Norm Preserving Extensions

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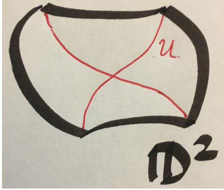
Prehistory - late 20th century

- Solve a Pick problem on bidisk $\mathbb{D}^2$ (find function with smallest H^∞ norm satisfying finitely many interpolation conditions)
- Either solution is unique, or there exists one dimensional variety $\mathcal{U}$ on which all solutions coincide
- All solutions satisfy $\|\phi\|_{\mathcal{U}} = \|\phi\|_{\mathbb{D}^2}$.



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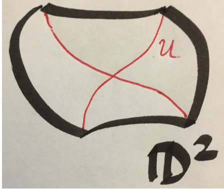

Does this say $\mathcal{U}$ is special, or $\phi|_{\mathcal{U}}$ is special?
Does every function in $H^\infty(\mathcal{U})$ extend to a function in $H^\infty(\mathbb{D}^2)$ of same norm?

Ω pseudo-convex domain in $\mathbb{C}^d$, $\quad O(\Omega) :=$ holomorphic functions on Ω
V analytic subset of Ω
(locally defined as common zero set of functions in $O(\Omega)$)

Def: $f : V \to \mathbb{C}$ is holomorphic if $\forall \lambda \in V,\ \exists \varepsilon > 0$ and $h \in O(\mathbb{B}(\lambda, \varepsilon))$ with $h|_{V \cap \mathbb{B}(\lambda,\varepsilon)} = f|_{V \cap \mathbb{B}(\lambda,\varepsilon)}$

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A1: Yes always - H. Cartan, 1950

Isometric extension property

Q2: Which $V \subseteq \mathbb{D}^2$ have isometric extension property (IEP):
$\forall\, f \in H^\infty(V)\ \exists\, \phi \in H^\infty(\mathbb{D}^2)$, norm-preserving extension

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Example

$\Omega = \mathbb{D}^2,\ V = \{z \in \mathbb{D}^2 : z_1 z_2 = 0\}$
$f(z_1, 0) = z_1,\ f(0, z_2) = z_2$.
$\|D\phi(0)\| = \|(1,1)\| = \sqrt{2}$. Contradicts Schwarz's Lemma.

Singularities bad

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In general, answer to Q2 not known. But for nice sets (eg algebraic sets)

Thm. [Agler-M 2003]

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If V is retract, $\phi := f \circ r$ gives norm-preserving extension.

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Thm. [Heath-Suffridge 1981]

All retracts of $\mathbb{D}^d$ are graphs
$\{(z, \Psi(z)) : z \in \mathbb{D}^m, \Psi : \mathbb{D}^m \to \mathbb{D}^{d-m} \text{ holomorphic}\}$

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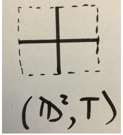
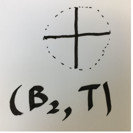
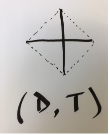
No if

- Ω is symmetrized bidisk (not convex) (Agler-Lykova-Young 17)
- $D = \{|z_1| + |z_2| < 1\}$ (convex, not strictly convex)







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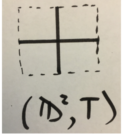
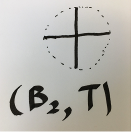
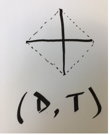


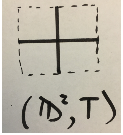


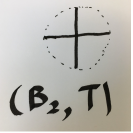


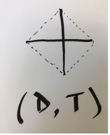
Schwarz lemma for balanced set Ω

Suppose $\phi : \Omega \to \mathbb{D}$ and $\phi(0) = 0$.
Then $D\phi(0) : \Omega \to \mathbb{D}$

Ω is balanced if $\lambda \in \Omega \Rightarrow z\lambda \in \Omega \; \forall \; z \in \overline{\mathbb{D}}$







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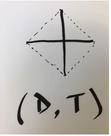
Schwarz Lemma goes from enemy to friend

$f(z_1, 0) = z_1, \;\; f(0, z_2) = z_2 \qquad\qquad \Rightarrow D\phi(0) = Df(0) = (1, 1)$
$(1, 1)$ does not map $\mathbb{D}^2$ or B_2 to $\mathbb{D}$, but does map D to $\mathbb{D}$



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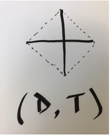
Thm: (D, T) has IEP



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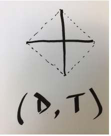
Let $g \in H^\infty(T)$, $\|g\| \leq 1$ and suppose $g(0) = 0$.

$$E(g) = \phi(z_1, z_2) := g(z_1, 0) + g(0, z_2)$$

Win by Schwarz!

$$|g(z_1, 0) + g(0, z_2)| \leq |g(z_1, 0)| + |g(0, z_2)| \leq |z_1| + |z_2| < 1$$



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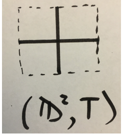
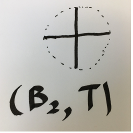
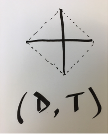
If $f(0) = a$, use $m_a \circ E(m_a \circ f)$, where $m_a(z) = \frac{a-z}{1-\bar{a}z}$.







