Infinitesimal Freeness and Non-commutative Functions

Jamie Mingo (Queen's University, Canada)

(*joint work with Pei-Lun Tseng, Queen's*)

2TART, August 18, 2020

infinitesimal distributions, a summary from:

Biane, Goodman, Nica (2003); Belinschi, Shlyakhtenko (2009); Fevrier, Nica (2009), Fevrier (2013), Shlyakhtenko (2018) [spike models], M (2019)

- let $\{X_n\}_{n=1}^{\infty}$ be identically distributed symmetric Bernoulli random variables with mean 0 and variance 1
- let $S_N = \frac{1}{\sqrt{N}}(X_1 + \cdots + X_N)$
- $\{S_N\}_N$ converges to S, a Gaussian random variable with mean 0 and variance 1
- let μ_N be the distribution of S_N and μ the distribution of S
- $N(\mu_N - \mu)$ converges to a distribution with moments $m'_{2n-1} = 0$ and $m'_{2n} = (2(n-2)-1)!!\binom{2n}{4}$
- we say the pair (μ, μ') is an *infinitesimal distribution*
- *infinitesimal probability space* $= (\mathcal{A}, \varphi, \varphi')$ where $\mathcal{A}$ is a unital algebra over $\mathbf{C}$, $\varphi, \varphi' : \mathcal{A} \overset{\text{linear}}{\mapsto} \mathbf{C}$, $\varphi(1) = 1$, $\varphi'(1) = 0$
- if (μ, μ') is an infinitesimal distribution let $\mathcal{A} = \mathbf{C}[x]$, $\varphi(x^n) = \int t^n \, d\mu(t)$ and $\varphi'(x^n) = \int t^n \, d\mu'(t)$

free cumulants (Speicher)

$$\kappa_1(X_1) = \mathrm{E}(X_1), \qquad G(z) = \mathrm{E}((z - X)^{-1})$$

$$\kappa_2(X_1, X_2) = \mathrm{E}(X_1 X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) \qquad G^{\langle -1 \rangle}(x) = \frac{1}{z} + R(z)$$

$$\kappa_{|\,\sqcup}(X_1, X_2, X_3) = \kappa_1(X_1)\,\kappa_2(X_2, X_3) \qquad R(z) = \kappa_1 + \kappa_2 z + \cdots$$

$$\mathrm{E}(X_1\, X_2\, X_3) = \kappa_{\sqcup\sqcup} + \kappa_{|\,\sqcup} + \kappa_{\sqcup\!|\!\sqcup} + \kappa_{\sqcup\,|} + \kappa_{|\,|\,|}$$

$$\begin{aligned}\mathrm{E}(X_1\, X_2\, X_3\, X_4) = {} & \kappa_{\sqcup\sqcup\sqcup} + \kappa_{|\,\sqcup\sqcup} + \kappa_{\sqcup\!|\!\sqcup\sqcup} + \kappa_{\sqcup\sqcup\!|\!\sqcup} + \kappa_{\sqcup\sqcup\,|} \\ & + \kappa_{\sqcup\,\sqcup} + \kappa_{\sqcup\!\sqcup\!\sqcup} \\ & + \kappa_{|\,|\,\sqcup} + \kappa_{|\,\sqcup\!|\!\sqcup} + \kappa_{\sqcup\!|\,|\!\sqcup} + \kappa_{|\,\sqcup\,|} + \kappa_{\sqcup\!|\!\sqcup\,|} + \kappa_{\sqcup\,|\,|} \\ & + \kappa_{|\,|\,|\,|}\end{aligned}$$

$$\boxed{\mathrm{E}(X_1 \cdots X_n) = \sum_{\pi \in NC(n)} \kappa_\pi(X_1, \ldots, X_n)}$$

where $NC(n)$ is the set of non-crossing partitions of $[n]$ $= \{1, 2, \ldots, n\}$ $\Big[$crossing partition: $\{(1,3), (2,4,5,6)\} = \sqcup\!\sqcup\!\sqcup\sqcup$

