Boundary Value Problems in Euclidean Space for Bosonic Laplacians

Chao Ding
joint work with Phuoc-Tai Nguyen and John Ryan

Masaryk University, Czech Republic

OTWIA 2020, 13/08/2020

Outline

- Introduction to the conformally invariant operators in higher spin spaces in Clifford analysis
- The higher spin Laplace operator (bosonic Laplacians)
- Dirichlet problem in the upper-half space and the unit ball
- Mean-value property, Cauchy's estimates, Liouville's Theorem

Classical Clifford analysis

Classical Clifford analysis started as a generalization of aspects of the analysis of functions with one complex variable to higher dimensional Euclidean spaces. One of the most important topics of this theory is the study of the Dirac operator D_x in the Euclidean space. That is a fist order conformally invariant differential operator, which generalizes the role of the Cauchy-Riemann operator, acting on Clifford algebra (spinor)-valued functions. Moreover, this operator is related to the Laplace operator by $D_x^2 = -\Delta_x$. Hence, Clifford analysis is also considered as a refinement of harmonic analysis. More details can be found in [Brackx, Delanghe, Sommen, 1982, Delanghe, Sommen, Souček, 1992].

classical Clifford analysis vs. higher spin theory in Clifford analysis

- Classical Clifford analysis is the study of Clifford (spinor)-valued functions on $\mathbb{R}^m$. For instance, Dirac operator acts on $C^\infty(\mathbb{R}^m, \mathcal{C}l_m)$, Laplacian acts on $C^\infty(\mathbb{R}^m, \mathbb{C})$.
- Higher spin spaces in Clifford analysis is the study of functions on $\mathbb{R}^m$ taking values in irreducible representations of $Spin(m)$. For instance, Rarita-Schwinger operator acts on $C^\infty(\mathbb{R}^m, \mathcal{M}_k)$, the higher spin Laplace operators (bosonic Laplacians) act on $C^\infty(\mathbb{R}^m, \mathcal{H}_k)$.

Note: Here, $\mathcal{M}_k$ stands for the space of spinor-valued homogeneous monogenic (null solutions of the Dirac operator) polynomials of degree k, $\mathcal{H}_k$ stands for the space of complex-valued homogeneous harmonic polynomials of degree k.

Conformally invariant operators in higher spin theory

Conformally invariant differential operators in higher spin spaces

- [Bureš, et. al, 2001, Dunkl et. al., 2013]
  1st order, Rarita-Schwinger operator, acting on $(\mathbb{R}^m \times \mathbb{R}^m, Cl_m; \mathcal{M}_k)$.
  (Cauchy theorem, Cauchy integral formulas and Borel-Pompeiu formula).
- [Eelbode, Roels, 2014, De Bie et. al., 2015] 2nd order, the higher spin Laplace operators (bosonic Laplacians), acting on $(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{C}; \mathcal{H}_k)$.
  (Fundamental solutions, polynomial solutions, ellipticity).
- [Ding,Walter,Ryan, 2016] Higher order(≥ 3) conformally invariant differential operators in higher spin spaces. (Intertwining operator, convolution type operator, fundamental solutions).

Known results of these differential operators

- Rarita-Schwinger operator has Cauchy integral formula, Cauchy theorem and a Borel-Pompeiu formula, see [Dunkl et. al., 2013].
- The generalized Maxwell operator and the higher spin Laplace operator (bosonic Laplacians) have a Borel-Pompeiu formula and a Green type integral formula. See [Ding,Ryan,2016].
- The higher order differential operators (odd order) has Borel-Pompeiu formula and a Cauchy type integral formula, see [Ding,2019].
- Descriptions for the polynomial solutions of the generalized Maxwell equation in $\mathbb{R}^3$ and $\mathbb{R}^{3,1}$ have been obtained in [Ding,Bock,Gürlebeck,2018, Ding,Bock,Gürlebeck,2019].

Why called bosonic Laplacians?