- What is this terrifying anomaly?

Shift Perspective

Prob A: Given Ω, find all V s.t. (Ω, V) has IEP

Prob B: Given V find all Ω s.t. (Ω, V) has IEP

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Prob A: Given Ω, find all V s.t. (Ω, V) has IEP
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What conditions must V satisfy for $\{\Omega : (\Omega, V) \text{ has IEP}\}$ non-empty?

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Most sets (eg T, the two crossed disks) are not retracts of anything.

Absent some form of convexity, retracts seem to have little to do with Isometric Extension Property

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Thm 2 [Agler-Kosinski-M]

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Let Ω be pseudoconvex domain in $\mathbb{C}^2$ with $T \subset \Omega$.
Then (Ω, T) has IEP iff T is relatively closed in Ω and $\exists$ pseudoconvex set G in $\mathbb{C}^2$ and a function $\tau \mapsto C_\tau$ from $\mathbb{T}^2$ into $\mathrm{Hol}(G)$ so that

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Says just need to be able to extend each $\tau \cdot \lambda = \tau_1 \lambda_1 + \tau_2 \lambda_2$

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Example

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$$\Omega := \{z \in \mathbb{D}^2 : (|z_1| + |z_2|)|1 + z_1 z_2| < 1\}$$

(Ω, T) has IEP, $\Omega \not\subset D$ and $D \not\subset \Omega$.

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No maximal domain (without balanced)

Question (Rudin, 1969)

If (Ω, V) has bounded extension property (every $f \in H^\infty(V)$ extends to $H^\infty(\Omega)$, but with perhaps larger norm), is there a bounded linear operator?

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Thm 4 [Agler-Kosinski-M]

There is no isometric linear extension operator from $H^\infty(T)$ to $H^\infty(D)$.

Can do it linearly with smaller domain

Thm 5 [Agler-Kosinski-M]

There is a domain Ω containing T and an isometric linear extension operator from $H^\infty(T)$ to $H^\infty(\Omega)$.

Can, using operator theory, analyze some other sets

Example

$$\mathcal{V} = \{z \in \mathbb{D}^3 : z_3^2 = z_1 z_2\}$$

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What is an isometric envelope?

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$$\mathcal{V} = \{z \in \mathbb{D}^3 : z_3^2 = z_1 z_2\}$$

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$\mathcal{G}$ is convex, and $(\mathcal{G}, \mathcal{V})$ has IEP.

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If Ω is balanced, $(\Omega, \mathcal{V})$ has IEP iff $\Omega \subseteq \mathcal{G}$.

Challenge

Prove Theorem 6 without using operator theory!

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Question 3 - Rule this out

Suppose

$$V = \{z \in \mathbb{D}^3 : z_1 + z_2 + z_3 = z_1 z_2 + z_1 z_3 + z_2 z_3\}$$

Does $(\mathbb{D}^3, V)$ have IEP?

Thank You!

What does convexity have to do with retracts?

Thm (KM): If Ω is strictly convex and 2-dimensional, then (Ω, V) has IEP iff V is retract.

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A geodesic map is a holomorphic $k : \mathbb{D} \to \Omega$ with a left inverse $c : \Omega \to \mathbb{D}$. (Also called Kobayashi extremal)

A set $G \subseteq \Omega$ is geodesically complete if, whenever k is a geodesic map and $k(\lambda_1), k(\lambda_2) \in G$, then $k(\mathbb{D}) \subseteq G$. (Or $k(\lambda_1)$ and tangent vector)

Step 1: If Ω is strictly convex and V has IEP, then V is geodesically complete.

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Step 1: If Ω is strictly convex and V has IEP, then V is geodesically complete.

Step 2: $k(\mathbb{D})$ is a retract (since $r = k \circ c$ is retraction)

If V is one dimensional, it is one geodesic. If Ω is 2-dimensional, 0 and 2 dimensional cases are trivial.