infinitesimal cumulants

- let (μ, μ') be an infinitesimal distribution let $m_n = \int t^n \, d\mu(t)$ and $m'_n = \int t^n \, d\mu'(t)$
- if $\pi = \{V_1, \dots, V_k\}$ is a partition we let $m_\pi = \prod_{V \in \pi} m_{|V|}$
- $m_n = \sum_{\pi \in NC(n)} \kappa_\pi$ (*and* $\kappa_n = \sum_{\pi \in NC(n)} \text{Möb}(\pi, 1_n) m_\pi$)
- $\partial m_n = \partial m_{1_n} = m'_n$
- if $\pi = \{V_1, \dots, V_k\}$ is a partition we extend by the derivation rule
$$\partial m_\pi = \sum_{V \in \pi} m'_{|V|} \prod_{W \neq V} m_{|W|}$$
- define $\kappa'_n = \sum_{\pi \in NC(n)} \text{Möb}(\pi, 1_n) \partial m_\pi$
- or equivalently $m'_n = \sum_{\pi \in NC(n)} \partial \kappa_\pi$
- e.g. $m_3 = \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3$
and $m'_3 = \kappa'_3 + 3\kappa'_1\kappa_2 + 3\kappa_1\kappa'_2 + 3\kappa'_1\kappa_1^2$

freeness & the utility of free cumulants (*Nica-Speicher*)

- $(\mathcal{A}, \varphi)$ is a unital algebra and a linear functional $\varphi : \mathcal{A} \to \mathbf{C}$ with $\varphi(1) = 1$
- $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ are *free* with respect to φ if whenever $a_1, \ldots, a_n \in \mathcal{A}$
  (*i*) $\varphi(a_i) = 0$ for $1 \leqslant i \leqslant n$,
  (*ii*) $a_i \in \mathcal{A}_{j_i}$ and $j_1 \neq j_2 \neq \cdots \neq j_n$,
  then $\varphi(a_1 \cdots a_n) = 0$
- random variables are free if the unital subalgebras they generate are free
- random variables $x_1, \ldots, x_s$ are *free* iff *mixed cumulants vanish*: $\forall n$, $\kappa_n(x_{i_1}, \ldots, x_{i_n}) = 0$ unless $i_1 = \cdots = i_n$
- if x_1 and x_2 are free then $\kappa_n^{(x_1+x_2)} = \kappa_n^{(x_1)} + \kappa_n^{(x_2)}$ for all n
- 'most' distributions are determined by their cumulants

heuristic concepts of freeness

- if x_1 and x_2 are free and we know the distributions of x_1 and x_2 then there is a universal rule for determining the joint distribution of $\{x_1, x_2\}$, i.e. all mixed moments

$$\left\{ \varphi(x_{i_1} x_{i_2} \cdots x_{i_n}) \mid \text{ for all } i_1, \ldots i_n \in \{1, 2\} \right\}$$

- we do the same for *infinitesimal freeness*

  if x_1 and x_2 are infinitesimally free and we know the infinitesimal distributions of x_1 and x_2 then there is a universal rule for determining the joint infinitesimal distribution of $\{x_1, x_2\}$, i.e. all mixed moments: regular and infinitesimal

$$\left\{ \varphi(x_{i_1} x_{i_2} \cdots x_{i_n}), \varphi'(x_{i_1} x_{i_2} \cdots x_{i_n}) \mid \text{ for all } i_1, \ldots i_n \in \{1, 2\} \right\}$$

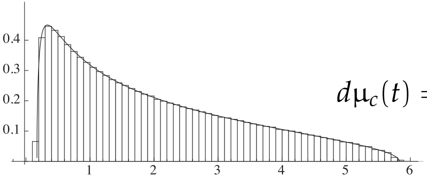
- if $\varphi(x_{i_j}) = 0$ for $j = 1, \ldots, n$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ then $\varphi'(x_{i_1} x_{i_2} \cdots x_{i_n}) = 0$ for n even and for $n = 2m + 1$ odd we have

$$\varphi'(x_{i_1} \cdots x_{i_n}) = \sum_{j=1}^{n} \varphi(x_{i_1} \cdots \varphi'(x_j) \cdots x_{i_n}).$$

Wishart random matrices and the Marchenko-Pastur Law

- $X_{N,M} = \frac{1}{N} G_{N,M} G^*_{N,M}$, $G = (g_{ij})$ an $N \times M$ matrix of independent complex $N(0,1)$ random variables, $W =$ *Wishart matrix* with shape parameter (M, N)

