Connections Between Boundary Triples and Self-Adjoint Perturbation Theory

Michael Bush, Dale Frymark* and Constanze Liaw

Stockholm University

Joint Mathematics Meetings
January 17, 2020

Motivating Example

- Let $\mathbf{L}_\theta$ be a Sturm–Liouville symmetric operator on $L^2[(a,b),w]$ acting via

$$\ell_\theta[f](x) := (-p(x)f'(x))' + q(x)f(x)$$

Assume $\mathbf{L}_\theta$ requires a single boundary condition at the left endpoint a.

Motivating Example

- Let $\mathbf{L}_\theta$ be a Sturm–Liouville symmetric operator on $L^2[(a,b),w]$ acting via

$$\ell_\theta[f](x) := (-p(x)f'(x))' + q(x)f(x)$$

  Assume $\mathbf{L}_\theta$ requires a single boundary condition at the left endpoint a.

- The boundary condition can be written as

$$f(a)\cos(\theta) + f^{[1]}(a)\sin(\theta) = 0 \qquad 0 \leq \theta < \pi,$$

  where $f^{[1]}(x) = p(x)f'(x)$ denotes the first "quasi-derivative" of $f(x)$. Equivalently,

$$f^{[1]}(a) = -\cot(\theta)f(a).$$

Setup of Boundary Conditions

Integrating by parts yields

$$\begin{aligned}\langle f, \ell_\theta[f]\rangle &= \int_a^b f(x)[(-p(x)f'(x))' + q(x)f(x)]dx \\ &= \int_a^b [p(x)f'(x)^2 + q(x)f(x)^2]dx + f^{[1]}(a)f(a) \\ &\equiv \tau(f,f) - \cot(\theta)f(a)^2,\end{aligned}$$

Let A be the $\theta = \pi/2$ Neumann boundary condition operator. Then,

$$A_\alpha = A + \alpha\langle \delta_a, \cdot\rangle \delta_a,$$

where $\ell_\theta = A_{-\cot(\theta)}$ and δ_a is the Dirac delta function at $x = a$.

Breakout #1: Hilbert Scales

Definition (Albeverio–Kurasov)

For $s \geq 0$, define the space $\mathcal{H}_s(A)$ to be $\mathcal{D}(A^{s/2})$ with norm equal to the graph norm of the operator

$$||\varphi||_s = ||(A+1)^{s/2}\varphi||_{\mathcal{H}}.$$

The space $\mathcal{H}_s$ equipped with the norm $||\cdot||_s$ is complete, and the adjoint spaces formed by the linear bounded functionals will be denoted by $\mathcal{H}_{-s} = \mathcal{H}_s^*$. The *scale of Hilbert spaces associated with the self-adjoint operator* A is the collection of these $\mathcal{H}_s(A)$ spaces when $s \in \mathbb{Z}$.

- The spaces also have the following nesting property:

$$\cdots \subset \mathcal{H}_2(A) \subset \mathcal{H}_1(A) \subset \mathcal{H} = \mathcal{H}_0(A) \subset \mathcal{H}_{-1}(A) \subset \mathcal{H}_{-2}(A) \subset \ldots$$

Classification of Endpoints

- The perturbation being well-defined heavily relies on how the coefficient functions behave at the endpoints.

Classification of Endpoints

- The perturbation being well-defined heavily relies on how the coefficient functions behave at the endpoints.

- There are only two kinds of endpoints that require boundary conditions, regular and limit circle:
  - The endpoint a is called *regular* if $1/p(x), q(x), w(x) \in L^1[(a,c), dx]$ for all $c \in (a,b)$.
  - The endpoint a is in the *limit circle case* if for every $z \in \mathbb{C}$, all solutions u of $(\ell - z)u = 0$ are in $L^2[(a,b), w]$ near a.
- An endpoint a in the *limit point case* does not require a boundary condition at a.

