Random Interpolating Sequences for Dirichlet spaces

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Outline of the talk

- The scale of weighted Dirichlet spaces
- Interpolating sequences and more
- A probabilistic setting
- Some characterizations

Joint work with A. Hartmann, K. Kellay and B. Wick.

Weighted Dirichlet spaces

In this talk we are interested in the scale of weighted Dirichlet spaces. For $s \leq 1$, $\mathcal{D}_s$ is the space of $f \in \mathcal{O}(\mathbb{D})$ (holomorphic functions in the unit disc) such that the norm

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-s} dA(z)$$

is finite. For $f \in \mathcal{O}(\mathbb{D})$ with Taylor coefficients $\{a_n\}$, the quantity

$$\|f\|_s^2 := \sum_{n \geq 0} (n+1)^s |a_n|^2$$

defines an equivalent norm. For $s = 1, 0, -1$ we have the Dirichlet, Hardy and Bergman spaces respectively. For some equivalent inner product the kernel is given by

$$k^s(z,w) = \begin{cases} \frac{1}{(1-z\overline{w})^{1-s}}, & \text{s<1} \\ \log \frac{e}{1-z\overline{w}}, & \text{s=1.} \end{cases}$$

Interpolating Sequences

Let $\mathcal{Z} := \{z_i\} \subseteq \mathbb{D}$ a sequence, consider the associated restriction operator

$$T_{\mathcal{Z}} : \mathcal{D}_s \dashrightarrow \ell^2$$
$$f \mapsto \left\{ \frac{f(z_i)}{\|k_{z_i}\|} \right\}$$

The dashed arrow means that a priori $T_{\mathcal{Z}}$ is not defined everywhere.

- If $T_{\mathcal{Z}}$ is surjective we say that it is *simply interpolating* (SI) (also onto interpolating exists in the literature).
- If $T_{\mathcal{Z}}$ is surjective *and* bounded, $\mathcal{Z}$ is called *universally interpolating* (UI).

Interpolation in RKHS

Some remarks on the definitions

1. The boundedness of $T_{\mathcal{Z}}$ is equivalent to say that the measure

$$d\mu_{\mathcal{Z}} := \sum_{z\in\mathcal{Z}} \frac{\delta_z}{\|k_z\|^2}$$

is *Carleson* for $\mathcal{D}_s$, i.e. $\mathcal{D}_s \subseteq L^2(\mathbb{D}, d\mu_{\mathcal{Z}})$.

2. If $\mathcal{Z}$ is simply interpolating, there exists (closed graph theorem) a bounded right inverse $S_{\mathcal{Z}} : \ell^2 \to \mathcal{D}_s$, hence for any $\{a_n\}$ there exists $f \in \mathcal{D}_s$ such that

$$f(z_i) = a_i\|k_{z_i}\|, \;\; \|f\| \lesssim \|\{a_i\}\|_{\ell^2}.$$

Interpolation in RKHS

3. Applying the above remark to the sequences $e_j := \{\delta_{i,j}\}_i$, a necessary condition for simple interpolation is that there exist $\{f_i\} \subseteq \mathcal{D}_s$ such that $f_i(z_j) = \delta_{ij}\|k_{z_i}\|, \|f_i\| \lesssim 1$. Or in the language of base theory, the system $g_i := k_{z_i}/\|k_{z_i}\|$ is *uniformly minimal*. We then say that $\{z_i\}$ is *strongly separated*.
4. A weaker condition is that there exist $f_{ij} \in \mathcal{D}_s$ such that $f_{ij}(z_i) = 1, f_{ij}(z_j) = 0, \|f_{ij}\|\|k_{z_i}\| \lesssim 1$. We shall say that the sequence is *weakly separated*.This condition has a *geometric interpretation*.
5. Finally we say that a sequence is a *zero sequence* for $\mathcal{D}_s$ if there exists $f \in \mathcal{D}_s$ which vanishes exactly on z_i. This is equivalent

Weak Separation

The weak separation condition is different depending on s. For $s < 1$ the weak separation condition is equivalent to separation with respect to the hyperbolic metric

$$\exists \varepsilon > 0, d_h(z_i, z_j) > \varepsilon, \forall i \neq j.$$

Where d_h is the hyperbolic metric.

