An Operator Theorist does Combinatorics: Numerical Semigroups and Positivity

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Operator Theory with its Applications

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Abstract

How did an operator theorist get involved in combinatorics? How do the two fields interact? Using tools from complex, harmonic, and functional analysis, probability theory, algebraic combinatorics, and computer-aided design, we answer virtually all asymptotic questions about factorization lengths in numerical semigroups. This yields uncannily accurate predictions that agree with numerical computations. We also present positivity results for certain multivariate polynomials, potential applications to AF algebras, and generalizations via several complex variables.

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Combinatorics:
What's up with That?

Combinatorics: how did it come to this?

My background

Function-related operator theory (shift operators, Toeplitz operators, Hardy spaces, etc.), matrix analysis, and so forth.

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Fruitful interplay

We mix techniques and problems from different areas. This leads to new problems inspired by work in other areas.

Numerical Semigroups

Numerical semigroups

Definition (Numerical semigroup)

Let $n_1, n_2, \ldots, n_k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ with $k \geqslant 3$ and

$$0 < n_1 < n_2 < \cdots < n_k \quad \text{and} \quad \gcd(n_1, n_2, \ldots, n_k) = 1.$$

Then

$$S = \langle n_1, n_2, \ldots, n_k \rangle = \{a_1 n_1 + \cdots + a_k n_k : a_i \in \mathbb{N}\}$$

is the *numerical semigroup* with *generators* $n_1, n_2, \ldots, n_k$.

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McNugget Monoid

$\langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, 24, 26, 27, 29, 30, 32, \ldots\}$.

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Theorem (Frobenius Coin Problem)

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$. Then $n \in S$ for sufficiently large n.

Factorizations

Definition (Factorization)

A *factorization* of $n \in S = \langle n_1, n_2, \ldots, n_k \rangle$ is an expression

$$n = a_1 n_1 + a_2 n_2 + \cdots + a_k n_k,$$

in which $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$. The *length* of $\mathbf{a}$ is

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Example

The number 42 has exactly three factorizations in $S = \langle 6, 9, 20 \rangle$:

$$\begin{aligned} 42 &= 1 \cdot 6 + 4 \cdot 9 + 0 \cdot 20 & \qquad \|(1, 4, 0)\| &= 5 \\ &= 7 \cdot 6 + 0 \cdot 9 + 0 \cdot 20 & \|(7, 0, 0)\| &= 7 \\ &= 4 \cdot 6 + 2 \cdot 9 + 0 \cdot 20 & \|(4, 2, 0)\| &= 6. \end{aligned}$$

Length multisets

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Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$. Then $\mathsf{L}[\![n]\!]$ denotes the multiset (set with multiplicities taken into account) of factorizations of n.

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Example

The number $132 \in \langle 6, 9, 20 \rangle$ has many factorizations:

$$(2, 0, 6),\ (0, 8, 3),\ (3, 6, 3),\ (6, 4, 3),\ (9, 2, 3),\ (12, 0, 3),\ (1, 14, 0),\ (4, 12, 0),\ (7, 10, 0),\ (10, 8, 0),\ (13, 6, 0),\ (16, 4, 0),\ (19, 2, 0),\ (22, 0, 0)$$

Thus,

$$\mathsf{L}[\![132]\!] = \{\!\{8, 11, 12, 13, 14, 15, 15, 16, 17, 18, 19, 20, 21, 22\}\!\}.$$

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Thus,

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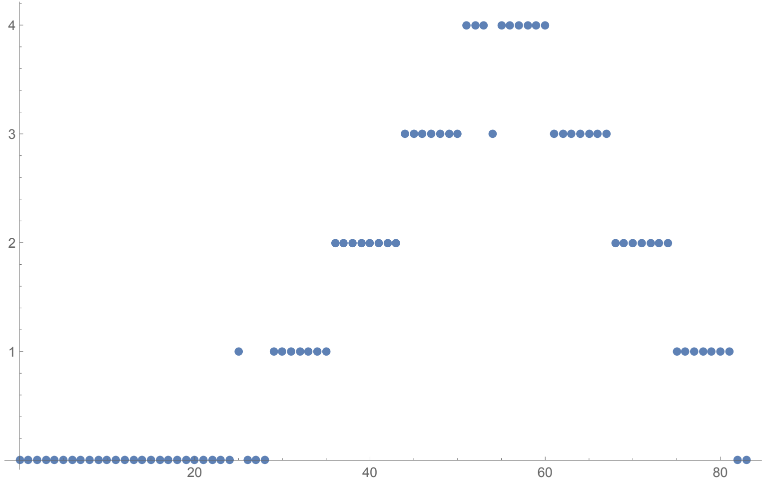
Hopelessly general question

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$. Describe the behavior of $\mathsf{L}[\![n]\!]$ as $n \to \infty$.

The numbers are not encouraging



Histogram of length multiset $\mathsf{L}[\![500]\!]$ in $\langle 6, 9, 20\rangle$.