Goal

- Can we extend the previous setup for regular boundary conditions to cover limit-circle endpoints? In particular, can we write all possible self-adjoint extensions of the minimal operator as a rank-one perturbation of a self-adjoint extension?
- What about two limit-circle endpoints and writing a rank-two perturbation?

The Jacobi Differential Operator

Let $0 < \alpha, \beta < 1$, and consider the classical Jacobi differential expression given by

$$\ell_{\alpha,\beta}[f](x) = -\frac{1}{(1-x)^{\alpha}(1+x)^{\beta}}[(1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x)]'$$

on the maximal domain

$$\mathcal{D}_{\max}^{(\alpha,\beta)} = \{f \in L^2_{\alpha,\beta}(-1,1) \mid f, f' \in AC_{\mathrm{loc}}; \ell_{\alpha,\beta}[f] \in L^2_{\alpha,\beta}(-1,1)\},$$

where the Hilbert space $L^2_{\alpha,\beta}(-1,1) := L^2\left[(-1,1),(1-x)^{\alpha}(1+x)^{\beta}\right]$. Two boundary conditions are required.

The Jacobi Differential Operator

- The associated sesquilinear form is defined, for $f, g \in \mathcal{D}_{\max}^{(\alpha,\beta)}$, via the Greens formula. Integration by parts easily yields the explicit expression

$$[f,g](\pm 1) := \lim_{x \to \pm 1^{\mp}} (1-x)^{\alpha+1}(1+x)^{\beta+1}[g'(x)f(x) - f'(x)g(x)].$$

- The minimal domain is

$$\mathcal{D}_{\min}^{(\alpha,\beta)} := \{f \in \mathcal{D}_{\max}^{(\alpha,\beta)} \mid [f,g]|_{-1}^{1} = 0 \ \forall g \in \mathcal{D}_{\max}^{(\alpha,\beta)}\}$$

Boundary Triples

A boundary triple is comprised of two maps $\Gamma_0, \Gamma_1 : \mathcal{D}_{\max}^{(\alpha,\beta)} \to \mathbb{C}^2$ and written with the boundary space as $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$. Boundary triples have two properties:

- For all $f, g \in \mathcal{D}_{\max}^{(\alpha,\beta)}$, we have
$$[f,g]\Big|_{-1}^{1} = \langle \Gamma_1 f, \Gamma_0 g\rangle - \langle \Gamma_0 f, \Gamma_1 g\rangle.$$
- The maps Γ_0, Γ_1 are surjective onto their ranges.

Self-adjoint extensions are in one-to-one correspondence with 2×2 self-adjoint relations, θ. Boundary conditions are then given by $\theta\Gamma_0 = \Gamma_1$.

Solutions

Define

$$u(x) := \begin{Bmatrix} 1 & \text{near } x = 1 \\ -1 & \text{near } x = -1 \end{Bmatrix}$$

$$v(x) := \begin{Bmatrix} \dfrac{((1-x)/2)^{-\alpha}}{\alpha 2^{\alpha+\beta+1}} F(-\alpha, \beta+1; 1-\alpha; (1-x)/2) & \text{near } 1 \\ \dfrac{((1+x)/2)^{-\beta}}{\beta 2^{\alpha+\beta+1}} F(-\beta, \alpha+1; 1-\beta; (1+x)/2) & \text{near } -1 \end{Bmatrix}.$$

Solutions

Define

$$u(x) := \left\{\begin{matrix} 1 & \text{near } x = 1 \\ -1 & \text{near } x = -1 \end{matrix}\right\}$$

$$v(x) := \left\{\begin{matrix} \dfrac{((1-x)/2)^{-\alpha}}{\alpha 2^{\alpha+\beta+1}} F(-\alpha, \beta+1; 1-\alpha; (1-x)/2) & \text{near } 1 \\ \dfrac{((1+x)/2)^{-\beta}}{\beta 2^{\alpha+\beta+1}} F(-\beta, \alpha+1; 1-\beta; (1+x)/2) & \text{near } -1 \end{matrix}\right\}.$$