While for $s = 1$ the condition is quite different. It is equivalent to the following geometric condition. There exists $\varepsilon > 0$ such that

$$d_h(z_i, z_j) > \varepsilon(d_h(z_i, 0) + 1)$$

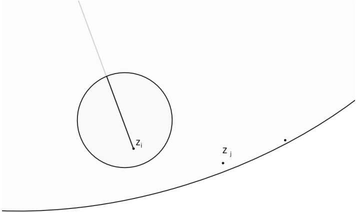


Figure 1: Weak Separation In the Dirichlet Space

Some examples

The (SI) and (UI) sequences were first studied in the Hardy space H^2, and its multiplier algebra H^∞ by Carleson (1958) and Shapiro & Shields (1961). The results collectively can be summarized as

$$(UI) \iff (SI) \iff (WS) \ \& \ d\mu_{\mathcal{Z}} \text{ is a Carleson measure.}$$

For the weighted Dirichlet spaces Bishop and Marshall & Sundberg proved that

$$(UI) \iff (WS) \ \& \ d\mu_{\mathcal{Z}} \text{ is a Carleson measure.}$$

Although surprisingly enough

$$(SI) \not\Longrightarrow (UI)$$

when $s = 1$. For $0 < s < 1$ this is still an open problem.

Random Interpolation

In some way random sequences give us a sense of which situations are "generic". One possible way to consider random sequences are the so called Steinhaus sequences. Let Θ_n a sequence of independent uniformly distributed random variables in $[0, 2\pi]$ and a (deterministic) sequence of radii $\{r_n\} \subseteq (0, 1]$. Then the sequence of random variables

$$\Lambda_n = r_n e^{i\Theta_n}$$

is called *Steinhaus sequence.* Notice that being interpolating (in any sense) is a *tail* event. Therefore Kolmogorov 0-1 theorem applies.

Definition

Suppose X_i are independent random variables. If F is an event in the $\sigma-$algebra generated by X_i it is called tail if it is independent of any finite subset of the r.v. X_i.

Hence there exists a condition on r_n which determines whether Λ_n is interpolating with probability 0 or 1. Same applies for weak separation, and the Carleson condition on $d\mu_{\mathcal{Z}}$.

Random Interpolation in H^2

Let $N_n := \#\{r_k : 1 - 2^{-n} \le r_k < 1 - 2^{-n-1}\}$

Theorem

- *(Cochran 1990) Λ is a.s. hyperbolically separated (resp. not separated) iff $\sum_{n\ge 1} 2^{-n} N_n^2 < +\infty$ (resp. $= +\infty$).*
- *(Rudowicz, 1994) If $\sum_{n\ge 1} 2^{-n} N_n^2 < +\infty$ then $d\mu_\Lambda$ is a.s. Carleson for H^2.*

Steinhaus interpolating sequences are characterized this way in view of the resutls of Carleson, Shapiro & Shields.

Theorem (C. Hartmann, Kellay, Wick, 2019)

If $\sum_{n\ge 1} 2^{-n} N_n^\beta < +\infty$, for $\beta > 1$ then $d\mu_\Lambda$ is Carleson a.s. The Blaschke condition for $\beta = 1$ is not sufficient.

Random Sequences in $D_s, 0 < s < 1/2$

$D_{1/2}$ is a strange point in the scale of spaces D_s where the theory bifurcates.

Theorem (CHKW, 2019)

Let $0 < s < 1/2$, then

$$\mathbb{P}(\Lambda \text{ is UI for } D_s) = \begin{cases} 1, \\ 0 \end{cases} \quad \textit{iff} \quad \begin{cases} \sum_{n\geq 1} 2^{-n} N_n^2 < \infty \\ \sum_{n\geq 1} 2^{-n} N_n^2 = \infty. \end{cases}.$$

Same is true if universal interpolation is replaced by simple interpolation (and of course just separation by Cochran's theorem).

Random Interpolation in $D_s, 1/2 \leq s < 1$

Theorem (CHKW)

Let $1/2 \leq s < 1$, *then*

$$\mathbb{P}(\Lambda \textit{ is UI for } D_s) = \begin{cases} 1, \\ 0 \end{cases} \quad \textit{if} \quad \begin{cases} \sum_{n\geq 1}(1-|r_n|)^{1+s} < \infty \\ \sum_{n\geq 1}(1-|r_n|)^{1+s} = \infty. \end{cases}.$$

The same is true if universal interpolation is replaced by simple interpolation.
The condition appearing above is Carleson's sufficient condition for a sequence to be zero sequence for D_s.