- the eigenvalue distribution of X (which is random) converges to the non-random distribution μ_c (where $M/N \to c$, $a = (1 - \sqrt{c})^2$, and $b = (1 + \sqrt{c})^2$)



$$d\mu_c(t) = (1 - c)\delta_0 + \frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$$

for μ_c, $\boxed{\kappa_n = c}$ for all n

infinitesimal complex Marchenko-Pastur law

M-Nica 2004

- $X_1, \ldots, X_s$ independent complex Wishart matrices, limit eigenvalue distribution of X_i has parameter c: $\kappa_\pi = c^{\#(\pi)}$
- asymptotically, the X_i's have the limit distribution of a free family: $\ker(i) \in \mathcal{P}(n)$ is the partition in which $r \sim_{\ker(i)} s$ iff $i(r) = i(s)$

$$\lim_N \mathrm{E}(\mathrm{tr}(X_i^n)) = \sum_{\pi \in NC(n)} c^{\#(\pi)}, \qquad \lim_N \mathrm{E}(\mathrm{tr}(X_{i_1} \cdots X_{i_n})) = \sum_{\substack{\pi \in NC(n) \\ \pi \leqslant \ker(i)}} c^{\#(\pi)}$$

- *suppose* $c' = \lim_N N\left(\frac{M}{N} - c\right)$ exists then

$$\lim_N N\left(\mathrm{E}(\mathrm{tr}(X_{i_1} \cdots X_{i_n})) - \sum_{\substack{\pi \in NC(n) \\ \pi \leqslant \ker(i)}} c^{\#(\pi)}\right) = \sum_{\substack{\pi \in NC(n) \\ \pi \leqslant \ker(i)}} \#(\pi) c^{\#(\pi)-1} c'$$

- asymptotically, the X_i's have the joint distribution of an infinitesimally free family with $\kappa_n' = c'$ for all n

Functional Equations, [M, 2019]

- $(\mathcal{A}, \varphi, \varphi')$ infinitesimal probability space
- $\mathcal{B} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbf{C} \right\}$, 2 dimensional algebra over $\mathbf{C}$
- $\tilde{\varphi} : \mathcal{A} \mapsto \mathcal{B}$, $\tilde{\varphi}(x) = \begin{pmatrix} \varphi(x) & \varphi'(x) \\ 0 & \varphi(x) \end{pmatrix}$
- $M_n = \begin{pmatrix} m_n & m_n' \\ 0 & m_n \end{pmatrix}$, $K_n = \begin{pmatrix} \kappa_n & \kappa_n' \\ 0 & \kappa_n \end{pmatrix}$, $Z = \begin{pmatrix} z & w \\ 0 & z \end{pmatrix}$
- $R(Z) = K_1 + K_2 Z + K_3 Z^2 + \cdots = \begin{pmatrix} R(z) & wR'(z) + r(z) \\ 0 & R(z) \end{pmatrix}$
- $G(Z) = \dfrac{1}{Z} + \dfrac{m_1}{Z^2} + \dfrac{m_2}{Z^3} + \cdots = \begin{pmatrix} G(z) & wG'(z) + g(z) \\ 0 & G(z) \end{pmatrix}$
- $G(Z)^{-1} + R(G(Z)) = Z = G\Big(Z^{-1} + R(Z)\Big)$
- $r(z) = \kappa_1' + \kappa_2' z + \kappa_3' z^2 + \cdots = -g(G^{\langle -1 \rangle}(z))(G^{\langle -1 \rangle})'(z)$
- $g(z) = \dfrac{m_1'}{z^2} + \dfrac{m_2'}{z^3} + \cdots = -r(G(z))G'(z)$

examples of infinitesimal transforms, [M, 2019]

Recall:

- $r(z) = \kappa_1' + \kappa_2' z + \kappa_3' z^2 + \cdots$, infinitesimal R-transform
- $g(z) = \dfrac{m_1'}{z^2} + \dfrac{m_2'}{z^3} + \cdots$, infinitesimal Cauchy transform

Infinitesimal GOE: $\kappa_2 = 1$, $\kappa_n = 0$ otherwise, and $\kappa_n' = 1$ for n even and 0 otherwise