Two operations are then generated via these particular solutions:

$$f^{[0]}(x) := [f, u](x) \text{ and } f^{[1]}(x) := [f, v](x).$$

Boundary Triple for Jacobi

Explicitly, we have

$$f^{[0]}(-1) = [f,u](-1) = \lim_{x\to -1^+} -(1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x),$$

$$f^{[0]}(1) = [f,u](1) = \lim_{x\to 1^-} (1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x),$$

$$f^{[1]}(-1) = [f,v](-1) = \lim_{x\to -1^+} -f(x) - \frac{(1+x)f'(x)}{\beta},$$

$$f^{[1]}(1) = [f,v](1) = \lim_{x\to 1^-} f(x) - \frac{(1-x)f'(x)}{\alpha}.$$

A boundary triple for $\mathcal{D}_{\max}^{(\alpha,\beta)}$ is given by $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f := \begin{pmatrix} f^{[0]}(-1) \\ f^{[0]}(1) \end{pmatrix}, \quad \Gamma_1 f := \begin{pmatrix} f^{[1]}(-1) \\ -f^{[1]}(1) \end{pmatrix}, \quad f \in \mathcal{D}_{\max}^{(\alpha,\beta)}.$$

Key Extensions

- The operators $\mathbf{A}_0$ and $\mathbf{A}_\infty$ will act via $\ell_{\alpha,\beta}[\cdot]$ on the domains

$$\operatorname{dom}\mathbf{A}_0 = \{f \in \operatorname{dom}\mathcal{D}_{\max}^{(\alpha,\beta)} \ : \ f^{[1]}(-1) = f^{[1]}(1) = 0\}, \text{ and}$$
$$\operatorname{dom}\mathbf{A}_\infty = \{f \in \operatorname{dom}\mathcal{D}_{\max}^{(\alpha,\beta)} \ : \ f^{[0]}(-1) = f^{[0]}(1) = 0\},$$

  respectively.
- We want to find a compatible boundary pair for our boundary triple.

Breakout #2: Semi-bounded Forms

- Let S be a semi-bounded self-adjoint operator with lower bound $m(S)$. Recall that there is a a natural way to identify S with a closed semi-bounded form $\mathfrak{t}$ in $\mathcal{H}$ with lower bound $m(\mathfrak{t}) = m(S)$. Let $\varphi, \psi \in \mathrm{dom}(\mathfrak{t})$ and $\gamma < m(S)$, where

$$\mathrm{dom}(\mathfrak{t}_S) = \mathrm{dom}(S-\gamma)^{1/2},$$
$$\mathfrak{t}_S[\varphi,\psi] = \langle (S-\gamma)^{1/2}\varphi, (S-\gamma)^{1/2}\psi\rangle + \gamma\langle\varphi,\psi\rangle.$$

The space $\mathrm{dom}(\mathfrak{t}_S)$ endowed with the inner product

$$\langle\varphi,\psi\rangle_{\mathfrak{t}_{S-\gamma}} := \mathfrak{t}[\varphi,\psi] - \gamma\langle\varphi,\psi\rangle, \text{ for } \varphi,\psi \in \mathrm{dom}(\mathfrak{t}),$$

is a Hilbert space, denoted $\mathcal{H}\left(\mathfrak{t}_{S-\gamma}\right)$.

Boundary Pairs

Definition

A pair $\left\{\mathbb{C}^2, \Lambda\right\}$ is called a **boundary pair** for $\mathbf{L}_{\min}^{(\alpha,\beta)}$ corresponding to $\mathbf{A}_\infty$ if $\Lambda \in \mathbf{B}\left(\mathcal{H}\left(\mathfrak{t}_{\mathbf{A}_0}\right), \mathbb{C}^2\right)$ satisfies

$$\ker(\Lambda) = \operatorname{dom}\left(\mathfrak{t}_{\mathbf{A}_\infty}\right) \quad \text{and} \quad \operatorname{ran}(\Lambda) = \mathbb{C}^2.$$