Recall that bosonic Laplacians $\mathcal{D}_k$ act on functions $f(\boldsymbol{x}, \boldsymbol{u}) \in (\mathbb{R}^m \times \mathbb{R}^m, \mathbb{C}; \mathcal{H}_k)$. When we let $k = 1$ and replace $\mathbb{R}^m$ with $\mathbb{R}^{3,1}$, the equation $\mathcal{D}_k f = 0$ can turn into $\Box A_\mu - \partial_\mu \partial^\gamma A_\gamma = 0$, where A_μ is the four-potential used in quantum mechanics (electrodynamics), see [Ding,Bock,Gürlebeck,2018, Eelbode, Roels, 2014]. This equation is indeed a relativistic wave equation (Maxwell equation with Lorentz gauge condition) describes a massless spin-1 field or a type of bosons.

Goals

- Introduce a Poisson kernel for bosonic Laplacians in the upper-half space, which solves the Dirichlet problem of bosonic Laplacians in the upper-half space.
- Analogs of these results in the unit ball case can be obtained by applying a Cayley transformation.
- Applications to the study of null solutions of bosonic Laplacians. (Mean-value property, Cauchy's estimate, Liouville's Theorem, etc.)

Clifford Algebras

Denote the m-dimensional Euclidean real space by $\mathbb{R}^m$. A standard basis of $\mathbb{R}^m$ is $\{e_1, \ldots, e_m\}$. The real Clifford algebra $\mathcal{C}l_m$ is generated from $\mathbb{R}^m$ by considering the relationship

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \cdots, j_r\} \subset \{1, 2, \cdots, m\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq m$.

Anti-involutions and reflection

Hence for any element $a \in \mathcal{C}l_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$.
We will need the following anti-involutions:
Reversion:
$\widetilde{e_{j_1} \cdots e_{j_r}} = e_{j_r} \cdots e_{j_1}$. Also $\widetilde{ab} = \tilde{b}\tilde{a}$ for $a, b \in Cl_m$.

Suppose that $\boldsymbol{a} \in \mathbb{S}^{m-1} \subseteq \mathbb{R}^m$, if we consider $\boldsymbol{axa}$, we may decompose

$$\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{a}\|} + \boldsymbol{x}_{\boldsymbol{a}\perp},$$

where $\boldsymbol{x}_{\boldsymbol{a}\|}$ is the projection of $\boldsymbol{x}$ onto $\boldsymbol{a}$ and $\boldsymbol{x}_{\boldsymbol{a}\perp}$ is the rest, perpendicular to a. Hence $\boldsymbol{x}_{\boldsymbol{a}\|}$ is a scalar multiple of $\boldsymbol{a}$ and we have

$$\boldsymbol{axa} = \boldsymbol{ax}_{\boldsymbol{a}\|}\boldsymbol{a} + \boldsymbol{ax}_{\boldsymbol{a}\perp}\boldsymbol{a} = -\boldsymbol{x}_{\boldsymbol{a}\|} + \boldsymbol{x}_{\boldsymbol{a}\perp}.$$

So the action $\boldsymbol{axa}$ describes a reflection of $\boldsymbol{x}$ across the hyperplane perpendicular to $\boldsymbol{a}$.

Bosonic Laplacians in $\mathbb{R}^m$

Consider functions $f(\boldsymbol{x}, \boldsymbol{u}) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}; \mathcal{H}_k)$, where $\mathcal{H}_k$ stands for homogeneous polynomials with degree-k w.r.t. the variable $\boldsymbol{u}$. This means that for each fixed $\boldsymbol{x} \in \mathbb{R}^m$, $f(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{H}_k$. Bosonic Laplacians in $\mathbb{R}^m$, denoted by $\mathcal{D}_k$, is given by [Eelbode, Roels, 2014]

$$\mathcal{D}_k : \ C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{C}; \mathcal{H}_k) \longrightarrow C^\infty(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{C}; \mathcal{H}_k),$$

$$\mathcal{D}_k = \Delta_{\boldsymbol{x}} - \frac{4\langle \boldsymbol{u}, \nabla_{\boldsymbol{x}}\rangle\langle \nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{x}}\rangle}{m+2k-2} + \frac{4|\boldsymbol{u}|^2\langle \nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{x}}\rangle^2}{(m+2k-2)(m+2k-4)},$$

where $\langle \ , \ \rangle$ is the standard inner product in $\mathbb{R}^m$.
Note: When $k = 1$, the differential operator is called the generalized Maxwell operator, see [Eelbode, Roels, 2014].

