The graph isomorphism game

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OTWIA
August 13, 2020

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Outline

Non-local games: a primer

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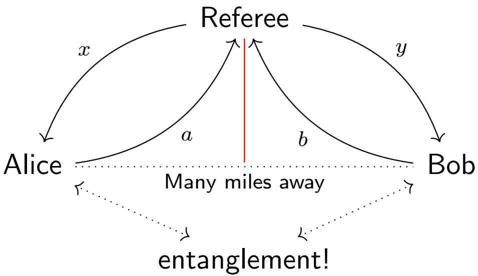

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Recall that Alice and Bob *cannot communicate* once the game begins. (But they can share an entanglement resource space!)

Probabilistic models

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- We consider when $p(a,b|x,y) = \langle E_{a,x}F_{b,y}\psi, \psi\rangle$, where ψ is a unit vector in a Hilbert space $\mathcal{H}$, $\{E_{a,x}\}_{a=1}^{m}$ and $\{F_{b,y}\}_{b=1}^{m}$ are projection-valued measures for each x, y, and $[E_{a,x}, F_{b,y}] = 0$ for all a, b, x, y.

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- We could model (a subset of) these probability distributions using a finite-dimensional tensor product model (giving $C_q(m,k)$), a tensor product model (giving $C_{qs}(m,k)$), an approximate tensor product model (giving $C_{qa}(m,k)$), a commuting model (giving $C_{qc}(m,k)$), or a local model (giving $C_{loc}(m,k)$).

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- By a lot of theorems, we know that

$$C_{loc}(m,k) \subseteq C_q(m,k) \subseteq C_{qs}(m,k) \subseteq C_{qa}(m,k) \subseteq C_{qc}(m,k),$$

and *all* of these containments are (in general) strict.

The synchronicity condition

- Key property: $\text{Iso}(G, H)$ is a **synchronous game**: if Alice and Bob receive inputs x, y and $x = y$, then Alice and Bob's outputs a and b *must* be equal. That is, if they win with probability 1, then

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- For $t \in \{loc, q, qs, qa, qc\}$, one defines

$$C_t^s(m, k) = \{(p(a, b|x, y)) \in C_t(m, k) : p \text{ is synchronous}\}.$$

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- Surprisingly, $C_q^s(m, k) = C_{qs}^s(m, k)$ (Kim-Paulsen-Schafhauser, '18), but all the other analogous containments are (in general) strict.

Synchronous realizations

Theorem (Paulsen, Severini, Stahlke, Todorov, Winter, '16)

$(p(a,b|x,y)) \in C_{qc}^s(m,k) \iff$ *there are projection-valued measures* $\{E_{a,x}\}_{a=1}^k$ *for each* x *in a tracial* C^**-algebra* $(\mathcal{A},\tau)$*, such that*

$$p(a,b|x,y) = \tau(E_{a,x}E_{b,y}).$$

Moreover,

- $(p(a,b|x,y)) \in C_{loc}^s(m,k) \iff \mathcal{A}$ *can be taken to be abelian;*
- $(p(a,b|x,y)) \in C_{q}^s(m,k) \iff \mathcal{A}$ *can be taken to be finite-dimensional;*
- *(Kim-Paulsen-Schafhauser, '18)* $(p(a,b|x,y)) \in C_{qa}^s(m,k) \iff \mathcal{A}$ *can be taken to be* $\mathcal{R}^\omega$ *(ultrapower of hyperfinite* II_1*-factor).*

Quantum permutations

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- $p = (p(a,b|x,y)) = \tau(E_{a,x}E_{b,y}) \in C_t^s(|V(G) \sqcup V(H)|, |V(G) \sqcup V(H)|)$ wins the game $\text{Iso}(G,H)$ with probability 1 if and only if $U = (E_{g,h})_{g \in V(G),\, h \in V(H)}$ is a quantum permutation such that

$$(A_G \otimes 1)U = U(A_H \otimes 1),$$

where A_G (resp. A_H) is the adjacency matrix of G (resp. H).

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- All of these cases can be thought of as representations of a universal $*$-algebra for $\mathrm{Iso}(G,H)$.

Algebra of the Game

Definition (Kim-Paulsen-Schafhauser, '18)

We define $\mathcal{A}(\mathrm{Iso}(G,H))$ to be the free (unital) $*$-algebra generated by entries $e_{g,h}$ ($g \in V(G)$, $h \in V(H)$) such that

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- We say that $G \simeq_{alg} H$ if $\mathcal{A}(\mathrm{Iso}(G,H))$ exists.

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- Eg. There's a graph *homomorphism* game, with an associated $*$-algebra $\mathcal{A}(\mathrm{Hom}(G,H))$. But $\mathcal{A}(\mathrm{Hom}(K_5,K_4)) \neq 0$ (Helton-Meyer-Paulsen-Satriano '19), so there's an (algebraic) 4-coloring of K_5!

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- It's not known if there are any graphs G, H with $G \simeq_{qc} H$ but $G \simeq_{qa} H$. (It's possible that Connes' embedding has a negative answer *and* qa/qc isomorphisms are the same for all G, H!)
- What about $\simeq_{alg} \implies \simeq_{qc}$? (What could go wrong?)
- Eg. There's a graph *homomorphism* game, with an associated $*$-algebra $\mathcal{A}(\mathrm{Hom}(G,H))$. But $\mathcal{A}(\mathrm{Hom}(K_5,K_4)) \neq 0$ (Helton-Meyer-Paulsen-Satriano '19), so there's an (algebraic) 4-coloring of K_5!
- What's the problem? In a C^*-algebra, if $p_1, \ldots, p_n$ are self-adjoint idempotents and $\sum_{i=1}^n p_i = 1$, then $p_i p_j = 0$ if $i \neq j$. In a unital $*$-algebra, this only works if $n \leq 3$, and is *false* if $n \geq 4$. (In $\mathcal{A}(\mathrm{Hom}(K_5,K_4))$, there are $p_1, \ldots, p_4$ with $p_i^2 = p_i = p_i^*$ and $\sum p_i = -1$.)

$\simeq_{alg} \implies \simeq_{qc}$

Theorem (Chirvasitu-Brannan-Eifler-H.-Paulsen-Su-Wasilewski, '19)

If G and H are graphs with $G \simeq_{alg} H$, then $G \simeq_{qc} H$.

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Theorem (Chirvasitu-Brannan-Eifler-H.-Paulsen-Su-Wasilewski, '19)

If G and H are graphs with $G \simeq_{alg} H$, then $G \simeq_{qc} H$.

- This also means that if $G \simeq_{C^*} H$ (i.e. there's a representation $\mathcal{A}(\mathrm{Iso}(G, H)) \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$), then $G \simeq_{qc} H$.

Related: Lovasz's theorem on homomorphism counts

Theorem (Lovasz, 1967)

If G and H are graphs, then $G \simeq H$ if and only if $|\mathit{Hom}(Z, G)| = |\mathit{Hom}(Z, H)|$ for all graphs Z.

Related: Lovasz's theorem on homomorphism counts

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Let G and H be graphs. Then $G \simeq_{qc} H$ if and only if $|Hom(Z, G)| = |Hom(Z, H)|$ for all planar graphs Z.

Thus, $G \simeq_{alg} H \iff G \simeq_{qc} H \iff$
$|\text{Hom}(Z, G)| = |\text{Hom}(Z, H)|$ for all planar graphs Z.