Factoring multi-Toeplitz operators

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Hardy space H^2 in the disk $|z| < 1$:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad \sum |a_n|^2 < \infty$$

f has boundary values $f(e^{i\theta})$ a.e. θ
$|f|^2 \in L^1(\mathbb{T})$

Theorem (Szegő 1922)

Let $\phi \in L^1$, $\phi \geq 0$. TFAE:

- $\phi = |f|^2$ *for some* $f \in H^2$
- $\int \log \phi \, dm > -\infty$

Distance formula and the "soft" Szegő condition:

Theorem (Szegő 1922)

Let $\phi \in L^1$, $\phi \geq 0$. TFAE:

- $\int \log\phi \, dm > -\infty$
- $\inf_{q(0)=0} \int |1-q|^2 \phi \, dm > 0$

In fact

$$\inf(\cdots) = \exp\left(\int \log \phi \, dm\right)$$

Remark: $\inf(\cdots) > 0$ is equivalent to

$$p \to p(0) \text{ is bounded for the norm } \langle p, p\rangle_\phi = \int p\overline{p}\phi \, dm$$

The Toeplitz operator with symbol $\phi \in L^\infty$:

$\langle T_\phi g, h\rangle := \int g\overline{h}\phi dm$

Then $\phi = |f|^2$ is equivalent to

$$T_\phi = M_f^* M_f \quad \text{(and } f(z) \text{ is bounded in } |z| < 1)$$

Abstract formulation (Rosenblum/Rovnyak 1971): $S =$ multiplication by z on H^2 (the unilateral shift)

Fact (Halmos): T is a Toeplitz operator if and only if

$$S^*TS = T \quad \text{(say } T \text{ is } S\text{-Toeplitz)}$$

Fact: an operator A is equal to M_f if and only if

$$AS = SA \quad \text{(say } A \text{ is } S\text{-analytic)}$$

Abstract Szegő problem: when does a bounded, ≥ 0 Toeplitz opeartor T factor as

$$T = A^*A$$

for some S-analytic A?

Noncommutative:
We replace the shift S with isometries with orthogonal ranges:

$$L = (L_1, L_2, \ldots, L_d), \quad L_i^* L_j = \delta_{ij} I$$

Construction: form a Hilbert space $\mathbb{H}^2$ with o.n. basis indexed by all (nc) words in d letters, of all lengths ≥ 0:

For example when $d = 2$:

$$\xi_\varnothing, \quad \xi_1, \quad \xi_2, \quad \xi_{11}, \quad \xi_{12}, \quad \xi_{21}, \quad \xi_{22}, \quad \xi_{111}, \quad \xi_{112}, \ldots$$

$$L_1 \xi_w = \xi_{1w}, \quad L_2 \xi_w = \xi_{2w}$$

(or on the right:)

$$R_1 \xi_w = \xi_{w1}, \quad R_2 \xi_w = \xi_{w2}$$

Definitions (Popescu 1995+):

Say T is L-Toeplitz if

$$L_i^* T L_j = \begin{cases} T & i = j \\ 0 & i \neq j \end{cases}$$

Say A is L-analytic if

$$AL_i = L_i A, \quad i = 1, \dots, d$$

nc Szegő problem: if $T \geq 0$, L-Toeplitz, when can we factor

$$T = A^* A$$

for some L-analytic A?

Popescu 1995: $T = A^*A$ iff (...Lowdenslager-like condition, as in Rosenblum-Rovnyak...)(...but often difficult to check...)

Popescu 2006: Let $T \geq 0$ be L-Toeplitz. TFAE:

- (Factorable minorant) $T \geq G^*G$ for some L-analytic $G \neq 0$
- (soft Szegő condition) $inf_{q(0)=0} \langle T(1-q), (1-q) \rangle > 0$

soft Szegő is equivalent to:

$$p \to p(0) \text{ is a bpe for the } T\text{-inner product}$$

nc point evaluations:

$Z = (Z_1, \ldots Z_d)$ - $n \times n$ matrices, any n

$\|Z\|^2 := \|Z_1 Z_1^* + \cdots Z_d Z_d^*\|$

For $f \in \mathbb{H}^2$:

$$f = \sum c_w \xi_w, \quad \sum |c_w|^2 < \infty$$

The "point evaluations"

$$f \to f(Z) = \sum c_w Z^w \in M_{n \times n}$$

are bounded for the $\mathbb{H}^2$ norm when $\|Z\| < 1$ (Popescu).

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3) *(0 is a bpe) $p \to p(0)$ is bounded for the T-inner product*
4) *(every $\|Z\| < 1$ is a bpe) $p \to p(Z)$ is bounded for the T-inner product, for all $\|Z\| < 1$*

What the proof looks like when $d = 1$: the reproducing kernel picture

Lemma (kernel picture)

T factors as F^*F if and only if $\langle T\cdot, \cdot\rangle$ is a reproducing kernel Hilbert space over $|z| < 1$, in which case

$$k^T(z, w) = \frac{f(z)f(w)^*}{1 - zw^*}$$

and $f = 1/F$.

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In nc setting, use Ball-Marx-Vinnikov nc reproducing kernel machinery the key step:

1) perturb: $T + \epsilon I = F_\epsilon^* F_\epsilon$

2) what happens when $\epsilon \to 0^+$? two possibilities:
   - if $T = F^* F$ then $F_\epsilon(z) \to F(z)$ in $|z| < 1$
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Lemma (continuity lemma)

For each fixed $|z| < 1$:

$$\lim_{\epsilon \to 0} |F_\epsilon(z)| > 0$$

if and only if

$$p \to p(z) \textit{ is a bpe for } \langle T\cdot, \cdot\rangle$$

So:

$$
\begin{aligned}
0 \text{ is a bpe} \quad &\iff \lim |F_\epsilon(0)| > 0 \text{ (continuity lemma)} \\
&\iff \lim |F_\epsilon(z)| > 0 \text{ for all } z \text{ (Hurwitz theorem)} \\
&\iff \text{every } z \text{ is a bpe (continuity lemma again)}
\end{aligned}
$$