2TALK: Complete spectral sets and numerical range & Maximum determinant positive definite Toeplitz completions

Hugo J. Woerdeman

Drexel University, Philadelphia, USA

Talk is based on the following papers:

K. R. Davidson, V. I. Paulsen and H. J. Woerdeman, *Complete spectral sets and numerical range*, Proc. AMS. **146** (2018), 1189–1195.

Stefan Sremac, Henry Wolkowicz, and Hugo J. Woerdeman, *Maximum determinant positive definite Toeplitz completions*. Operator Theory: Adv. Appl. **271** (2018), 421–441.

Complete spectral sets and numerical range

Hugo J. Woerdeman

Drexel University, Philadelphia, USA

Joint work with Kenneth R. Davidson and Vern I. Paulsen

General setting

Given

- $T \in \mathbb{C}^{n \times n}$
- f analytic on spectrum $\sigma(T)$ of T (could be matrix valued)
- $|||\cdot|||$ a norm (unitary similarity invariant)

Want

- Estimate $|||f(T)|||$ based on information on f and T

Observe

- If $T = U\,\mathrm{diag}(\lambda_i)_{i=1}^n U^*$ normal and a 'diagonal' norm, then $|||f(T)||| = |||\mathrm{diag}(f(\lambda_i))_{i=1}^n||| = \max_i |f(\lambda_i)| =: \|f\|_{\sigma(T)}$.

Notation

- For $K \subset \mathbb{C}$ denote $\|f\|_K = \sup_{z \in K} \|f(z)\|$

Classical result

- [Von Neumann, '51]: If $\|T\| \leq 1$ and f analytic on closed unit disk $\overline{\mathbb{D}}$, then $\|f(T)\| \leq \|f\|_{\overline{\mathbb{D}}}$. Here $\|\cdot\|$ is spectral norm

Previous results

- $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ *numerical range* of T
- $w(T) = \sup_{z\in W(T)} |z|$ *numerical radius* of T

Theorem ([Drury, '08])

If $W(T) \subseteq \overline{\mathbb{D}}$, $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *analytic, then* $W(f(T)) \subset \mathrm{teardrop}(f(0))$.

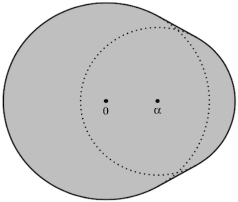

FIGURE 1. teardrop(α)

Corollary

$$w(f(T)) \leq \frac{5}{4}\|f\|_{\overline{\mathbb{D}}}$$

Previous results, continued

Theorem ([Crouzeix, '07])

There is a universal constant C so that

$$\|f(T)\| \leq C\|f\|_{W(T)}$$

This constant satisfies $2 \leq C \leq 11.08$

Theorem ([Crouzeix and Palencia, '17])

$$2 \leq C \leq 1 + \sqrt{2}$$

Conjecture ([Crouzeix, '07])

$$C = 2$$

Terminology

- Let $C \geq 1$. $K \subset \mathbb{C}$ is called a *C-spectral set* for T if
$$\|f(T)\| \leq C\|f\|_K \tag{1}$$
for all $f \in \mathcal{R}(K)$ = uniform closure of all rational functions with poles off K.
- K is called a *complete C-spectral set* for T if (1) holds for all matrix valued f with entries in $\mathcal{R}(K)$.

Theorem ([Crouzeix and Palencia, '17])

$W(T)$ is a complete $(1+\sqrt{2})$-spectral set for T

- K is called a *(complete) C'-numerical radius set* for T if
$$w(f(T)) \leq C'\|f\|_K \tag{2}$$
for all (matrix valued) f (with entries) in $\mathcal{R}(K)$

Main result

Note

- Since
$$w(A) \leq \|A\| \leq 2w(A),$$
- K is a C-spectral set $\implies$ K is a C-numerical radius set
- K is a C-numerical radius set $\implies$ K is a $2C$-spectral set
- One can insert 'complete' everywhere in the above

Theorem ([Davidson, Paulsen, W])