Lengths (horizontal) versus multiplicities (vertical)

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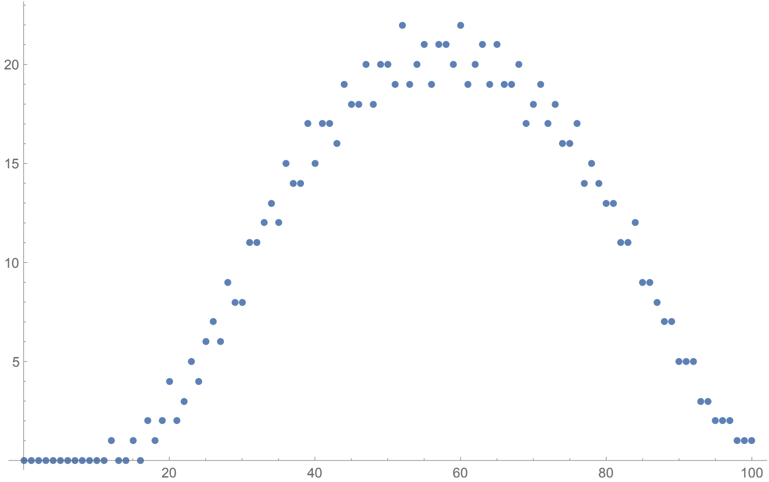

Histogram of length multiset $\mathsf{L}[\![500]\!]$ in $\langle 5, 6, 18, 45\rangle$.

Lengths (horizontal) versus multiplicities (vertical)

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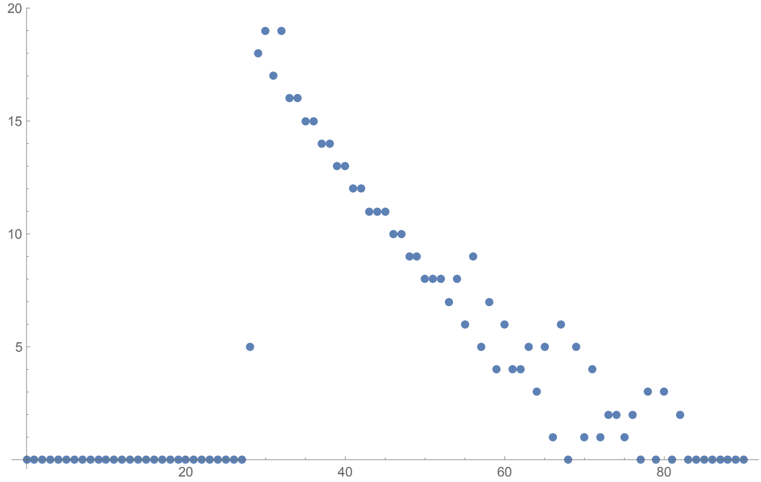

Histogram of length multiset $\mathsf{L}[\![1000]\!]$ in $\langle 11, 34, 35, 36\rangle$.

Lengths (horizontal) versus multiplicities (vertical)

Too much to ask?

Wish list

We want asymptotics for the following statistics of $\mathsf{L}[\![n]\!]$ as $n \to \infty$:

- $|\mathsf{L}[\![n]\!]|$
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- Mean
- Median
- Mode
- Variance & Std. Dev.
- pth moments
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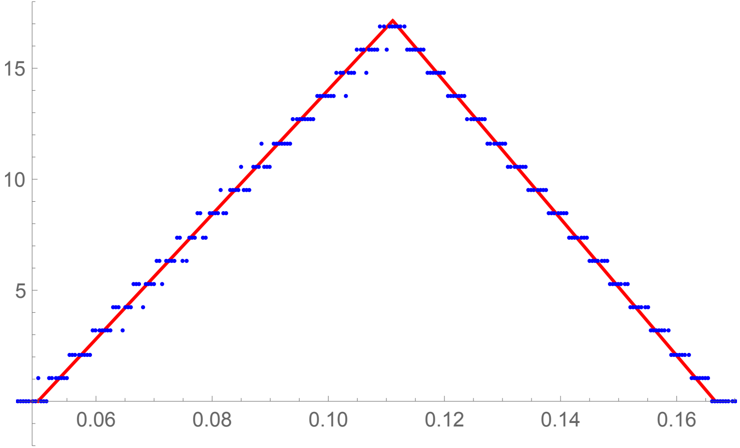
Mission Accomplished

We answer all of these questions very explicitly and do much more. It just needs a little algebraic combinatorics, complex analysis, harmonic analysis, functional analysis, and probability theory.

The McNugget triangle



$n = 2{,}000$

Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 6, 9, 20 \rangle$. For $i \in \mathbb{N}$, a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

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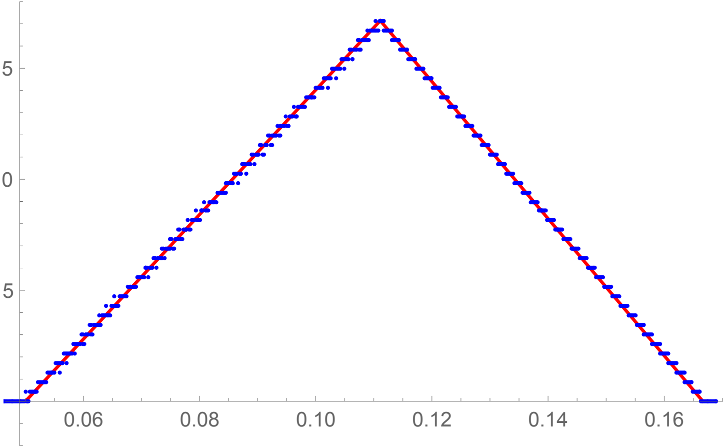