Let $\boldsymbol{x} = (\boldsymbol{x}', y) = (x_1, \cdots, x_{m-1}, y) \in \mathbb{R}^m_+$ with $y > 0$, $\boldsymbol{t} =: (\boldsymbol{t}', 0) = (t_1, \cdots, t_{m-1}, 0)$, we have our main theorem as follows.

Dirichlet problem

Theorem

(A Dirichlet Problem in the upper-half space)
Suppose $f(\boldsymbol{x}', \boldsymbol{u})$ *is continuous and bounded on* $\mathbb{R}^{m-1} \times \mathbb{R}^m$ *and* $f(\boldsymbol{x}', \boldsymbol{u}) \in \mathcal{H}_k$ *for each fixed* $\boldsymbol{x}' \in \mathbb{R}^{m-1}$*. Define* g *on* $\overline{\mathbb{R}^m_+} \times \mathbb{R}^m$ *by*

$$g(\boldsymbol{x}, \boldsymbol{v}) = \begin{cases} P_H[f](\boldsymbol{x}, \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{R}^m_+, \\ f(\boldsymbol{x}', \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{R}^{m-1}. \end{cases}$$

where

$$P_H[f](\boldsymbol{x}, \boldsymbol{v}) = \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} P_H(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}) f(\boldsymbol{t}', \boldsymbol{u}) dS(\boldsymbol{u}) d\boldsymbol{t}'.$$

Then g *is continuous on* $\overline{\mathbb{R}^m_+}$ *with respect to* $\boldsymbol{x}$ *and* $\mathcal{D}_k g = 0$ *in* $\mathbb{R}^m_+ \times \mathbb{R}^m$*.*

Note: $P_H(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v})$ is called the **Poisson kernel**, which will be introduced shortly.

A particular non-trivial null solution

Let $Z_k(\boldsymbol{u}, \boldsymbol{v})$ be the reproducing kernel for the zonal spherical harmonics in the sense that

$$P_k(\boldsymbol{v}) = \int_{\mathbb{S}^{m-1}} Z_k(\boldsymbol{u}, \boldsymbol{v}) f(\boldsymbol{u}) dS(\boldsymbol{u}), \text{ for all } f(\boldsymbol{v}) \in \mathcal{H}_k.$$

Since $\mathcal{D}_k$ is a second order differential operator with respect to $\boldsymbol{x}$, then $yZ_k(\boldsymbol{u}, \boldsymbol{v})$ is a trivial solution of $\mathcal{D}_k$. Futher, in [De Bie et. al., 2015], it shows that $\mathcal{D}_k$ is a second order conformally invariant differential operator in the higher spin theory. In particular, it is conformally invariant under the following special conformal transformation

$$\begin{aligned} K: \ & C^2(\mathbb{R}^m, \mathcal{H}_k) \longrightarrow C(\mathbb{R}^m, \mathcal{H}_k) \\ & f(\boldsymbol{x}, \boldsymbol{u}) \mapsto K[f](\boldsymbol{x}, \boldsymbol{u}) := |\boldsymbol{x}|^{2-m} f\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}, \frac{\boldsymbol{xux}}{|\boldsymbol{x}|^2}\right). \end{aligned}$$

Hence, we apply K to the trivial solution $yZ_k(\boldsymbol{u}, \boldsymbol{v})$ to obtain a non-trivial solution $\frac{y}{\|\boldsymbol{x}\|^m} Z_k\left(\frac{\boldsymbol{xux}}{|\boldsymbol{x}|^2}, \boldsymbol{v}\right)$.

Critical Lemma

Now, we introduce a critical lemma.

Lemma

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{m-1}$, there exists a non-zero constant $c_{m,k}$ such that

$$c_{m,k} \int_{\mathbb{R}^{m-1}} \frac{y}{\|\boldsymbol{x}\|^m} Z_k\left(\frac{\boldsymbol{x}\boldsymbol{u}\boldsymbol{x}}{\|\boldsymbol{x}\|^2}, \boldsymbol{v}\right) d\boldsymbol{x}' = Z_k(\boldsymbol{u}, \boldsymbol{v}),$$

where $c_{m,k} = \dfrac{2(m+2k-2)}{(m-2)\omega_m}$ and ω_m stands for the area of the unit sphere $\mathbb{S}^{m-1}$.

