Finite-dimensional approximations on the bidisc

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Operator Theory with its Applications
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Based on joint work in progress with Adam Dor On.

The disc algebra

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Just how much finite-dimensionality does $\mathrm{A}(\mathbb{D})$ enjoy?

Some associated C*-algebras

Example

By definition, we have that $\mathrm{A}(\mathbb{D}) \subset \mathrm{C}(\overline{\mathbb{D}})$. In fact, we have that $\mathrm{C}^*(\mathrm{A}(\mathbb{D})) = \mathrm{C}(\overline{\mathbb{D}})$. Moreover, $\bigoplus_{z\in\overline{\mathbb{D}}} \varepsilon_z : \mathrm{C}(\overline{\mathbb{D}}) \to \prod_{z\in\overline{\mathbb{D}}} \mathbb{C}$ is a completely isometric homomorphism.

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Example

By the maximum modulus principle, the restriction map $\rho : A(\mathbb{D}) \to C(\mathbb{T})$ is completely isometric. We have that $C^*(\rho(A(\mathbb{D}))) = C(\mathbb{T})$ and $\bigoplus_{z\in\mathbb{T}} \varepsilon_z : C(\mathbb{T}) \to \prod_{z\in\mathbb{T}} \mathbb{C}$ is a completely isometric homomorphism.

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Example

Consider the Hardy space

$$H^2(\mathbb{D}) = \left\{ \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

Let $\Phi : \mathrm{A}(\mathbb{D}) \to B(H^2(\mathbb{D}))$ be defined as $\Phi(f) = M_f$. Then, $\mathfrak{T} = \mathrm{C}^*(\Phi(\mathrm{A}(\mathbb{D})))$ is the *Toeplitz algebra* and it is not RFD (it contains the ideal of compact operators).

The maximal C*-algebra

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Is $\mathrm{C}^*_{\max}(\mathrm{A}(\mathbb{D}))$ residually finite-dimensional? Yes!

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The disc algebra $\mathrm{A}(\mathbb{D})$ *is residually finite-dimensional, and so is its maximal* C*-*algebra* $\mathrm{C}^*_{\max}(\mathrm{A}(\mathbb{D}))$.

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Ingredients:

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- if T is a contraction on $\mathcal{H}$ and $\mathcal{K} \subset \mathcal{H}$ is a subspace, then $P_{\mathcal{K}} T|_{\mathcal{K}}$ is a contraction.

$\square$

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Theorem (Courtney–Sherman 2019, Ji–Natarajan–Vidick–Wright–Yuen 2020)

The universal C^**-algebra generated by a pair of doubly commuting contractions is not residually finite-dimensional.*

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Recall: contractions T_1 and T_2 are doubly commuting if $T_1T_2 = T_2T_1$ and $T_1T_2^* = T_2^*T_1$

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Theorem (Exel–Loring 1992)

A C^**-algebra is residually finite-dimensional if and only if every* $*$*-representation admits an Exel–Loring approximation (of any kind).*

Some implications

Theorem (C.–Dor On 2020)

Let $\mathcal{A}$ be an operator algebra. Then, we have the following implications.

$$\mathrm{C}^*_{\max}(\mathcal{A}) \text{ is residually finite-dimensional.}$$
$$\Updownarrow$$
Every completely contractive representation of $\mathcal{A}$ admits an Exel–Loring C^*-approximation.
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Refined question

Does every completely contractive representation of $\mathrm{A}(\mathbb{D}^2)$ admit an Exel–Loring $*$-approximation?

Examples of Exel–Loring approximations

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$\mathcal{H}$ reproducing kernel Hilbert space on some set X, $\mathcal{A} \subset \mathrm{Mult}(\mathcal{H})$ subalgebra
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$F \subset X$ finite, $\mathcal{K}_F = \mathrm{span}\{k_x : x \in F\} \subset \mathcal{H}$ finite-dimensional subspace

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The compression $\theta : \mathrm{A}(\mathbb{D}) \to B(\mathcal{K}_\varphi)$ admits an Exel–Loring C*-approximation. But an approximating net must be rather complicated: there is no finite-dimensional subspace of $\mathcal{K}_\varphi$ which is semi-invariant for $\mathrm{A}(\mathbb{D})$.

