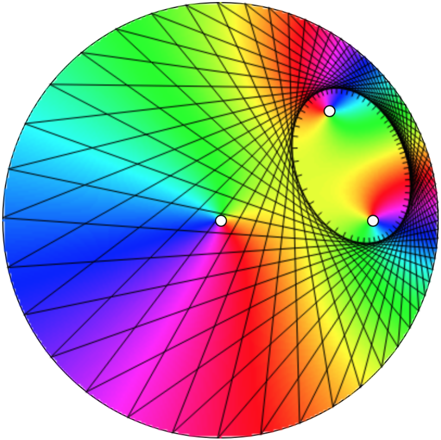
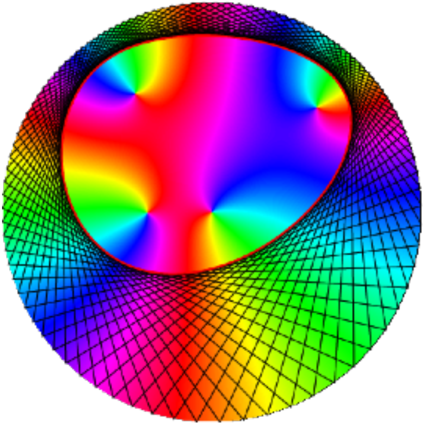


Figure: $W(S_B)$



Theorem

Let B be a finite Blaschke product of degree $n \geq 2$. Then 0 lies in the interior of $W(S_B)$ if and only if the curve of geodesic centers $\mathcal{Z}$ is a compact closed convex curve containing the unit circle. In this case, the unit circle will be a Poncelet curve relative to $\mathcal{Z}$.

In theory, this gives you a formula for the curve.

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In theory, this gives you a formula for the curve. In practice, it gives you a formula for the curve of centers of geodesics. In general, it shows that it's easier to find a formula for the dual than the original curve.

Application to function theory

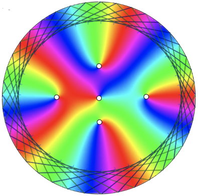
Let B be a Blaschke product of degree $n-1$ and $\widehat{B}(z) := zB(z)$.

Lemma

Let $z_1, \ldots, z_n \in \mathbb{T}$ be the n points satisfying $\widehat{B}(z_j) = \widehat{B}(z_k)$ and enumerated according to argument on $\mathbb{T}$. With indices considered modulo n:

- *0 lies in the interior of $W(S_B)$ if and only if no set of n points on $\mathbb{T}$ identified by $\widehat{B}$ contains sequential opposite points;*
- *0 lies on the boundary of $W(S_B)$ if and only if there is exactly one set of n points on $\mathbb{T}$ identified by $\widehat{B}$ and containing two opposite sequential points.*
- *0 lies outside $W(S_B)$ if and only if there exist (at least) two sets of n points on $\mathbb{T}$ identified by $\widehat{B}$ each of which contains two opposite sequential points;*

A picture is worth...



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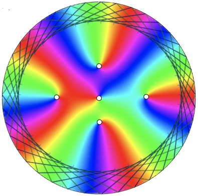

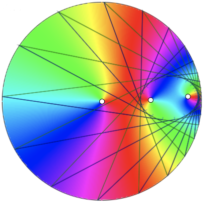

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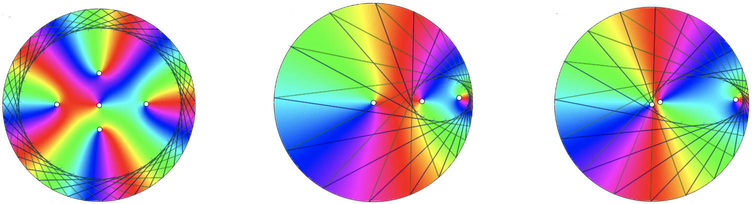

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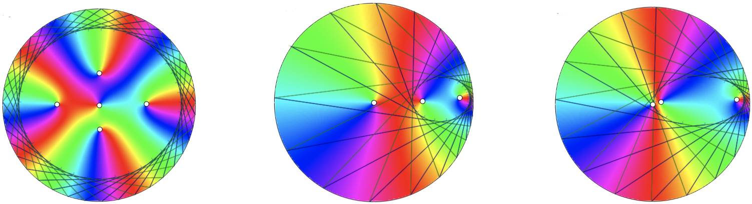
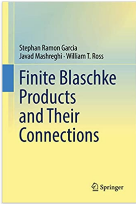


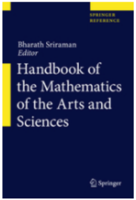
Inside, Outside, On

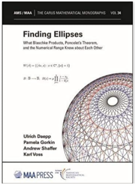
Further reading

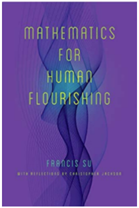
1. Envelopes: Bickel, G., Tran, 2020.
2. Operator theory, Mirman, Gau and Wu, for infinite Blaschke products: Chalendar+G+Partington, for inner functions and more general operators: Bercovici+Timotin; for several variables: Bickel+G.
3. Algebraic/Projective Geometry: Masayo Fujimura, Interior and Exterior Curves, 2019 (also 2017 and 2013).
4. Geometry, David Singer, 2006
5. Orthogonal Polynomials: Martínez-Finkelshtein, Simanek, Simon
6. Geometric analysis: Richard Schwartz, Serge Tabachnikov, Monthly, 2020
7. Halbeisen, Hungerbühler, Closed chains of conics carrying Poncelet curves, 2017

Books for further reading









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