Differentiation and antidifferentiation in free analysis

Meric Augat – Washington University in St. Louis

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Overview

- Free sets and free functions
- Differentiation of free functions
- What properties must a derivative satisfy?
- When is a free linear map the derivative of a free function?

Our main results are free analogs of the following well-known classical results:

- if f is twice differentiable then $\mathrm{curl}(\nabla f) = 0$,
- "conversely", if F is a differentiable vector field on a simply connected region and $\mathrm{curl}(F) = 0$, then $F = \nabla f$.

Main ideas

Without defining anything, the derivative in free analysis is the noncommutative directional derivative.

That is, $Df(X)[H]$ is the derivative of a function f, at a matrix point X, in the direction of H.

We have the following simple examples of derivatives

$$p(X_1, X_2) = 2X_2 + 3X_1X_2$$
$$Dp(X_1, X_2)[H_1, H_2] = 2H_2 + 3H_1X_2 + 3X_1H_2$$

$$q(X_1, X_2) = X_1X_2X_1$$
$$Dq(X_1, X_2)[H_1, H_2] = H_1X_2X_1 + X_1H_2X_1 + X_1X_2H_1$$

What about these?

Naturally, we can ask when a function that "looks like" a derivative, is actually a derivative.

For example, are the following functions derivatives?

1. $2X_1^2 H_2 + X_1 X_2 H_1$
2. $X_1 H_2 - H_1 X_2$
3. $X_1^2 H_1 - H_1 X_1^2$

Free Sets

Let $M(\mathbb{C}) = (M_n(\mathbb{C}))_{n=1}^{\infty}$ and let $M(\mathbb{C})^{\mathrm{g}} = (M_n(\mathbb{C})^{\mathrm{g}})_{n=1}^{\infty}$.

We say $\Omega = (\Omega[n])_{n=1}^{\infty} \subset M(\mathbb{C})^{\mathrm{g}}$ is a **free set** if

1. $X \in \Omega[n]$ and $Y \in \Omega[m] \implies X \oplus Y \in \Omega[n+m]$
2. $X \in \Omega[n]$ and $S \in \mathrm{GL}_n(\mathbb{C}) \implies S^{-1}XS \in \Omega[n]$

where

$$\begin{aligned} X \oplus Y &= (X_1 \oplus Y_1, \ldots, X_{\mathrm{g}} \oplus Y_{\mathrm{g}}) \\ &= \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_{\mathrm{g}} & 0 \\ 0 & Y_{\mathrm{g}} \end{pmatrix} \right), \end{aligned}$$

and

$$S^{-1}XS = (S^{-1}X_1S, \ldots, S^{-1}X_{\mathrm{g}}S).$$

Free Sets II.

1. $M(\mathbb{C})^{\mathrm{g}}$ is a free set
2. $\left(\mathrm{GL}_n(\mathbb{C})\right)_{n=1}^{\infty}$ is a free set since $\det(X \oplus Y) = \det(X)\det(Y)$
3. $\left(\left\{(A,B) \in M_n(\mathbb{C})^2 \,:\, \det(AB - BA) \neq 0\right\}\right)_{n=1}^{\infty}$ is a free set for the same reason (also note this set is empty on $\mathbb{C}^2$).
4. $\left(\{X \in M_n(\mathbb{C}) \,:\, \det(X) = 2\}\right)_{n=1}^{\infty}$ is **not** a free set.

Free Functions

Let $\Omega \subset M(\mathbb{C})^{\mathtt{g}}$ be a free set and let $f[n] : \Omega[n] \to M_n(\mathbb{C})^{\mathtt{h}}$.
Setting $f = (f[n])_{n=1}^{\infty}$ we write $f : \Omega \to M(\mathbb{C})^{\mathtt{h}}$.

We say f is a **free function** if

1. $f(X \oplus Y) = f(X) \oplus f(Y)$ $(f[n+m](X \oplus Y) = f[n](X) \oplus f[m](Y))$
2. $f(S^{-1}XS) = S^{-1}f(X)S$ $(f[n](S^{-1}XS) = S^{-1}f[n](X)S)$

whenever $X \in \Omega[n], \quad Y \in \Omega[m] \quad$ and $S \in \mathrm{GL}_n(\mathbb{C})$.

1. $X \mapsto X^2$ is a free map on $M(\mathbb{C})$.
2. $f(X,Y) = XY^{-1}X$ is a free map on $\Big(M_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})\Big)_{n=1}^{\infty}$.

Free Functions II

We say a free set Ω is **open** if each $\Omega[n]$ is open.