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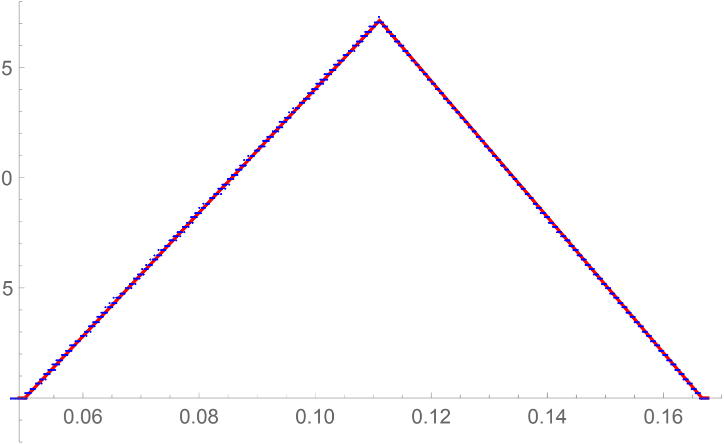

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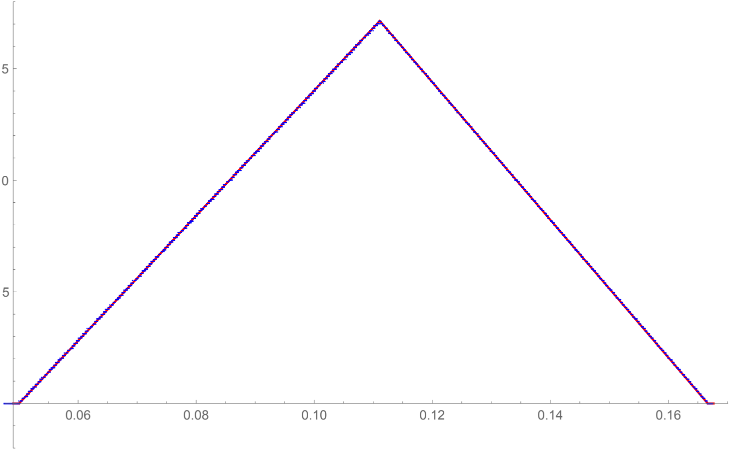

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Getting explicit

Example

For $S = \langle 11, 34, 35, 36 \rangle$, the *length distribution function* is

$$F(x) = 1413720 \begin{cases} 0 & \text{if } x \leqslant \frac{1}{36}, \\ \frac{1}{50}(36x-1)^2 & \text{if } \frac{1}{36} \leqslant x \leqslant \frac{1}{35}, \\ \frac{1}{600}\left(-15073x^2 + 886x - 13\right) & \text{if } \frac{1}{35} \leqslant x \leqslant \frac{1}{34}, \\ \frac{1}{13800}(11x-1)^2 & \text{if } \frac{1}{34} \leqslant x \leqslant \frac{1}{11}, \\ 0 & \text{if } x > \frac{1}{11}. \end{cases}$$

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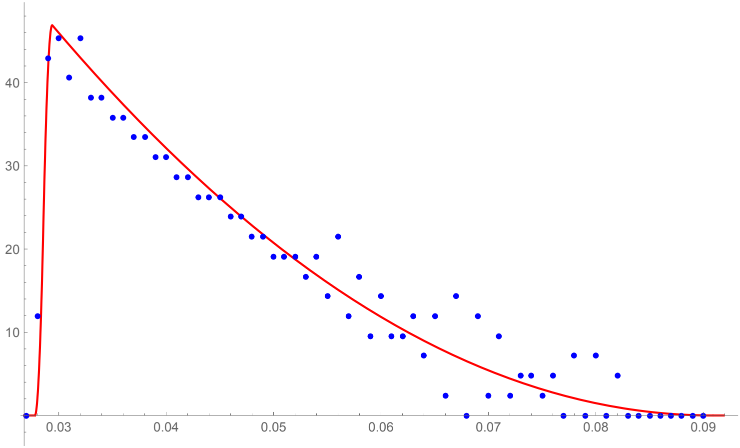
From this we can deduce things like

$$\text{Median}\,\mathrm{L}[\![n]\!] \;\sim\; \frac{1}{11}\left(1 - \sqrt[3]{\frac{115}{714}}\right) n.$$

Perfect fit



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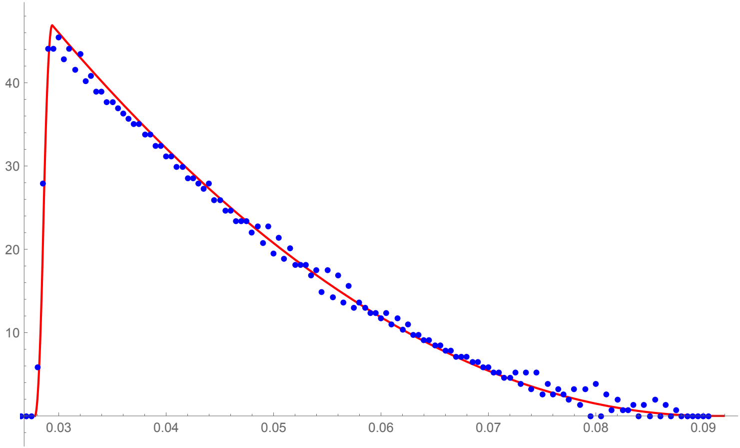