Note: One can notice that when $k = 0$, this reduces to a property of the Poisson kernel of Laplacian.

Poisson kernel

Now, we set

$$P_H(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}) = c_{m,k} \frac{y}{\|\boldsymbol{x} - \boldsymbol{t}\|^m} Z_k\left(\frac{(\boldsymbol{x} - \boldsymbol{t})\boldsymbol{u}(\boldsymbol{x} - \boldsymbol{t})}{|\boldsymbol{x} - \boldsymbol{t}|^2}, \boldsymbol{v}\right).$$

The function P_H is called the **Poisson kernel** for bosonic Laplacians in the upper-half space. Notice that, with the critical lemma, we already know that

$$\int_{\mathbb{R}^{m-1}} P_H(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}) d\boldsymbol{t}' = Z_k(\boldsymbol{u}, \boldsymbol{v}).$$

For $1 \leq p < \infty$, let $L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$ stand for the space of Borel measurable functions f on $\mathbb{R}^{m-1} \times \mathbb{B}^m$ for which

$$\|f\|_p = \left(\int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} |f(\boldsymbol{x}, \boldsymbol{u})|^p dS(\boldsymbol{u}) d\boldsymbol{x}\right)^{1/p} < +\infty.$$

$L^\infty(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$ consists of the Borel measurable functions f on $\mathbb{R}^{m-1} \times \mathbb{B}^m$ for which $\|f\|_\infty < +\infty$, where $\|\cdot\|_\infty$ stands for the essential supremum norm on $\mathbb{R}^{m-1} \times \mathbb{B}^m$.

Poisson integral

The **Poisson integral** of $f \in L^p(\mathbb{R}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$, for any $p \in [1, \infty]$, is given by

$$P_H[f](\boldsymbol{x}, \boldsymbol{v}) = \int_{\mathbb{R}^{m-1}} \int_{\mathbb{S}^{m-1}} P_H(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}) f(\boldsymbol{t}, \boldsymbol{u}) dS(\boldsymbol{u}) d\boldsymbol{t}.$$

By taking derivatives under the integral signs, one can see that $P_H[f] \in \ker \mathcal{D}_k$. Further, it solves the following Dirichlet problem.

Theorem (Dirichlet problem in the upper-half space)

Suppose $f(\boldsymbol{x}', \boldsymbol{u})$ is continuous and bounded on $\mathbb{R}^{m-1} \times \mathbb{B}^m$ and $f(\boldsymbol{x}', \boldsymbol{u}) \in \mathcal{H}_k$ for each fixed $\boldsymbol{x}' \in \mathbb{R}^{m-1}$. Define g on $\overline{\mathbb{R}^m_+} \times \overline{\mathbb{B}^m}$ by

$$g(\boldsymbol{x}, \boldsymbol{v}) = \begin{cases} P_H[f](\boldsymbol{x}, \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{R}^m_+, \\ f(\boldsymbol{x}', \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{R}^{m-1}. \end{cases}$$

Then g is continuous on $\overline{\mathbb{R}^m_+}$ with respect to $\boldsymbol{x}$ and $\mathcal{D}_k g = 0$ in $\mathbb{R}^m_+ \times \mathbb{B}^m$.

Cayley transform from $\mathbb{B}^m$ to $\mathbb{R}^m_+$

Let $\mathbb{B}^m$ be the open unit ball in $\mathbb{R}^m$ and φ is the Cayley transform given as follows.

$$\begin{aligned} \varphi : \qquad & \mathbb{B}^m \longrightarrow \mathbb{R}^m_+, \\ & \boldsymbol{x} \mapsto \boldsymbol{z} = -\frac{1}{2}(\boldsymbol{x} + \boldsymbol{e}_m)(\boldsymbol{e}_m\boldsymbol{x} + 1)^{-1}. \end{aligned} \tag{1}$$

In particular, if $\zeta \in \mathbb{S}^{m-1}$ then $\varphi(\zeta) \in \mathbb{R}^{m-1}$. One can apply this Cayley transform to the result in the upper-half space to obtain an analog in the unit ball.