K is a complete C-spectral set for T

$$\Leftrightarrow$$

K is a complete $\frac{1}{2}(C + \frac{1}{C})$-numerical radius set for T

Observe

$$C' = \frac{1}{2}(C + \frac{1}{C}) \Leftrightarrow C = C' + \sqrt{C'^2 - 1}$$

Proof

Since K is complete C-spectral for T, by [Paulsen, '84] there exists invertible S with $\|S\|\|S^{-1}\| = C$ so that K is 1-spectral for STS^{-1}. We can choose S so that

$$\frac{1}{C} I \leq S^* S \leq CI.$$

Thus

$$\|f(STS^{-1})\| \leq \|f\|_K.$$

Scale f so that $\|f\|_K = 1$. Now

$$\begin{bmatrix} I & f(STS^{-1}) \\ f(STS^{-1})^* & I \end{bmatrix} \geq 0 \text{ implies } \begin{bmatrix} (S^*S)^{-1} & f(T) \\ f(T)^* & S^*S \end{bmatrix} \geq 0.$$

[Ando, '73] gives $w(f(T)) \leq \frac{1}{2}\|(S^*S)^{-1} + S^*S\| \leq \frac{1}{2}(C + \frac{1}{C})$.

Proof continued

Let f be so that $\|f\|_K = 1$ and $C = \|f(T)\|$ (or close to). Put

$$g(z) = \begin{bmatrix} \frac{1}{C} I & (1 - \frac{1}{C^2}) f(z) \\ 0 & \frac{1}{C} I \end{bmatrix}.$$

Then $\|g\|_K = \frac{1}{2}\left(1 - \frac{1}{C^2} + \sqrt{(1 - \frac{1}{C^2})^2 + 4\frac{1}{C^2}}\right) = 1$, and

$$w(g(T)) = \frac{1}{C} + \frac{1}{2}(C - \frac{1}{C}) = \frac{1}{2}(C + \frac{1}{C}).$$

Corollary ([Davidson, Paulsen, W])

For matrix valued f analytic on $\overline{\mathbb{D}}$ and $w(T) \le 1$,

$$w(f(T)) \le \frac{5}{4}\|f\|_{\overline{\mathbb{D}}}.$$

Use [Okubo and Ando, '75] showing $w(T) \le 1$ implies $\|STS^{-1}\| \le 1$ for S with $\|S\|\|S^{-1}\| = 2$. Note $\frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4}$.

Reformulation of Crouzeix's conjecture

Corollary

The following are equivalent

- $W(T)$ *is a complete 2-spectral set for* T *(Crouzeix conj).*
- $W(T)$ *is a complete* $\frac{5}{4}$*-numerical radius set for* T*.*

In the paper '*Crouzeix's Conjecture and Related Problems*' by Kelly Bickel, Pamela Gorkin, Anne Greenbaum, Thomas Ransford, Felix Schwenninger, Elias Wegert (arXiv 2006.04901) the authors study properties of extremal functions and associated vectors in the context of the Crouzeix conjecture.

Question. Does the alternative viewpoint, replacing the spectral norm by the numerical radius, provide new insight into this analysis?

Operator algebra formulation

Let $\mathcal{A}$ be a unital operator algebra and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a bounded linear map. It induces coordinatewise maps $\Phi^{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{(n)})$; and one defines the *completely bounded norm* by

$$\|\Phi\|_{cb} = \sup_{n\geq 1} \|\Phi^{(n)}\|.$$

We also define a *complete numerical radius norm* on such maps

$$\|\Phi\|_{wcb} := \sup_{n\geq 1} \sup_{A\in\mathcal{M}_n(\mathcal{A}),\ \|A\|\leq 1} w(\Phi^{(n)}(A)).$$

Our main result in this context is the following:

Theorem ([Davidson, Paulsen, W])

For Φ a unital completely bounded homomorphism

$$\|\Phi\|_{wcb} = \frac{1}{2}\left(\|\Phi\|_{cb} + \|\Phi\|_{cb}^{-1}\right).$$

C. Badea, M. Crouzeix, and H. Klaja's follow up result

Let $w_\rho(T)$ denote the ρ-operator radius introduced by Sz.-Nagy and Foias. Let $T \in \mathcal{B}(\mathcal{H})$. Define $T \in \mathcal{C}_\rho$ if and only if there exists a unitary $U \in \mathcal{B}(\mathcal{K})$ with $\mathcal{H} \subseteq \mathcal{K}$ so that $T^n = \rho P U^n P^*$, $n \in \mathbb{N}$, where P is the orthogonal projection $\mathcal{K} \to \mathcal{H}$. Define now

$$w_\rho(T) = \inf\{\lambda > 0 \ : \ \lambda^{-1} T \in \mathcal{C}_\rho\}.$$

Then $w_1(T) = \|T\|$, $w_2(T) = w(T)$, $\lim_{\rho\to\infty} w_\rho(T) = r(T)$.