Boundary Pairs

Definition

A pair $\left\{\mathbb{C}^2, \Lambda\right\}$ is called a **boundary pair** for $\mathbf{L}_{\min}^{(\alpha,\beta)}$ corresponding to $\mathbf{A}_\infty$ if $\Lambda \in \mathbf{B}\left(\mathcal{H}\left(\mathfrak{t}_{\mathbf{A}_0}\right), \mathbb{C}^2\right)$ satisfies

$$\ker(\Lambda) = \operatorname{dom}\left(\mathfrak{t}_{\mathbf{A}_\infty}\right) \text{ and } \operatorname{ran}(\Lambda) = \mathbb{C}^2.$$

A boundary pair is only **compatible** with a boundary triple if extra conditions are verified. In our example, it suffices to additionally require that

$$\Gamma_0 f = \Lambda f \text{ for all } f \in \mathcal{D}_{\max}^{(\alpha,\beta)}.$$

Form Domain Setup

- Choose fixed points c and d so that

$$-1 < c < a_0 < b_0 < d < 1.$$

We arbitrarily choose $c = -\frac{3}{4}$ and $d = \frac{3}{4}$.

Form Domain Setup

- Choose fixed points c and d so that
$$-1 < c < a_0 < b_0 < d < 1.$$
We arbitrarily choose $c = -\frac{3}{4}$ and $d = \frac{3}{4}$.
- We introduce the first-order differential operator that acts on functions $f \in AC(-1, -3/4]$ via
$$N_{v_{-1}} f := \sqrt{p(x)} v_{-1}(x) \left(\frac{f(x)}{v_{-1}(x)} \right)',$$
where $\sqrt{p(x)} = (1-x)^{\frac{\alpha+1}{2}} (1+x)^{\frac{\beta+1}{2}}$.

Form Domain

- The closed semi-bounded form associated with $\mathbf{A}_0$ is denoted by $\mathfrak{t}_0$, defined on

$$\mathfrak{D} := \left\{ f \in L^2_{\alpha,\beta}(-1,1) \ : \ f \in AC(-1,1)\ , \sqrt{p} f' \in L^2\left(-\frac{3}{4}, \frac{3}{4}\right), \right.$$
$$\left. N_{v_{-1}} f \in L^2\left(-1, -\frac{3}{4}\right), N_{v_1} f \in L^2\left(\frac{3}{4}, 1\right) \right\}.$$

Form Domain

- The closed semi-bounded form associated with $\mathbf{A}_0$ is denoted by $\mathfrak{t}_0$, defined on

$$\mathfrak{D} := \left\{ f \in L^2_{\alpha,\beta}(-1,1) \ : \ f \in AC(-1,1) \ , \sqrt{p} f' \in L^2\left(-\frac{3}{4}, \frac{3}{4}\right), \right. \\ \left. N_{v_{-1}} f \in L^2\left(-1, -\frac{3}{4}\right), N_{v_1} f \in L^2\left(\frac{3}{4}, 1\right) \right\}.$$

- The mapping Γ_0 is thus extended to $\mathfrak{D}$:

$$\Lambda f := \begin{pmatrix} f^{[0]}(-1) \\ f^{[0]}(1) \end{pmatrix}, \qquad f \in \mathfrak{D}$$

so that $\{\mathbb{C}^2, \Lambda\}$ can be shown to be a boundary pair for $\mathbf{L}_{\min}$ compatible with the boundary triple $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$.

Perturbation Setup

- Define two weighted distributions that mimic the operation Λ via

$$\langle f, \widetilde{\delta}_{-1} \rangle = \lim_{x \to -1^+} -(1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x),$$
$$\langle f, \widetilde{\delta}_{1} \rangle = \lim_{x \to 1^-} (1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x),$$

on the domain $\mathfrak{D}$.