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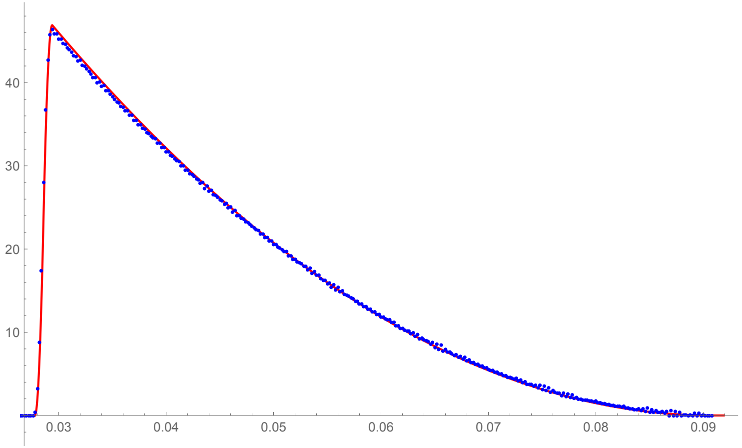

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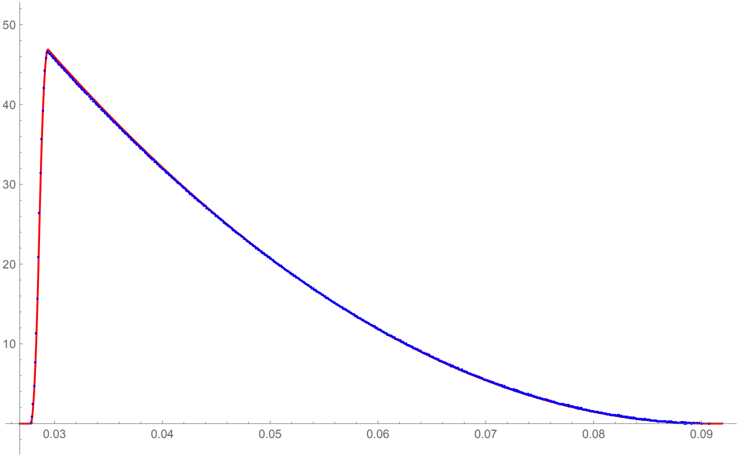

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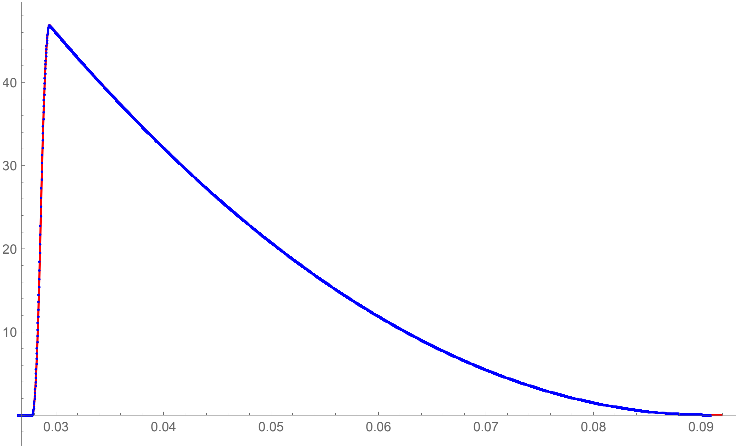

$n = 50{,}000$

Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 11, 34, 35, 36 \rangle$. For $i \in \mathbb{N}$, a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

A close call

Statistic	Actual	Predicted
Min / Max $\mathsf{L}[\![10^5]\!]$	2778/9082	2777.78/9090.91
Mean $\mathsf{L}[\![10^5]\!]$	4417.31	4416.76
Median $\mathsf{L}[\![10^5]\!]$	4145	4144.69
Mode $\mathsf{L}[\![10^5]\!]$	2939	2939.03
StDev $\mathsf{L}[\![10^5]\!]$	1207.84	1207.14
Skew $\mathsf{L}[\![10^5]\!]$	0.8594802	0.8594804
HarMean $\mathsf{L}[\![10^5]\!]$	4130.30	4130.03
GeoMean $\mathsf{L}[\![10^5]\!]$	4266.46	4266.06

Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$, the multiset of factorization lengths of 100,000, in $S = \langle 11, 34, 35, 36 \rangle$.

Quasipolynomiality and Moments

Complete homogeneous symmetric polynomials

Definition

The *complete homogeneous symmetric polynomial* of degree p in the k variables $x_1, x_2, \ldots, x_k$ is

$$h_p(x_1, x_2, \ldots, x_k) \quad = \sum_{1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_p \leqslant k} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_p},$$

the sum of all degree p monomials in $x_1, x_2, \ldots, x_k$.

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the sum of all degree p monomials in $x_1, x_2, \ldots, x_k$.

Example (CHS polynomials in two variables)

$$h_0(x_1, x_2) = 1,$$

$$h_1(x_1, x_2) = x_1 + x_2,$$

$$h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2.$$

Quasipolynomality and asymptotics

Definition

A *quasipolynomial* of degree d is a function $f : \mathbb{Z} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_1(n)n + c_0(n),$$

in which $c_1(n), c_2(n), \ldots, c_d(n)$ are periodic functions of n. A *quasirational function* is a quotient of two quasipolynomials.

