Spectral operators in finite von Neumann algebras

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The Spectral Theorem for Normal Operators

Theorem

If $T \in B(\mathcal{H})$ is normal, there exists a projection-valued spectral measure E_T such that

$$T = \int_{\sigma(T)} \lambda \, dE_T(\lambda)$$

- Projection: $P \in B(\mathcal{H}), P^* = P, P^2 = P$
- Idempotent: $Q \in B(\mathcal{H}), Q^2 = Q$

Spectral Measures

A *projection-valued spectral measure* is a mapping $B \mapsto E(B)$ that assigns to every Borel set B a projection $E(B) \in B(\mathcal{H})$

1. $E(\mathbb{C}) = 1$,
2. for all B_1, B_2, $E(B_1 \cap B_2) = E(B_1)E(B_2)$,
3. for all pairwise disjoint Borel sets B_i,

$$E(\bigcup_{i=1}^{\infty} B_i)x = \sum_{i=1}^{\infty} E(B_i)x, \; \forall x \in \mathcal{H}$$

Spectral Measures

A bounded **idempotent**-valued spectral measure is a mapping $B \mapsto E(B)$ that assigns to every Borel set B an **idempotent** $E(B) \in B(\mathcal{H})$

1. $E(\mathbb{C}) = 1$,
2. for all B_1, B_2, $E(B_1 \cap B_2) = E(B_1)E(B_2)$,
3. for all pairwise disjoint B_i
$$E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i),$$
where the sum converges in the strong operator topology.
4. $\sup_{B \text{ Borel}} \|E(B)\| < \infty$.

Spectral measures

Dunford: T is called a **spectral operator** if there exists an idempotent-valued spectral measure E such that

- $E(B)T = TE(B)$, for every Borel set B.
- The spectrum of T restricted to the range of $E(B)$ is contained in $\overline{B}$

S is of **scalar type** if S is spectral, and

$$S = \int_{\sigma(S)} \lambda E(d\lambda)$$

Theorem (Dunford, '58)

$S \in B(\mathcal{H})$ is a scalar type operator iff there exists A invertible in $B(\mathcal{H})$, such that $A^{-1}SA$ is a normal operator. Moreover, $A \in \{S\}''$

Theorem (Dunford, '58)

T is a spectral operator iff it is similar to the sum of a commuting normal operator and a quasinilpotent operator.

(analogous to the Jordan Canonical Form for matrices)

Decomposable operators

Definition (Foais)

An operator $T \in B(\mathcal{H})$ is said to be decomposable if it has a spectral capacity. A spectral capacity for T is a map

$$\{\textit{closed sets in } \mathbb{C}\} \ni K \mapsto \mathcal{E}(K) \in \{\textit{closed subspaces of } \mathcal{H}\}$$

so that

1. $\mathcal{E}(\emptyset) = \{0\}$, $\mathbb{C} = \mathcal{H}$
2. $\mathcal{E}(\overline{U_1}) + \mathcal{E}(\overline{U_2}) + \ldots + \mathcal{E}(\overline{U_n}) = \mathcal{H}$ *(algebraic sum) for all open covers* $\{U_1, \ldots U_n\}$ *of* $\mathbb{C}$
3. $E(\cap_{k=1}^{\infty} K_n) = \cap_{k=1}^{\infty} E(K_n)$
4. $\sigma(T\restriction_{\mathcal{E}(K)}) \subseteq K$

Spectral $\implies$ decomposable: $\mathcal{E}(K) = E(K)\mathcal{H}$

Finite Von Neumann algebras

$\mathcal{M} \subset B(\mathcal{H})$ is a finite von Neumann algebra if:

- $\mathcal{M}$ is a *-subalgebra.
- $\mathcal{M}$ is closed under the strong/weak operator topology.
- There is a 'nice' trace $\tau : \mathcal{M} \to \mathbb{C}$:
    - τ is linear.
    - $\tau(AB) = \tau(BA)$.
    - $\tau(1) = 1$
    - $\tau(A^*A) \geq 0$
    - reasonable continuity properties