Theorem

Let $K \geq 1$ and $\rho \geq 1$, and set

$$\tilde{K} = \frac{K^2 + 1 + \sqrt{(K^2+1)^2 - 4\rho(2-\rho)K^2}}{2\rho K}.$$

For every unital completely bounded homomorphism Φ

$$\|\Phi\|_{cb} = K \Leftrightarrow \|\Phi\|_{w_\rho cb} = \tilde{K}.$$

Their proof is different, and addresses the non-complete case.

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"And now for something completely different."
Monty Python

Maximum determinant positive definite Toeplitz completions

Hugo J. Woerdeman

Drexel University, Philadelphia, USA

Joint work with Stefan Sremac and Henry Wolkowicz

Toeplitz matrix completions

Given *Toeplitz real symmetric partial matrix*, e.g.

$$\mathcal{T} = \begin{bmatrix} 8 & 4 & ? & 1 \\ 4 & 8 & 4 & ? \\ ? & 4 & 8 & 4 \\ 1 & ? & 4 & 8 \end{bmatrix}.$$

We are interested in *positive definite completions*, e.g.

$$T = \begin{bmatrix} 8 & 4 & 0 & 1 \\ 4 & 8 & 4 & 1 \\ 0 & 4 & 8 & 4 \\ 1 & 1 & 4 & 8 \end{bmatrix}, T^\star = \begin{bmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 1 & 2 & 4 & 8 \end{bmatrix} = \begin{bmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{bmatrix}^{-1}.$$

Among all positive definite completions there is a unique one with *maximum determinant*; it is the unique one with zeroes in the inverse in positions corresponding to unknowns [Grone, Johnson, Sa, Wolkowicz, 1984].

Question: For what patterns is this unique completion Toeplitz?

Previous results

A *pattern* $P \subseteq \{1, \ldots, n-1\}$ indicates the prescribed diagonals in the lower triangular part. E.g., $n = 4$,

$$P = \{1,3\} \Leftrightarrow \begin{bmatrix} * & * & ? & * \\ * & * & * & ? \\ ? & * & * & * \\ * & ? & * & * \end{bmatrix}, \quad P = \{1\} \Leftrightarrow \begin{bmatrix} * & * & ? & ? \\ * & * & * & ? \\ ? & * & * & * \\ ? & ? & * & * \end{bmatrix}.$$

- [Dym & Gohberg, 1981] Banded Toeplitz partial positive definite matrices have a Toeplitz maximum determinant completion. Here $P = \{1, 2, \ldots, r\}$.
- [Naevdal, 1997] If the unknown diagonal is one but last one and a positive semidefinite completion exists, then a Toeplitz one as well. Here $P = \{1, 2, \ldots, n-3, n-1\}$.
- [Ming & Ng, 2005] **Conjecture**: in the cycle case if a positive semidefinite completion exists, then a Toeplitz one as well. Here $P = \{k, n-k\}$.

Trench '64, Gohberg-Semencul '72

if $T = (c_{i-j})_{i,j=0}^{n-1} > 0$ then

$$T^{-1} = \begin{bmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & p_{n-1} & \cdots & p_1 \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 0 \end{bmatrix}.$$

Also, $p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1}$ has no roots in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ [Szegö, 1920]

Observation

If $P = kP'$, then the partial Toeplitz matrix with pattern P is permutation similar to a k block diagonal of (almost) the same partial Toeplitz matrices with pattern P'.

E.g., $n = 7, P = \{2, 4\}$, then

$$\begin{bmatrix} a & ? & b & ? & c & ? & ? \\ ? & a & ? & b & ? & c & ? \\ b & ? & a & ? & b & ? & c \\ ? & b & ? & a & ? & b & ? \\ c & ? & b & ? & a & ? & b \\ ? & c & ? & b & ? & a & ? \\ ? & ? & c & ? & b & ? & a \end{bmatrix} \sim \begin{bmatrix} a & b & c & ? & ? & ? & ? \\ b & a & b & c & ? & ? & ? \\ c & b & a & b & ? & ? & ? \\ ? & c & b & a & ? & ? & ? \\ ? & ? & ? & ? & a & b & c \\ ? & ? & ? & ? & b & a & b \\ ? & ? & ? & ? & c & b & a \end{bmatrix}.$$

For the maximum determinant the unknowns outside the block diagonal are all 0.