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Theorem (G–Omar–O'Neill–Yih, 2019)

Let $S = \langle n_1, n_2, \ldots, n_k \rangle$. For $p \in \mathbb{N}$,

$$\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p = \frac{p!\, h_p(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k})}{(k+p-1)!(n_1 n_2 \cdots n_k)} n^{k+p-1} + w_p(n),$$

in which $w_p(n)$ is a quasipolynomial of degree at most $k+p-2$ whose coefficients have period dividing $\operatorname{lcm}(n_1, n_2, \ldots, n_k)$.

Quasirational quantities and large n asymptotics

1. **Number of Factorizations.**

$$|\mathsf{L}[\![n]\!]| \;\sim\; \frac{n^{k-1}}{(k-1)!(n_1 n_2 \cdots n_k)}.$$

2. **Moments.**

$$\frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p \;\sim\; \binom{p+k-1}{p}^{-1} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right) n^p$$

3. **Mean.**

$$\frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \ell \;\sim\; \frac{n}{k}\left(\frac{1}{n_1} + \frac{1}{n_2} + \cdots \frac{1}{n_k}\right)$$

4. **Variance.**

$$\sigma^2(n) \;\sim\; \frac{n^2}{k^2(k+1)}\left((k-1)\sum_{i=1}^{k} \frac{1}{n_i^2} - 2\sum_{i<j} \frac{1}{n_i n_j}\right).$$

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The generating function

Proof. Let $S = \langle n_1, n_2, \ldots, n_k \rangle$ with $\gcd(n_1, n_2, \ldots, n_k) = 1$. Let

$$g(z, w) := \prod_{i=1}^{k} \frac{1}{1 - wz^{n_i}}$$

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$$\begin{aligned} g(z, w) &:= \prod_{i=1}^{k} \frac{1}{1 - wz^{n_i}} \\ &= \prod_{i=1}^{k} (1 + wz^{n_i} + w^2 z^{2n_i} + \cdots) \end{aligned}$$

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Powering up

From

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A long and grueling residue computation yields

$$\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p \;=\; \frac{p!\,h_p(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k})}{(k+p-1)!(n_1 n_2 \cdots n_k)} n^{k+p-1} + w_p(n). \qquad \square$$

Beyond Vandermonde

Exponential generating function

Theorem (G–Omar–O'Neill–Yih, 2019)

Let $x_1, x_2, \ldots, x_k \in \mathbb{C}\backslash\{0\}$ *be distinct. For* $z \in \mathbb{C}$,

$$\sum_{p=0}^{\infty} \frac{h_p(x_1, x_2, \ldots, x_k)}{(p+k-1)!} z^{p+k-1} \;=\; \sum_{r=1}^{k} \frac{e^{x_r z}}{\prod_{j \neq r}(x_r - x_j)}.$$

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Vandermonde determinants

$$\det \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{bmatrix}}_{V(x_1, x_2, \ldots, x_k)} = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

Being manipulative

Proof. Observe that

$$\det V(x_1, x_2, \ldots, x_k) \sum_{r=1}^{k} \frac{e^{x_r z}}{\prod_{j \neq r}(x_r - x_j)}$$

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Since

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This follows from Jacobi's bialternant formula applied to the partition $\lambda = (p, 0, \ldots, 0)$. $\square$

A Measured Approach

From the discrete to the continuous

The usual setting

Fix a numerical semigroup $S = \langle n_1, n_2, \ldots, n_k \rangle$ with $k \geqslant 3$. For convenience, we sometimes write $x_i = 1/n_i$ for $i = 1, 2, \ldots, n$.

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The probability measures ν_n

Let δ_x denotes the point mass at x. For $n \in \mathbb{N}$, let

$$\nu_n = \frac{1}{\mathsf{L}[\![n]\!]} \sum_{\ell \in \mathsf{L}[\![n]\!]} \delta_{\frac{\ell}{n}}.$$

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Strategy

We prove that the singular probability measures ν_n converge weakly (in the topology of the dual of $C[0,1]$) to an absolutely continuous probability measure ν which governs $\mathsf{L}[\![n]\!]$. The probability density F of this limit measure is the "length distribution function."

Diagram chasing

$$
\begin{tikzcd}
\left(m_p(\nu_n)\right)_{p=0}^{\infty} \arrow[r, "\text{convergence of moments}"] \arrow[d, leftrightarrow, "\text{moment sequence}"'] & \left(m_p(\nu)\right)_{p=0}^{\infty} \arrow[d, leftrightarrow, "\text{moment sequence}"] \\
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Convergence of moments

Lemma

The pth moment

$$m_p(\nu_n) = \int_{\mathbb{R}} t^p \, d\nu_n(t)$$

of $\nu_n = \frac{1}{\mathsf{L}[\![n]\!]} \sum_{\ell \in \mathsf{L}[\![n]\!]} \delta_{\ell/n}$ *satisfies*

$$\lim_{n\to\infty} m_p(\nu_n) \;=\; \binom{p+k-1}{p}^{-1} h_p\left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\right).$$

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Proof.