For a normal operator $T \in \mathcal{M}$, there is a probability measure associated with the projection-valued measure, given by

$$\mu_T(B) = \tau(E_T(B))$$

(Brown, '83): For every operator $T \in \mathcal{M}$, there is an associated "good" probability measure μ_T on $\mathbb{C}$.
Constructed by taking the Laplacian of the log of the 'determinant' of T:

$$\mu_T = \frac{1}{2\pi}\nabla^2 \tau\left(\log(|A - \lambda|)\right)$$

μ_T is supported inside $\sigma(T)$, but the support can be much smaller.

Haagerup-Schultz projections

Haagerup and Schultz constructed a set of invariant projections for T, which behave well with the Brown measure:

Theorem (Haagerup, Schultz '09)

Let $T \in \mathcal{M}$. For any Borel set $B \subset \mathbb{C}$, there exists a unique projection $p = P(T, B) \in \mathcal{M}$ such that

1. $Tp = pTp$
2. $\tau(p) = \mu_T(B)$
3. *The Brown measure of T restricted to $p\mathcal{H}$ is supported inside B.*
4. *p is the maximal projection such that the above holds.*

In general, not a projection-valued *measure*, since the projections need not commute.

Theorem

$\mu_T = \delta_0$ iff
$\lim ((T^n)^* T^n)^{1/2n} = 0$ in the strong operator topology

Such operators are called 'sot-quasnilpotent operators'.
Compare to quasinilpotents:

$$\sigma(T) = \{0\} \iff ||T^{*n} T^n||^{1/2n} \to 0$$

Angles between subspaces

Definition

Let V, W be closed non-zero subspaces of a Hilbert space $\mathcal{H}$, with $V \cap W = \{0\}$. Then, the angle between them is

$$\alpha(V,W) := \inf\left\{\cos^{-1}\left(|\langle v,w\rangle|\right) \mid v \in V,\ w \in W,\ ||v|| = ||w|| = 1\right\}$$

We say that $T \in \mathcal{M}$ has the **uniformly nonzero angle property** (or UNZA property), if there is $\kappa > 0$ such that for all Borel sets $B \subseteq \mathbb{C}$ with $P(T,B) \neq 0 \neq P(T,B^c)$, we have

$$\alpha(P(T,B)\mathcal{H}, P(T,B^c)\mathcal{H}) \geq \kappa$$

Question: Does NZA imply UNZA?

UNZA

UNZA allows us to construct lots of idempotents:

Lemma

Let V, W be closed subspaces of $\mathcal{H}$ with $V \cap W = \{0\}$ and $\overline{V + W} = \mathcal{H}$. Then the following are equivalent:

1. $\alpha(V, W) > 0$.
2. *$V + W$ is closed.*
3. *There exists a bounded idempotent $e \in B(\mathcal{H})$ such that*

$$e\mathcal{H} = V \quad \text{and} \quad (1 - e)\mathcal{H} = W.$$

Moreover,there is a continuous, strictly decreasing function $f : (0, 1] \to [1, \infty)$ such that

$$\|e\| \leq f\big(1 - \cos(\alpha(V, W))\big).$$

Lemma

Assume Let $T \in \mathcal{M}$ has the uniformly non-zero angle property. Then there exists an idempotent valued spectral measure E with the following properties,

(a) $E(B)\mathcal{H} = P(T, B)\mathcal{H}$ *and* $\ker E(B) = P(T, B^c)\mathcal{H}$,

(b) $TE(B) = E(B)T$,

(c) *The Brown measure of the restriction of T to the range of $E(B)$ is concentrated in B.*

where B is an arbitrary Borel subset of $\mathbb{C}$.

Theorem (Dykema, K.U.)