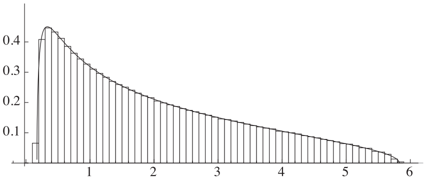
- $r(z) = \dfrac{z}{1-z^2}, \qquad g(z) = \dfrac{z - \sqrt{z^2-4}}{2} \cdot \dfrac{1}{z^2-4}.$

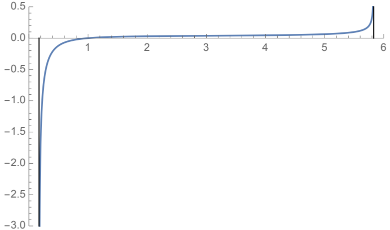
Infinitesimal complex Marchenko-Pastur: $\kappa_n = c$, $\kappa_n' = c'$ $\forall n$

- $R(z) = \dfrac{cz}{1-z}$ and $r(z) = \dfrac{c'z}{1-z}$, $P(z) = \sqrt{(z-a)(z-b)}$
- $G(z) = \dfrac{z+1-c+P(z)}{z}$

$$g(z) = \frac{-c'}{zP(z)} \frac{(1-c)^2 - (1+c)z - (1-c)P(z)}{z-1+c+P(z)}$$

infinitesimal density of complex Marchenko-Pastur





$$d\mu(x) = \begin{cases} (1-c)\delta_0 + \dfrac{\sqrt{(b-x)(x-a)}}{2\pi x} & 0 < c \leqslant 1 \\ \dfrac{\sqrt{(b-x)(x-a)}}{2\pi x} & 1 < c \end{cases}$$

$$d\mu'(x) = -c' \begin{cases} \delta_0 - \dfrac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\,dx & c < 1 \\ \frac{1}{2}\delta_0 - \dfrac{1}{2\pi\sqrt{x(4-x)}}\,dx & c = 1 \\ \quad - \dfrac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\,dx & c > 1 \end{cases}$$

Higher Order Infinitesimals (Fevrier 2013)

- Suppose we have moment sequences $\{m_n\}_{n\geqslant 0}$ (assume $m_0 = 1$), $\{m'_n\}_{n\geqslant 1}, \ldots, \{m_n^{(p-1)}\}_{n\geqslant 1}$ we make a $p \times p$ matrix, letting $\partial m_n^{(k)} = m_n^{(k+1)}$ we set

$$M_n = \begin{bmatrix} m_n & \frac{m_n^{(1)}}{1!} & \cdots & \frac{m_n^{(p-1)}}{(p-1)!} \\ 0 & m_n & \cdots & \frac{m_n^{(p-2)}}{(p-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} = \begin{bmatrix} m_n & \frac{\partial^1 m_n}{1!} & \cdots & \frac{\partial^{m-1} m_n}{(m-1)!} \\ 0 & m_n & \cdots & \frac{\partial^{m-2} m_n}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}.$$

and

$$K_n = \begin{bmatrix} \frac{\kappa_n^{(0)}}{0!} & \frac{\kappa_n^{(1)}}{1!} & \cdots & \frac{\kappa_n^{(m-1)}}{(m-1)!} \\ 0 & \frac{\kappa_n^{(0)}}{0!} & \cdots & \frac{\kappa_n^{(m-2)}}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\kappa_n^{(0)}}{0!} \end{bmatrix} = \begin{bmatrix} \kappa_n & \frac{\partial^1 \kappa_n}{1!} & \cdots & \frac{\partial^{m-1} \kappa_n}{(m-1)!} \\ 0 & \kappa_n & \cdots & \frac{\partial^{m-2} \kappa_n}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n \end{bmatrix}.$$

higher order functional equations (*new work*)

For $\pi = \{V_1, \ldots, V_k\}$:

$$K_\pi = K_{|V_1|} \cdots K_{|V_k|} = \begin{bmatrix} \frac{\kappa_\pi^{(0)}}{0!} & \frac{\kappa_\pi^{(1)}}{1!} & \cdots & \frac{\kappa_\pi^{(m-1)}}{(m-1)!} \\ 0 & \frac{\kappa_\pi^{(0)}}{0!} & \cdots & \frac{\kappa_\pi^{(m-2)}}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\kappa_\pi^{(0)}}{0!} \end{bmatrix}$$