We have

$$m_p(\nu_n) = \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} \left(\frac{\ell}{n}\right)^p.$$

Now use the asymptotic formulas for $\sum_{\ell \in \mathsf{L}[\![n]\!]} \ell^p$ and $|\mathsf{L}[\![n]\!]|$. □

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The sequence of characteristic functions

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converges (uniformly on compact sets in $\mathbb{C}$) to

$$\varphi(z) = \sum_{p=0}^{\infty} \binom{p+k-1}{p}^{-1} h_p(x_1, x_2, \ldots, x_k) \frac{(iz)^p}{p!}$$

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A measured approach

Lemma (Lévy's Continuity Theorem)

Let ν_n be probability measures on $[0,1]$ such that φ_{ν_n} converges pointwise on $\mathbb{R}$ to a continuous function φ. Then

1. $\varphi = \varphi_\nu$ *for some probability measure* ν *on* $[0,1]$*;*
2. $\nu_n \to \nu$ *(weak convergence of measures);*
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Lemma (Inversion theorem)

If $\int_{\mathbb{R}} |\varphi_\nu(t)|\, dt$ *is finite, then* $F := \check{\varphi_\nu}$ *is a bounded continuous function such that*

$$\nu(A) = \int_A F(t)\, dt$$

for all Borel sets $A \subseteq [0,1]$.

Fourier inversion

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&= \frac{(k-1)!}{i^{k-1}}\sum_{r=1}^{k}\frac{1}{\prod_{j\neq r}(x_r-x_j)}\widecheck{\left(\frac{e^{ix_r z}}{z^{k-1}}\right)}(x)
\end{aligned}
$$

Fourier inversion

The desired probability density function is

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&= \frac{(k-1)!}{i^{k-1}} \sum_{r=1}^{k} \frac{1}{\prod_{j\neq r}(x_r - x_j)} \left(\widecheck{\frac{e^{ix_r z}}{z^{k-1}}} \right)(x) \\
&= \frac{(k-1)!}{2(i^{k-1})^2 (k-2)!} \sum_{r=1}^{k} \frac{|x - x_r|(x - x_r)^{k-3}}{\prod_{j\neq r}(x_r - x_j)}
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&= \frac{(k-1) n_1 n_2 \cdots n_k}{2} \sum_{r=1}^{k} \frac{|1 - n_r x|(1 - n_r x)^{k-3}}{\prod_{j\neq r}(n_j - n_r)}. \qquad \square
\end{aligned}
$$

Life the Universe and Everything

The ultimate answer to the great question

Theorem (G–Omar–O'Neill–Yih, 2019)

Let $S = \langle n_1, n_2, \ldots, n_k \rangle$*, in which* $k \geqslant 3$*,* $\gcd(n_1, n_2, \ldots, n_k) = 1$*.*

1. *For real* $\alpha < \beta$*,*

$$\lim_{n \to \infty} \frac{|\{\ell \in \mathsf{L}[\![n]\!] : \ell \in [\alpha n, \beta n]\}|}{|\mathsf{L}[\![n]\!]|} = \int_\alpha^\beta F(t)\, dt,$$

where $F : \mathbb{R} \to \mathbb{R}$ *is the* C^{k-3} *probability density function*

$$F(x) := \frac{(k-1) n_1 n_2 \cdots n_k}{2} \sum_{r=1}^{k} \frac{|1 - n_r x|(1 - n_r x)^{k-3}}{\prod_{j \neq r}(n_j - n_r)}.$$

2. *For any continuous function* $g : (0,1) \to \mathbb{C}$*,*

$$\lim_{n \to \infty} \frac{1}{|\mathsf{L}[\![n]\!]|} \sum_{\ell \in \mathsf{L}[\![n]\!]} g\left(\frac{\ell}{n}\right) = \int_0^1 g(t) F(t)\, dt.$$

This stuff all works!

Example

Let $S = \langle 9, 11, 13, 15, 17 \rangle$. The length distribution function is

$$F(x) = \frac{109395}{32} \begin{cases} 0 & \text{if } x < \frac{1}{17}, \\ (17x-1)^3 & \text{if } \frac{1}{17} \leqslant x < \frac{1}{15}, \\ 3 - 129x + 1833x^2 - 8587x^3 & \text{if } \frac{1}{15} \leqslant x < \frac{1}{13}, \\ -3 + 105x - 1209x^2 + 4595x^3 & \text{if } \frac{1}{13} \leqslant x < \frac{1}{11}, \\ (1-9x)^3 & \text{if } \frac{1}{11} \leqslant x < \frac{1}{9}, \\ 0 & \text{if } \frac{1}{9} \leqslant x. \end{cases}$$

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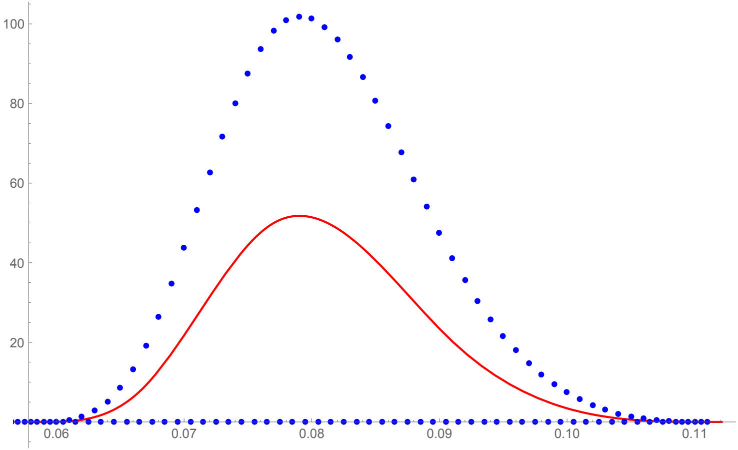
If n is even, then every element of $\mathsf{L}[\![n]\!]$ is even, and if n is odd, then every element of $\mathsf{L}[\![n]\!]$ is odd.