Let $T \in \mathcal{M}$. Then the following are equivalent:

(a) *T has the UNZA property,*

(b) *there exist $S, Q \in \mathcal{M}$ with $[S, Q] = 0$, S a scalar type operator and Q s.o.t.-quasinilpotent, such that $T = S + Q$,*

(c) *there exist $A, N, Q' \in \mathcal{M}$, with $[N, Q'] = 0$, N normal, Q' s.o.t-quasinilpotent, and A invertible, such that $ATA^{-1} = N + Q'$.*

Fact: For decomposable operators, the spectral capacity is given by the Haagerup-Schultz projections, and the support of μ_T is $\sigma(T)$.

Corollary

T is spectral if and only if T is decomposable and satisfies the UNZA property.

Corollary

$P(T, \cdot)$ defines a spectral measure if and only if $T = N + Q$ for some $N, Q \in \mathcal{M}$, where N is normal, Q is s.o.t.-quasinilpotent, and $NQ = QN$.

Non-spectral but decomposable operator

Let

$$T = \bigoplus_{n=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 0 & 1/n \end{pmatrix} \in \bigoplus_{n=1}^{\infty} M_2(\mathbb{C}) \subseteq \mathcal{R}$$

$$\sigma(T) = \{1, 1/2, 1/3...\} \cup \{0\}$$

The eigenvectors for the nth block are $(1,0)^t$ and $(1,1/n)^t$, and the angle between them goes to 0 and $n \to \infty$.
But T has countable spectrum, so it is decomposable.

Non-trivial examples of decomposable, non-spectral operators in a finite factor?

Free Probability

Look at free probability to construct examples of non-spectral operators -

- Non-commutative probability: Operators in $\mathcal{M}$ are the random variables, and τ is the 'expectation'.
- Two *-subalgebras $A, B \subset (\mathcal{M}, \tau)$ are *free* if mixed moments of centered elements vanish:
  For $a_i \in A$, $b_i \in B$,

$$\tau(a_i) = 0 = \tau(b_i) \implies \tau(a_1 b_1 a_2 \dots b_n) = 0$$

- Two operators are *-free if the *-algebras they generate are free.
- The *-distribution of an element X is the collection of traces of all non-commutative polynomials in X and X^*.

Non-commutative Gaussians

Non-commutative analogue of the real Gaussian:
A semicircular operator X is self-adjoint, with its measure given by the density function:

$$s(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\chi_{\{-2\leq t\leq 2\}}$$

Non-commutative analogue of the complex Gaussian:
Voiculescu's circular operator Z is the sum of two free semicircular operators

$$Z = \frac{1}{\sqrt{2}}(X_1 + iX_2)$$

The Brown measure of Z is the uniform measure on the disc of radius 1.
Z should be "highly non-normal", and hence non-spectral.

R-diagonal operators

- Z is an example of an R-diagonal operator.
- R-diagonal operators are of the form UH, where U is unitary, with $\tau(U^n) = 0, n \neq 0$, and H is positive, and U, H are *-free.
- R-diagonal operators have nice 2-norms: If x is R-diagonal,
  1. if x is invertible, then also x^{-1} is R-diagonal,
  2. for every $k \in \mathbb{N}$, $\|x^k\|_2^2 = \|x\|_2^{2k}$,
  3. the spectral radius of x equals $\|x\|_2$.
  4. $\text{supp}(\mu_x) = \sigma(x)$, and μ_x is radially symmetric.

DT-operators

Z is also an example of a DT-operator (diagonal + triangular)

- These arise as the limiting *-distributions of diagonal matrices + upper triangular gaussian random matrices.
- Can be realized as $Z = D + cT$, where D is a normal operator and T is the "upper triangular half" of a semicircular operator that is free from an abelian algebra containing D.
- DT-operators are R-diagonal iff their Brown measure is uniform on some annulus centered at 0.
- DT-operators have a nice upper triangular decomposition:

DT-operators decompose well:
(Dykema,Haagerup): Can realize Z in $M_N(\mathcal{M}) \cong M_N(\mathbb{C}) \otimes \mathcal{M}$ (with respect to the tracial state $\frac{1}{N}\mathrm{Tr}_N \otimes \tau$) as an upper triangular matrix

$$Z = \begin{pmatrix} a_1 & b_{12} & \cdots & \cdots & b_{1N} \\ 0 & a_2 & b_{23} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & a_{N-1} & b_{N-1,N} \\ 0 & \cdots & \cdots & 0 & a_N \end{pmatrix},$$