Let $Z = \begin{bmatrix} z_0 & z_1 & \cdots & z_{p-1} \\ 0 & z_0 & \cdots & z_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_0 \end{bmatrix}$ and $G(Z) = \sum_{n=0}^{\infty} M_n Z^{-(n+1)}$

- when $p = 2$ we have

$$G(Z) = \begin{bmatrix} G_0(z_0) & G_0'(z_0)z_1 + G_1(z_0) \\ 0 & G_0(z_0) \end{bmatrix} \text{ and } R(Z) = \begin{bmatrix} R_0(z_0) & R_0'(z_0)z_1 + R_1(z_0) \\ 0 & R_0(z_0) \end{bmatrix}$$

where $G_0(z_0) = \sum_{n=0}^{\infty} m_n z_0^{-(n+1)}$, $R_0(z_0) = \sum_{n=1}^{\infty} \kappa_n z_0^{n-1}$ and

$G_1(z_0) = \sum_{n=1}^{\infty} m_n' z_0^{-(n+1)}$, $R_1(z_0) = \sum_{n=1}^{\infty} \kappa_n' z_0^{n-1}$

$G(Z)^{-1} + R(G(Z)) = Z \Rightarrow G_1(z) + R_1(G_0(z))G_0'(z) = 0$ and $\cdots$

$$\cdots \quad R_1(z) + G_1(G_0^{\langle -1\rangle}(z))\,(G_0^{\langle -1\rangle})'(z) = 0$$

---

when $p = 3$?

$$Z = \begin{bmatrix} z_0 & z_1 & z_2 \\ 0 & z_0 & z_1 \\ 0 & 0 & z_0 \end{bmatrix}, M_n = \begin{bmatrix} m_n & m_n' & \frac{m_n''}{2} \\ 0 & m_n & m_n' \\ 0 & 0 & m_n \end{bmatrix} \; K_n = \begin{bmatrix} \kappa_n & \kappa_n' & \frac{\kappa_n''}{2} \\ 0 & \kappa_n & \kappa_n' \\ 0 & 0 & \kappa_n \end{bmatrix}$$

$$Z^{-n} = \begin{bmatrix} z_0^{-n} & -nz_0^{-(n+1)}z_1 & -nz_0^{-(n+1)}z_2 + \binom{n+1}{2} z_0^{-(n+2)} z_1^2 \\ 0 & z_0^{-n} & -nz_0^{-(n+1)}z_1 \\ 0 & 0 & z_0^{-n} \end{bmatrix}$$

$$G(Z) = \sum_{n=0}^{\infty} M_n Z^{-(n+1)}, \; G_i(z) = \sum_{n=0}^{\infty} \frac{m_n^{(i)}}{z^{n+1}} \text{ for } i = 0,1,2 \; (m_0' = m_0'' = 0)$$

$$G(Z) = \begin{bmatrix} G_0(z_0) & G_0'(z_0)z_1 + G_1(z_0) & G_0'(z_0)z_2 + G_1'(z_0)z_1 \\ & & +\frac{1}{2}(G_0''(z_0)z_1^2 + G_2(z_0)) \\ 0 & G_0(z_0) & G_0'(z_0)z_1 \\ 0 & 0 & G_0(z_0) \end{bmatrix}$$

$$R(Z) = \sum_{n=1}^{\infty} K_n Z^{n-1},\ R_i(z) = \sum_{n=0}^{\infty} \kappa_n^{(i)} z^{n-1} \text{ for } i = 0, 1, 2\ (\kappa_0' = \kappa_0'' = 0)$$

$$R(Z) = \begin{bmatrix} R_0(z_0) & R_0'(z_0)z_1 + R_1(z_0) & \begin{array}{c} R_0'(z_0)z_2 + R_1'(z_0)z_1 \\ +\frac{1}{2}(R_0''(z_0)z_1^2 + R_2(z_0)) \end{array} \\ 0 & R_0(z_0) & R_0'(z_0)z_1 \\ 0 & 0 & R_0(z_0) \end{bmatrix}$$

we have $G(Z)^{-1} + R(G(Z)) = Z$ implies $G_1(z) + R_1(G_0(z))G_0'(z) = 0$; is there a relationship between R_2 and G_2?

THM:

$$G_2(z) + R_2(G_0(z))G_0'(z) = \frac{d}{dz}\Big(R_2(G_0(z))^2 G_0(z)\Big).$$