Another example



$n = 2{,}000$

Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 9, 11, 13, 15, 17 \rangle$. For $i \in \mathbb{N}$, a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

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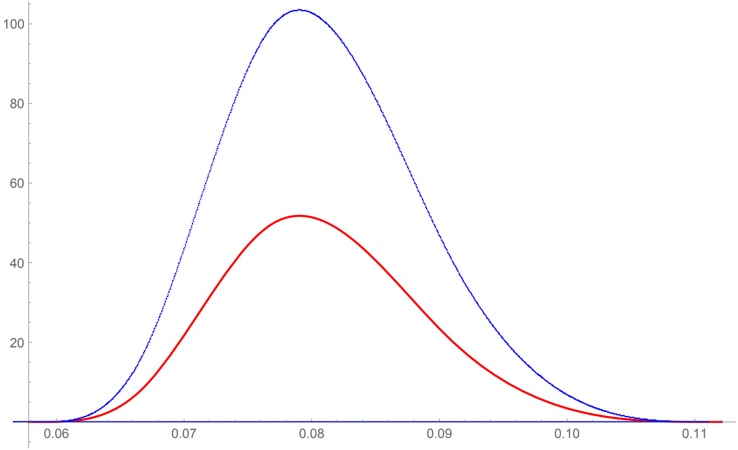

$n = 50{,}000$

Normalized histogram of the length multiset $\mathsf{L}[\![n]\!]$ (blue) and graph of the length distribution function $F(x)$ (red) for $S = \langle 9, 11, 13, 15, 17\rangle$. For $i \in \mathbb{N}$, a blue dot occurs above i/n at height equal to the multiplicity of i in $\mathsf{L}[\![n]\!]$.

On the money

Statistic	Actual	Predicted
Min /Max $\mathsf{L}[\![10^5]\!]$	11110/5884	11111.11/5882.35
Mean $\mathsf{L}[\![10^5]\!]$	8088.80	8088.67
Median $\mathsf{L}[\![10^5]\!]$	8038	8037.53
Mode $\mathsf{L}[\![10^5]\!]$	7904	7904.25
StDev $\mathsf{L}[\![10^5]\!]$	757.14	756.89
Skew $\mathsf{L}[\![10^5]\!]$	0.32812710	0.32812712
HarMean $\mathsf{L}[\![10^5]\!]$	8019.043	8018.96
GeoMean $\mathsf{L}[\![10^5]\!]$	8053.75	8053.64

Actual versus predicted statistics (rounded to two decimal places) for $\mathsf{L}[\![10^5]\!]$, the multiset of factorization lengths of 100,000, in $S = \langle 9, 11, 13, 15, 17\rangle$.

Where have I see this before?

My what a strange function!

The length distribution function

$$F(x) := \frac{(k-1)n_1 n_2 \cdots n_k}{2} \sum_{r=1}^{k} \frac{|1-n_r x|(1-n_r x)^{k-3}}{\prod_{j\neq r}(n_j - n_r)}$$

is piecewise polynomial of degree $k-2$ with nodes $\frac{1}{n_k}, \dots, \frac{1}{n_1}$ and support $[\frac{1}{n_k}, \frac{1}{n_1}]$. It is $(k-3)$-times continuously differentiable.

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A link to computer-aided design

This is a Curry–Schoenberg B-spline from computer aided design!

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Who cares?

These interpolate a finite set of values with piecewise polynomials subject to degree, smoothness, and support requirements.

Positive Thinking

Positivity of CHS polynomials

Theorem (D.B.Hunter, 1977)

For $p = 0, 1, 2, \ldots$ *and all* $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$,

$$h_{2p}(a_1, a_2, \ldots, a_n) > 0.$$

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Example

For $(x_1, x_2, x_3) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$,

$$\underbrace{x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_1 x_3}_{h_2(x_1, x_2, x_3)} \ > \ 0.$$

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Tao's Blog

"I was hunting for a probabilistic interpretation of h_d (this being another major way to prove positivity results, besides sum of squares methods and induction methods)."

CHS polynomials for nonintegral powers

Theorem (Böttcher–G–Omar–O'Neill, 2020)

Let $n \geqslant 2$, let $a_1 < a_2 < \cdots < a_n$, and let

$$F(x; a_1, a_2, \ldots, a_n) = \frac{n-1}{2} \sum_{j=1}^{n} \frac{|a_j - x|(a_j - x)^{n-3}}{\prod_{k \neq j}(a_j - a_k)}.$$

Then for suitable $z \in \mathbb{C}$, define

$$h_z(a_1, a_2, \ldots, a_n) = \binom{z+n-1}{n-1} \int_{\mathbb{R}} x^z F(x; a_1, a_2, \ldots, a_n)\, dx;$$

for $z = 0, 1, 2, \ldots$ this agrees with classical CHS polynomials.

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Freebie

Apply with $z = 2p$ and $p = 0, 1, 2, \ldots$ to obtain Hunter's positivity theorem as a trivial consequence since $F \geqslant 0$.