☞ A **free domain** is an open free set.

Let f be a free function on the free domain Ω. If each $f[n]$ is continuous (analytic), then we say f is **continuous** (**analytic**) .

If $X \in \Omega[n]$ and $H \in M_n(\mathbb{C})^{\mathfrak{g}}$, then $Df(X)[H]$ is the noncommutative directional derivative of f at X in the direction of H.

1. $f_1(X, Y) = XY + X^2 \Rightarrow Df_1(X, Y)[H, K] = HY + XK + HX + XH$
2. $f_2(X, Y) = 2X + Y^{-1} \Rightarrow Df_2(X, Y)[H, K] = 2H - Y^{-1}KY^{-1}$

Derivatives of Free Functions

We often use the following standard result in free analysis.

Theorem. Suppose f is a free map on the free domain Ω. If f is continuous then f is analytic and for all $X \in \Omega[n]$ and (sufficiently small) $H \in M_n(\mathbb{C})$,

$$f \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & Df(X)[H] \\ 0 & f(X) \end{pmatrix}.$$

Specifically, the derivative of an analytic free map can be found via point evaluation!

Free Curl

A free map T on $\Omega \times M(\mathbb{C})^{\mathrm{g}}$ is **free demilinear** if it is linear in half of its variables:

$$T(X, cH + K) = cT(X, H) + T(X, K).$$

- ⇨ Df is free demilinear for all analytic free maps f
- ⇨ $T_1(X, H) = X^2H - HX^2$ is free demilinear
- ⇨ $T_2(X, Y, H, K) = YH + HY - XK - KX$ is free demilinear.

If T is an analytic free demilinear map, then the **free-curl** of T is the difference $DT(X, H)[K, 0] - DT(X, K)[H, 0]$.

- ☞ if the free-curl of T is always zero, then we say T is **free-curl free**.

Main Results

Theorem (A' 2020)

Suppose f is an analytic free map on Ω. If T is the analytic demilinear free map on $\Omega \times M(\mathbb{C})^g$ defined by $T(X,H) = Df(X)[H]$ then T is free-curl free.

Theorem (A' 2020)

Suppose Ω is a free domain and each $\Omega[n]$ is connected. Suppose T is an analytic free demilinear map defined on $\Omega \times M(\mathbb{C})^g$. If T is free-curl free then there exists an analytic free map f on Ω such that $T = Df$.

Consequences

❶ We can now answer the question of "when is a free linear map a derivative?" with a simple and testable condition.

❷ This pair of theorems plus a little bit of extra work can be used to prove the existence of free pluriharmonic conjugates

- ∼ thanks to Rob Martin for pointing this out
- ∼ the existence of free pluriharmonic conjugates was previously shown in Pascoe's Free Monodromy paper

❸ If Df is a free rational function, then f is a free rational function.

❹ It may be possible to adapt the results to the NC operator setting.

Easier Direction

The proof of our first theorem is relatively elementary. We assume f is free analytic and $T(X, H) = Df(X)[H]$.

Consider the following computation

$$f\begin{pmatrix} X & K & H & 0 \\ 0 & X & 0 & H \\ 0 & 0 & X & K \\ 0 & 0 & 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & T(X,K) & \multicolumn{2}{c}{Df\begin{pmatrix} X & K \\ 0 & X \end{pmatrix}\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}} \\ 0 & f(X) & & \\ 0 & 0 & f(X) & T(X,K) \\ 0 & 0 & 0 & f(X) \end{pmatrix}$$

$$= \begin{pmatrix} f(X) & T(X,K) & T(X,H) & DT(X,H)[K,0] \\ 0 & f(X) & 0 & T(X,H) \\ 0 & 0 & f(X) & T(X,K) \\ 0 & 0 & 0 & f(X) \end{pmatrix}.$$

Similarly,

$$f\begin{pmatrix} X & H & K & 0 \\ 0 & X & 0 & K \\ 0 & 0 & X & H \\ 0 & 0 & 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & T(X,H) & T(X,K) & DT(X,K)[H,0] \\ 0 & f(X) & 0 & T(X,K) \\ 0 & 0 & f(X) & T(X,H) \\ 0 & 0 & 0 & f(X) \end{pmatrix}.$$

Easier Direction II

Let U be the block unitary that switches the second and third rows/columns. Hence

$$U^{-1}\begin{pmatrix} X & K & H & 0 \\ 0 & X & 0 & H \\ 0 & 0 & X & K \\ 0 & 0 & 0 & X \end{pmatrix} U = \begin{pmatrix} X & H & K & 0 \\ 0 & X & 0 & K \\ 0 & 0 & X & H \\ 0 & 0 & 0 & X \end{pmatrix}.$$