With

- $(a_k)_{k=1}^N$, $(b_{ij})_{1 \le i < j \le N}$ a $*$-free family in $(\mathcal{M}, \tau)$
- each b_{ij} circular with $\tau(b_{ij}^* b_{ij}) = \frac{1}{N}$,
- each a_j a $DT(\mu_j, \frac{1}{\sqrt{N}})$ operator for a Borel probability measure μ_j on $\mathbb{C}$,
- $\frac{1}{N}(\mu_1 + \cdots + \mu_N) = \mu$

The structure of Haagerup-Schultz projections

Proposition (Haagerup,Schultz)

Let $r > 0$*. Suppose* $\mathcal{M}$ *acts on the Hilbert space* $\mathcal{H}$ *and* $T \in \mathcal{M}$*. Then*

$$P(T, r\mathbb{D})\mathcal{H} =$$

$$\{\xi \in \mathcal{H} \mid \exists \xi_n \in \mathcal{H}, \lim_{n\to\infty} \|\xi_n - \xi\| = 0, \limsup_{n\to\infty} \|T^n \xi_n\|^{1/n} \leq r\}.$$

and

$$P(T, \mathbb{C} \setminus r\mathbb{D})\mathcal{H} =$$

$$\{\eta \in \mathcal{H} \mid \exists \eta_n \in \mathcal{H}, \lim_{n\to\infty} \|T^n \eta_n - \eta\| = 0, \limsup_{n\to\infty} \|\eta_n\|^{1/n} \leq \frac{1}{r}\}.$$

Lemma

Let $0 < r' < r < s < s'$ so that the annuli $A(r', r)$ and $A(s, s')$ have the same Lebesgue measure. Let T be a DT-operator with uniform measure μ on $A(r', r) \cup A(s, s')$.
Let $p_1 = P(T, A(r, r')), p_2 = P(T, A(s, s'))$.
Then,

$$\alpha(p_1, p_2) \leq \cos^{-1}\left(\frac{1}{\sqrt{2(s^2 - r^2) + 1}}\right)$$

Proof sketch

- Decompose
$$T = \begin{pmatrix} D_1 & Z \\ 0 & D_2 \end{pmatrix}$$
- Let
$$\xi = \sum_{k=0}^{\infty} D_1^k Z D_2^{-k-1}$$
- Use *-freeness of D_i, Z, characterisation of HS projections for discs, and 2-norm estimates from R-diagonality, to show $\begin{pmatrix} \xi \\ 1 \end{pmatrix} \in P(T, A(s, s'))$, $\begin{pmatrix} \xi \\ 0 \end{pmatrix} \in P(T, A(r', r))$
- Use R-diagonality of D_i to estimate the angle between $(\xi, 1)$ and $(\xi, 0)$.

Theorem (Dykema, K.U.)

Let Z be an operator which is both R-diagonal and DT (i.e. the circular free Poisson operators) Then Z does not have the uniformly non-zero angles property, and hence is not spectral.

Proof sketch:

- Partition the annulus $A_{\sqrt{c-1},\sqrt{c}}$ into N equally weighted mutually disjoint parts $E_1 ... E_N$, with E_1, E_2 equally weighted annuli.
- Use the upper triangular decomposition, and the previous trick to get angle estimates.
- Choose the annuli to be sufficiently close, to show the UNZA property fails.

Unresolved Questions

- Example of an operator with the NZA property that fails UNZA?
- Examples of non-R-diagonal DT-operators which fail the UZNA property?
- Conjecture (Kamil Szpojankowski): every freely infinitely divisible R-diagonal operator is non-spectral.
  X is f.i.d and R-diagonal iff X has the same *-distribution as MZ, where $M > 0$, Z circular, M, Z are $*$-free.