CHS polynomials of fractional degree

Example

For three variables,

$$h_z(a,b,c) = \frac{a^{z+2}(b-c) + b^{z+2}(c-a) + c^{z+2}(a-b)}{(a-b)(a-c)(b-c)}$$

CHS polynomials of fractional degree

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$$h_z(a,b,c) = \frac{a^{z+2}(b-c) + b^{z+2}(c-a) + c^{z+2}(a-b)}{(a-b)(a-c)(b-c)}$$

and hence

$$h_{\frac{1}{2}}(a,b,c) = \frac{a^{\frac{5}{2}}(b-c) + b^{\frac{5}{2}}(c-a) + c^{\frac{5}{2}}(a-b)}{(a-b)(a-c)(b-c)},$$

$$h_{-\frac{1}{2}}(a,b,c) = \frac{\sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{c} + \sqrt{b}\sqrt{c}}{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{c})(\sqrt{b}+\sqrt{c})},$$

$$h_{-3}(a,b,c) = (abc)^{-1},$$

$$h_{-4}(a,b,c) = \frac{ab+ac+bc}{a^2b^2c^2}.$$

New positivity results

Theorem (Böttcher–G–Omar–O'Neill, 2020)

Let

$$H(a_1, a_2, \ldots, a_n) = \sum_{j=0}^{m} c_j h_j(a_1, a_2, \ldots, a_n)$$

with real coefficients c_j *and let* $-\infty \leqslant r < s \leqslant \infty$. *Then*

$$H(a_1, \ldots, a_n) > 0 \ \textit{ for all } \ (a_1, \ldots, a_n) \in (r, s)^n \backslash \{\mathbf{0}\}$$

if and only if $H(a, a, \ldots, a) > 0$ *for all* $a \in (r, s) \backslash \{0\}$.

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if and only if $H(a, a, \ldots, a) > 0$ for all $a \in (r, s) \backslash \{0\}$.

Tip of the iceberg

More elaborate versions are available for expressions like

$$H(a_1, \ldots, a_n) = \sum_{j,k=1}^{m} c_{jk} h_j(a_1, \ldots, a_n) h_k(a_1, \ldots, a_n).$$

Future Work

A host of questions

Questions

Let $S = \langle n_1, n_2, \ldots, n_k \rangle$ be a numerical semigroup with length distribution function $F(x)$.

- Physical interpretation for the passage to the limit $n \to \infty$?

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- Semigroups in higher dimensions? Versions for PSD matrices?
- Noncommutative versions of positivity results?

Unital homomorphisms

Building blocks

The building blocks of (unital) AF C^*-algebras are (unital) embeddings of direct sums of matrix algebras.

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Simplest case

A unital homomorphism $\varphi : \mathrm{M}_{n_1} \oplus \cdots \oplus \mathrm{M}_{n_k} \to \mathrm{M}_n$ is determined up to unitary equivalence in M_n by $a_1, a_2, \ldots, a_k \in \mathbb{N}$ such that

$$a_1 n_1 + \cdots + a_k n_k = n.$$

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$$a_1 n_1 + \cdots + a_k n_k = n.$$

For large n, we understand the asymptotic properties of

$$\ell = a_1 + \cdots + a_k$$

as φ ranges over all such unital homomorphisms. Here ℓ is the total number of simple matrix algebras in the image of φ in M_n.

Potential application?

Fix $S = \langle n_1, n_2, \ldots, n_k \rangle$ and a large $C \in \mathbb{N}$ and let

$$\mathcal{U}_r = \bigoplus_{i=1}^{k} \mathsf{M}_{C^r n_i}.$$

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A unital homomorphism $\varphi_r : \mathcal{U}_r \to \mathcal{U}_{r+1}$ is determined up to unitary equivalence in $\mathcal{U}_{r+1}$ by $a_{ij} \in \mathbb{N}$ such that

$$\begin{array}{ccccccc}
a_{11}n_1 & + & \cdots & + & a_{1k}n_k & = & Cn_1 \\
a_{21}n_1 & + & \cdots & + & a_{2k}n_k & = & Cn_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{k1}n_1 & + & \cdots & + & a_{kk}n_k & = & Cn_k
\end{array}$$

AF C^*-algebras

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We know the asymptotic behavior of $\sum_{i,j} |a_{ij}| = \sum_i (\sum_j a_{ij})$. This may permit us to answer statistical questions about families of AF algebras generated from embeddings $\varphi_r : \mathcal{U}_r \to \mathcal{U}_{r+1}$.

Multivariable combinatorics

Higher-dimensional semigroups

Let $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k \in \mathbb{N}^d$ and

$$S = \langle \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k \rangle = \{a_1 \mathbf{n}_1 + \cdots + a_k \mathbf{n}_k : a_i \in \mathbb{N}\}.$$

The corresponding generating function is

$$g(z_1, z_2, \ldots, z_k, w) = \prod_{i=1}^{k} \frac{1}{1 - w z_1^{\mathbf{n}_i(1)} \cdots z_k^{\mathbf{n}_i(k)}}.$$

Singularities are no longer isolated poles, but varieties!

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Singularities are no longer isolated poles, but varieties!

New frontiers

There is a rapidly-developing theory of Analytic Combinatorics in Several Variables (ACSV) whose tools may address these problems.

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