Main result

For a positive definite completable partial Toeplitz matrix $\mathcal{T}$, we denote the unique maximum determinant positive definite completion by $T^\star$.

Theorem ([Sremac, Wolkowicz, W])

Let $\emptyset \neq P \subseteq \{1, \ldots, n-1\}$ denote a pattern. The following are equivalent.

1. *For every positive definite completable partial Toeplitz matrix $\mathcal{T}$ with pattern P, the matrix $T^\star$ is Toeplitz.*
2. *There exist $r, k \in \mathbb{N}$ such that P has one of the three forms:*
   - $P_1 := \{k, 2k, \ldots, rk\}$,
   - $P_2 := \{k, 2k, \ldots, (r-2)k, rk\}$, *where* $n = (r+1)k$,
   - $P_3 := \{k, n-k\}$.

Crucial characteristic of patterns

$$T^{\star -1} = \begin{bmatrix} p_0 & & & \\ p_1 & p_0 & & \\ \vdots & \vdots & \ddots & \\ p_{n-1} & p_{n-2} & \cdots & p_0 \end{bmatrix} \begin{bmatrix} \overline{p_0} & \overline{p_1} & \cdots & \overline{p_{n-1}} \\ & \overline{p_0} & \cdots & \overline{p_{n-2}} \\ & & \ddots & \vdots \\ & & & \overline{p_0} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & & & \\ \overline{p_{n-1}} & 0 & & \\ \vdots & \ddots & \ddots & \\ \overline{p_1} & \cdots & \overline{p_{n-1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & p_{n-1} & \cdots & p_1 \\ & 0 & \ddots & \vdots \\ & & \ddots & p_{n-1} \\ & & & 0 \end{bmatrix}.$$

can have the correct zero structure $\Leftrightarrow$

P is one of the three types of patterns.

Cycle case

- Let $\mathcal{O} \subset \mathbb{R}_{++} \times \mathbb{R}^2$ consist of all triples (t_0, t_k, t_{n-k}) so that the partial Toeplitz matrix with pattern $\{k, n-k\}$ and data $\{t_0, t_k, t_{n-k}\}$ is positive definite completable. Then $\mathcal{O}$ is an open convex set, and thus connected.
- Let $\mathcal{U} \subseteq \mathcal{O}$ consist of those triples (t_0, t_k, t_{n-k}) for which the corresponding maximum determinant completion is Toeplitz. Clearly $\mathcal{U} \neq \emptyset$ as $(t_0, 0, 0) \in \mathcal{U}$ for all $t_0 > 0$.
- $\mathcal{U}$ is closed in $\mathcal{O}$, as the Toeplitz condition is closed under taking limits.
- To show that $\mathcal{U}$ is also open, we introduce the set,

  $$\mathcal{P} := \{(p, q, r) : p + qz^k + rz^{n-k} \text{ has all roots satisfying } |z| > 1\}.$$

  There is a bijective continuous map between $\mathcal{U}$ and $\mathcal{P}$.
- $\mathcal{U}$ is nonempty, open and closed in $\mathcal{O}$, and thus $\mathcal{U} = \mathcal{O}$

Consequence

Theorem ([Sremac, Wolkowicz, W])

Consider the patterns

- $P_1 := \{k, 2k, \ldots, rk\}$,
- $P_2' := \{k, 2k, \ldots, (r-2)k, rk\}$,
- $P_3' := \{k, r\}$ *where* $n \geq k + r$.

If $\mathcal{T}$ *is an* $n \times n$ *positive semidefinite completable partial Toeplitz matrix with a pattern in the set* $\{P_1, P_2', P_3'\}$, *then* $\mathcal{T}$ *has a Toeplitz positive semidefinite completion.*

Idea of proof. Adjusting the size n reduces it to the previous theorem. Applying the previous theorem leads now to a banded pattern. Now use [Dym & Gohberg, 1981].

Note. This proves the conjecture of [Ming & Ng, 2005].

Question. Are these all patterns where a positive semidefinite completion implies the existence of a Toeplitz one?

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THANK YOU!