Perturbation Setup

- Define two weighted distributions that mimic the operation Λ via

$$\langle f, \widetilde{\delta}_{-1} \rangle = \lim_{x \to -1^+} -(1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x),$$
$$\langle f, \widetilde{\delta}_{1} \rangle = \lim_{x \to 1^-} (1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x),$$

  on the domain $\mathfrak{D}$.
- The coordinate mapping $\mathbf{B} : \mathbb{C}^2 \to \mathrm{Ran}(\mathbf{B}) \subset \mathcal{H}_{-1}(\mathbf{A}_0)$ that acts via multiplication by a column vector and its adjoint $\mathbf{B}^* : \mathrm{Ran}(\mathbf{B}) \to \mathbb{C}^2$ are thus given as

$$\begin{pmatrix} \widetilde{\delta}_a \\ \widetilde{\delta}_b \end{pmatrix}, \quad \mathbf{B}^* f = \begin{pmatrix} \langle f, \widetilde{\delta}_a \rangle_{\mathcal{H}} \\ \langle f, \widetilde{\delta}_b \rangle_{\mathcal{H}} \end{pmatrix}.$$

Perturbation

Theorem (BFL)

Let Θ be a linear relation in $\mathbb{C}^2$. Define $\mathbf{A}_\Theta$ as the singular rank-two perturbation:

$$\mathbf{A}_\Theta := \mathbf{A}_0 + \mathbf{B}\Theta\mathbf{B}^*.$$

Then every self-adjoint extension of the minimal operator $\mathbf{L}_{\min}$ can be written as $\boldsymbol{A}_\Theta$ for some Θ.

Breakout #3: Weyl m-Functions

- The **Weyl m-function** associated with the boundary triple $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is given by

$$\rho(\mathbf{A}_\infty) \ni \lambda \mapsto M_\infty(\lambda) = \Gamma_1 \left(\Gamma_0 \upharpoonright (\text{defect spaces for } \lambda)\right)^{-1}.$$

- For $\lambda \in \rho(\mathbf{A}_\infty) \cap \rho(\mathbf{A}_\Theta)$ we have

$$M_\Theta(\lambda) = (\Theta - M_\infty(\lambda))^{-1}.$$

- Explicit Weyl m-functions can be given for a wide variety of interesting cases, including separated and periodic boundary conditions along with special self-adjoint extensions.

Spectral Results

Sturm–Liouville operators with two limit-circle endpoints have only discrete spectrum. However, we can obtain very detailed information for our example:

- Explicit matrix-valued weights for point masses and the multiplicity of the eigenvalues, and
- Formulas for eigenvectors in $L^2(\mu^\Theta)$.

Nuances do exist. Although we can state the formula for an eigenvector of $\mathbf{A}_\Theta$ in $L^2(\mu^\Theta)$, it's precise location involves solving a transcendental equation.

Key References

- B. Simon, *Trace Ideals and Their Applications*, 2nd ed. American Mathematical Society, Providence, RI (2005).
- S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators*, London Mathematical Society Lecture Note Series, Vol. 271, Cambridge University Press, Cambridge, U.K. (2000).
- M. Naimark, *Linear Differential Operators Part I, II*, Frederick Ungar Publishing Co., New York, NY (1972).
- D. Frymark, *Boundary Triples and Weyl m-functions for Powers of the Jacobi Differential Operator*, J. Differential Equations **269** (2020), 7931–7974.
- J. Behrndt, S. Hassi, H. de Snoo, *Boundary Value Problems, Weyl Functions, and Differential Operators*, Monographs in Mathematics, Birkhäuser **108** (2020).
- F. Gesztesy, L. Littlejohn, R. Nichols, *On Self-Adjoint Boundary Conditions for Singular Sturm–Liouville Operators Bounded From Below*, J. Differential Equations **269** (2020) 6448–6491.
- C. Liaw and S. Treil, *Matrix measures and finite rank perturbations of self-adjoint operators*, Forthcoming in *J. Spectral Th.* Preprint available arXiv: 1806.08856v2.