(General) Optimal Polynomial Approximants

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Setting

Let $\mathcal{H}$ be a Hilbert space of analytic functions on the unit disk $\mathbb{D}$ with the following properties:

- The polynomials $\mathcal{P}$ are dense in $\mathcal{H}$.
- The forward shift operator S, $f(z) \mapsto zf(z)$, is bounded on $\mathcal{H}$.
- Point evaluation is bounded on $\mathbb{D}$.
  - For each $z \in \mathbb{D}$, there exists unique $k_z \in \mathcal{H}$ such that $\langle f, k_z \rangle_{\mathcal{H}} = f(z)$ for all $f \in \mathcal{H}$

Cyclic Functions

Definition

Say $f \in \mathcal{H}$ is *cyclic* (in $\mathcal{H}$ for S) if

$$[f] := \overline{\text{span}\{z^n f : n = 0, 1, 2, \ldots\}}^{\mathcal{H}}$$

is equal to $\mathcal{H}$.

Note:

- TFAE:
    1. f is cyclic
    2. $1 \in [f]$
    3. There exists $(p_n)_{n \geq 0} \subseteq \mathcal{P}$ such that $\|p_n f - 1\| \to 0$

Can we find $(p_n)_{n\geq 0}$ such that $\|p_n f - 1\|$ decays *optimally*?

Definition

Let $\mathcal{P}_n = \{p \in \mathcal{P} : \deg p \leq n\}$ and $f \in \mathcal{H}$. Say p_n^* is the nth optimal polynomial approximant to $1/f$ if p_n^* solves

$$\min_{p\in\mathcal{P}_n} \|pf - 1\|.$$

Note: $\mathcal{P}_n f$ is a closed subspace of $\mathcal{H}$, so $p_n^* f = \Pi_{\mathcal{P}_n f}(1)$.

What about other spaces?

For example, let $v \in L^1(\mathbb{T})$ be positive. Szegő says

$$P^2(v) := \overline{\operatorname{span}\{z^n : n = 0, 1, 2, \ldots\}}^{L^2(v)} \text{ is equal to all of } L^2(v)$$
$$\Updownarrow$$
$$\int_{\mathbb{T}} \log v = -\infty$$

If $\int_{\mathbb{T}} \log v > -\infty$, then there exists an H^2 cyclic function m such that $|m|^2 = v$ and $P^2(v) \cong \frac{1}{m} H^2$ with $\|f\|_{P^2(v)} = \|f/m\|_{H^2}$.

Putting $f = h/m \in P^2(v)$, it follows that

$$\|pf - 1\|_{P^2(v)} = \|ph - m\|_{H^2}.$$

General Approximants

Approximating functions other than 1

Definition (general)

Let $f, g \in \mathcal{H}$ with $\langle f, g \rangle \neq 0$. Say q_n^* is the nth optimal polynomial approximant to g/f if q_n^* solves

$$\min_{q \in \mathcal{P}_n} \|qf - g\|.$$

What can be said about (q_n^*)?

For example, can this sequence have a finite number of distinct elements? Can the sequence 'stop'?

Definition

Say $(q_n^*)_{n \geq 0}$ stabilizes at q_M^* if $q_n^* = q_M^*$ for all $n \geq M$.

Inner Functions

Definition

Say $f \in \mathcal{H}$ is $\underline{\mathcal{H}\text{-inner}}$ if $\langle f, z^k f \rangle_{\mathcal{H}} = \delta_{k0}$.

What do OPAs and inner functions know about each other?

Theorem (Bénéteau, Fleeman, Khavinson, Seco, Sola)

Suppose $\mathcal{H}$ is such that $k_0(z) = 1$. Let $f \in \mathcal{H}$ with $f(0) \neq 0$ and (p_n^*) the OPAs to $1/f$. Then f is $\mathcal{H}$-inner if and only if (p_n^*) stabilizes at p_0^*.

Connection between stabilization of general OPAs and inner functions?

Definition

$$\begin{aligned}\mathcal{K}_{Sf} :=& \{h \in \mathcal{H} : \langle h, z^k f \rangle = 0 \text{ for } k = 1, 2, 3, \ldots\} \\ =& \mathcal{H} \ominus [Sf]\end{aligned}$$

Theorem (F., '20)

Let $f, g \in \mathcal{H}$ with $\langle f, g \rangle \neq 0$. Let (q_n^*) be the OPAs to g/f. TFAE:

1. $g \in \mathcal{K}_{Sf}$ and $\Pi_{[f]}(g) = q_M^* f$
2. $q_M^* f \in \mathcal{K}_{Sf}$
3. $q_M^* f / \|q_M^* f\|$ is $\mathcal{H}$-inner and $\langle q_M^* f, z^k f \rangle = 0$ for $k = 1, \ldots, M$

Thank you!

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