Theorem (CHKW)

In $D_s, 1/2 \leq s < 1$ *a Steinhaus sequence is a.s. universally interpolating if and only if it is a.s. a zero sequence, i.e., there exists* $0 \neq f \in D_s$, *such that* $f(z_i) = 0, \forall i$.

Random Interpolation in the Dirichlet space, $s = 1$

Theorem (CHKW)

$$\mathbb{P}(\Lambda \textit{ is UI for } D_1) = \begin{cases} 1, \\ 0 \end{cases} \quad \textit{if} \quad \begin{cases} \sum_{n\geq 1}\left(\log\frac{1}{1-r_n}\right)^{-1} < \infty \\ \sum_{n\geq 1}\left(\log\frac{1}{1-r_n}\right)^{-1} = \infty. \end{cases}.$$

The same is true if universal interpolation is replaced by simple interpolation or zero sequence.

This result looks superficially the same as the result for $D_s, 1/2 \leq s < 1$ but there exists a fundamental difference. Weak separation is quite different in the Dirichlet space. While in $D_s, s < 1$ weak separation is equivalent to hyperbolic separation in the Dirichlet space a sequence $\mathcal{Z}$ is separated iff there exists $\varepsilon > 0$ such that

$$d(z_i, z_j) \geq \varepsilon(d(z_i, 0) + 1), \ \ \forall i \neq j.$$

Random Interpolation in the Dirichlet space, $s = 1$

Theorem (CHKW)

The Kolmogorov 0-1 law for weak separation in the Dirichlet space for a Steinhaus sequence generated by $\{r_n\}$ is given by the condition

$$\#\{r_n : d(r_n, 0) \leq N\} = \mathcal{O}(2^{\gamma N})$$

for some $\gamma < 1/2$.

An instance from the proofs

If we concentrate in the Dirichlet space, starting from a Steinhaus sequence which satisfies the Shapiro-Shields condition, i.e.

$$\sum_{n\geq 1}\Big(\log\frac{1}{1-r_n}\Big)^{-1}<+\infty,$$

we would like to show that it is a.s. weakly separated and Carleson. For the first one the argument is quite geometric and not so simple. For the Carleson condition, there exists the Stegenga characterization of Carleson measures. If $\mu \geq 0$ is a positive measure, the it is Carleson for the Dirichlet space iff $\exists K>0$ such that

$$\sum_{i=1}^{n}\mu(S(I_i))\leq Kc\Big(\bigcup_{i=1}^{n} I_i\Big),$$

An instance from the proofs

where $c(\cdot)$ is the logarithmic capacity, $S(I)$ is the Carleson box associated to an arc and I_i are disjoint closed arcs in $\mathbb{T}$.
The general approach is to consider the Whitney decomposition of the unit disc is a family of "rectangles" $\{\rho_\alpha\}_\alpha$, and the apply the Borel-Cantelli lemma for the random variables

$$X_\alpha := \mu_{\mathcal{Z}}(S(\rho_\alpha))/c(I_\alpha))$$

(This step is a bit more intricate than it looks like, and we employ the so called exponential moments method, a "trick" from probability theory.)
Once we have the one box condition (i.e. Stegenga condition for $n = 1$) we are done because of the radial symmetry of Steinhaus sequences and the symmetrization inequalities of capacity.

An instance from the proofs

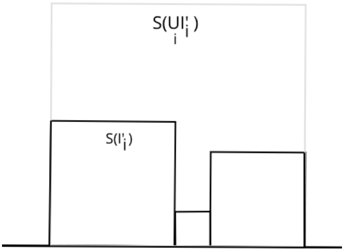
Because

$$\sum_{j=1}^{n} \mu_{\mathcal{Z}}(S_{I_j}) \sim_{distr.} \sum_{j=1}^{n} \mu_{\mathcal{Z}}(S_{I'_j})$$

where $|I_j| = |I'_j|$ and $I_{j'}$ are adjacent. Moreover,

$$\sum_{j=1}^{n} \mu_{\mathcal{Z}}(S_{I'_j}) \leq \mu_{\mathcal{Z}}\Big(S_{\cup I'_j}\Big) \leq_{a.s.} Kc\Big(\bigcup I'_j\Big) \leq c(\bigcup I_j)$$

the last inequality by symmetrization.



Thank you for your attention!