Building new approximations from old ones

Key lemma

Assume that $\theta : \mathcal{A} \to B(\mathcal{H}_\theta)$ admits an Exel–Loring approximation. Let $\mathcal{H} \subset \mathcal{H}_\theta$ be an invariant subspace for $\theta(\mathcal{A})$. Then, the representation

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Recall that

$$\mathrm{C}^*_{\max}(\mathcal{A}) \text{ is residually finite-dimensional.}$$
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where $(a_n), (b_n)$ non-negative sequences

Standing assumption: $a_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$

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- if T is a contraction on $\mathcal{H}$ and $\mathcal{K} \subset \mathcal{H}$ is a subspace, then $P_{\mathcal{K}} T|_{\mathcal{K}}$ is a contraction.

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Claim

Assume that $\theta(z_1)$ is a unilateral shift. Then, θ admits an Exel–Loring $*$-approximation.

Proof of the claim

- We have that $\theta(z_1) = S \otimes I_{\mathcal{E}}$ and $\theta(\mathrm{A}(\mathbb{D}^2)) \subset \mathrm{Mult}(H^2(\mathbb{D}) \otimes \mathcal{E})$.

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- If $\mathcal{E}$ is infinite-dimensional, write $\mathcal{E} = \cup \mathcal{E}_\alpha$ where $\mathcal{E}_\alpha \subset \mathcal{E}$ is a finite-dimensional subspace. Note that $S \otimes P_{\mathcal{E}_\alpha}$ and $(I \otimes P_{\mathcal{E}_\alpha})\theta(z_2)(I \otimes P_{\mathcal{E}_\alpha})$ is a pair of commuting contractions.
- By Ando's inequality, there is a completely contractive homomorphism $\theta_\alpha : \mathrm{A}(\mathbb{D}^2) \to B(H^2(\mathbb{D}) \otimes \mathcal{E}_\alpha)$ such that $\theta_\alpha(z_1) = S \otimes P_{\mathcal{E}_\alpha}$ and $\theta_\alpha(z_2) = (I \otimes P_{\mathcal{E}_\alpha})\theta(z_2)(I \otimes P_{\mathcal{E}_\alpha})$.

Proof of the claim

- We have that $\theta(z_1) = S \otimes I_{\mathcal{E}}$ and $\theta(\mathrm{A}(\mathbb{D}^2)) \subset \mathrm{Mult}(H^2(\mathbb{D}) \otimes \mathcal{E})$.
- If $\mathcal{E}$ is finite-dimensional, then for every $F \subset \mathbb{D}$ finite we let

$$\mathcal{K}_F = \mathrm{span}\{k_z \otimes v : z \in F, v \in \mathcal{E}\}.$$

  This is co-invariant for $\theta(\mathrm{A}(\mathbb{D}^2))$ and $(P_{\mathcal{K}_F}\theta(\cdot)|_{\mathcal{K}_F})$ is an Exel–Loring $*$-approximation for θ.
- If $\mathcal{E}$ is infinite-dimensional, write $\mathcal{E} = \cup \mathcal{E}_\alpha$ where $\mathcal{E}_\alpha \subset \mathcal{E}$ is a finite-dimensional subspace. Note that $S \otimes P_{\mathcal{E}_\alpha}$ and $(I \otimes P_{\mathcal{E}_\alpha})\theta(z_2)(I \otimes P_{\mathcal{E}_\alpha})$ is a pair of commuting contractions.
- By Ando's inequality, there is a completely contractive homomorphism $\theta_\alpha : \mathrm{A}(\mathbb{D}^2) \to B(H^2(\mathbb{D}) \otimes \mathcal{E}_\alpha)$ such that $\theta_\alpha(z_1) = S \otimes P_{\mathcal{E}_\alpha}$ and $\theta_\alpha(z_2) = (I \otimes P_{\mathcal{E}_\alpha})\theta(z_2)(I \otimes P_{\mathcal{E}_\alpha})$.
- We have that $(\theta_\alpha(a))$ converges to $\theta(a)$ in SOT* for every $a \in \mathrm{A}(\mathbb{D}^2)$.

Thank you!