Since f is free, $f(U^{-1}ZU) = U^{-1}f(Z)U$. Thus,

$$f\begin{pmatrix} X & H & K & 0 \\ 0 & X & 0 & K \\ 0 & 0 & X & H \\ 0 & 0 & 0 & X \end{pmatrix} = U^{-1} f\begin{pmatrix} X & K & H & 0 \\ 0 & X & 0 & H \\ 0 & 0 & X & K \\ 0 & 0 & 0 & X \end{pmatrix} U$$

Specifically, switching the rows and columns before evaluation is the same as switching the rows and columns after evaluation!

☞ The $(1, 4)$-entry is unaffected by the row/column switches.

Easier Direction III

Since

$$f\begin{pmatrix} X & K & H & 0 \\ 0 & X & 0 & H \\ 0 & 0 & X & K \\ 0 & 0 & 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & T(X,K) & T(X,H) & DT(X,H)[K,0] \\ 0 & f(X) & 0 & T(X,H) \\ 0 & 0 & f(X) & T(X,K) \\ 0 & 0 & 0 & f(X) \end{pmatrix}$$

and

$$\begin{aligned} Uf\left(\begin{smallmatrix} X & H & K & 0 \\ 0 & X & 0 & K \\ 0 & 0 & X & H \\ 0 & 0 & 0 & X \end{smallmatrix}\right)U^{-1} &= U\begin{pmatrix} f(X) & T(X,H) & T(X,K) & DT(X,K)[H,0] \\ 0 & f(X) & 0 & T(X,K) \\ 0 & 0 & f(X) & T(X,H) \\ 0 & 0 & 0 & f(X) \end{pmatrix}U^{-1} \\ &= \begin{pmatrix} f(X) & T(X,K) & T(X,H) & DT(X,K)[H,0] \\ 0 & f(X) & 0 & T(X,H) \\ 0 & 0 & f(X) & T(X,K) \\ 0 & 0 & 0 & f(X) \end{pmatrix}, \end{aligned}$$

we conclude $DF(X,H)[K,0] = DF(X,K)[H,0]$.

Path Independence

$\star$ In the classical setting, if F is a vector field on a simply connected region and $\mathrm{curl}(F) = 0$, then F is path independent.

$\star$ If T is an analytic free demilinear map on $\Omega \times M(\mathbb{C})^{\mathrm{g}}$ and $\gamma : [0,1] \to \Omega[n]$ is a smooth path, then we define

$$\int_0^1 T\Big(\gamma(t), \gamma'(t)\Big)\, dt$$

to be the result of entry-wise integration.

$\star$ We say T is **path independent** if whenever $n \in \mathbb{Z}^+$, $\gamma_1, \gamma_2 : [0,1] \to \Omega[n]$ are smooth, $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$, then

$$\int_0^1 T(\gamma_1(t), \gamma_1'(t))\, dt = \int_0^1 T(\gamma_2(t), \gamma_2'(t))\, dt.$$

Path Independence II

With an expected definition of path independence, we have the following result:

Proposition

Suppose Ω is a free domain and T is an analytic free demilinear map on $\Omega \times M(\mathbb{C})^g$. If T is free-curl free then T is path independent.

Notice that this Proposition makes no mention of geometry!

Converse

Recall our second theorem:

Theorem

Suppose Ω is a free domain and each $\Omega[n]$ is connected. Suppose T is an analytic free demilinear map defined on $\Omega \times M(\mathbb{C})^g$. If T is free-curl free then there exists an analytic free map f on Ω such that $T = Df$.

☆ In the classical setting, a potential function is found by choosing an "anchor point" and integrating from that anchor to each point in the domain.

☆ Since we have a graded set Ω, we must choose an "anchor point" at each n and then guarantee that the result is in fact a free map.

Converse II

The proof of the second Theorem is broken down into four main steps.

1. For each n, choose $Z_n \in \Omega[n]$ and let $\alpha_n(X) = \int_{Z_n}^{X} T$, integrated over any path from Z_n to X
2. Define $\beta_n(X)$ as Haar integral of $U^*\alpha_n(UXU^*)U$ over the unitary group and show it respects conjugation by unitaries
3. Use the entry-wise analyticity of β_n to show that β_n respects similarities
4. Use level-wise direct sums to find constants b_n such that $\Phi_n = \beta_n + b_n$ defines an analytic free map.

Thank You.