Poisson kernel in the unit ball case

Let $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{B}^m$, the Poisson kernel of bosonic Laplacians in the unit ball is given by

$$\begin{aligned} & P_B[h](\boldsymbol{x}, \boldsymbol{v}) \\ = \; & \frac{c_{m,k}}{2} \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} \frac{1 - |\boldsymbol{x}|^2}{|\boldsymbol{x} - \zeta|^m} Z_k\left(\frac{(\boldsymbol{x} - \zeta)\omega(\boldsymbol{x} - \zeta)}{|\boldsymbol{x} - \zeta|^2}, \boldsymbol{\nu} \right) h(\zeta, \boldsymbol{u}) dS(\boldsymbol{u}) dS(\zeta), \end{aligned}$$

where

$$\boldsymbol{\omega} = \frac{(\boldsymbol{e}_m \widetilde{\zeta + 1})\boldsymbol{u}(\boldsymbol{e}_m \zeta + 1)}{|\boldsymbol{e}_m \zeta + 1|^2} \quad \text{and} \quad \boldsymbol{\nu} = \frac{(\boldsymbol{e}_m \widetilde{\boldsymbol{x} + 1})\boldsymbol{v}(\boldsymbol{e}_m \boldsymbol{x} + 1)}{|\boldsymbol{e}_m \boldsymbol{x} + 1|^2}.$$

Dirichlet problem in the unit ball

Theorem (Dirichlet problem in $\mathbb{B}^m$ with continuous data)

Suppose $h \in C(\mathbb{S}^{m-1} \times \mathbb{B}^m, \mathcal{H}_k)$. *Define* h^* *on* $\mathbb{B}^m \times \mathbb{B}^m$ *by*

$$h^*(\boldsymbol{x}, \boldsymbol{v}) = \begin{cases} P_B[h](\boldsymbol{x}, \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{B}^m, \quad \boldsymbol{v} \in \mathbb{B}^m \\ h(\boldsymbol{x}, \boldsymbol{v}), & \text{if } \boldsymbol{x} \in \mathbb{S}^{m-1}, \quad \boldsymbol{v} \in \mathbb{B}^m. \end{cases}$$

Then h^* *is continuous on* $\overline{\mathbb{B}^m}$ *with respect to* $\boldsymbol{x}$, $\mathcal{D}_k h^* = 0$ *in* $\mathbb{B}^m \times \mathbb{B}^m$ *and*

$$\|h^*\|_{L^\infty(\mathbb{B}^m \times \mathbb{B}^m)} \leq a'_{m,k} \|h\|_{L^\infty(\mathbb{S}^{m-1} \times \mathbb{B}^m)}, \tag{2}$$

where $a'_{m,k}$ *is a positive constant only depending on* m *and* k.

Note: One can also generalize it to the Dirichlet problem with L^p data.

Motivation

Recall that the mean-value property (MVP) of harmonic functions can be obtained only with Green's identity. Unfortunately, the Green's identity for bosonic Laplacians is too complicated to handle. However, one can also obtain the MVP from the Poisson integral formula when evaluating at the origin. This motivates us that to obtain the MVP for null solutions of bosonic Laplacians we need to find an analog of the Poisson integral formula first, which is equivalent to has **the uniqueness of solutions to the Dirichlet problem of $\mathcal{D}_k$ in the unit ball**.

- In general, this is about uniqueness of solutions to Dirichlet problem of elliptic equation over sections of vector bundles.

Poisson integral formula

With an estimate for solutions to an elliptic operator over trivial vector bundles given in [Narasimhan,1985], we have the uniqueness of solutions to the Dirichlet problem for bosonic Laplcians. Further, the uniqueness provides us a Poisson integral formula.

Theorem (Poisson integral formula)

Let $f \in C^2(\mathbb{B}^m \times \mathbb{B}^m, \mathcal{H}_k) \cap C(\overline{\mathbb{B}^m} \times \overline{\mathbb{B}^m}, \mathcal{H}_k)$ and $\mathcal{D}_k f = 0$ in $\mathbb{B}^m \times \mathbb{B}^m$. Then for $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{B}^m$, we have

$$f(\boldsymbol{x}, \boldsymbol{v}) = \frac{c_{m,k}}{2} \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} \frac{1 - |\boldsymbol{x}|^2}{|\boldsymbol{x} - \zeta|^m} Z_k\left(\frac{(\boldsymbol{x} - \zeta)\omega(\boldsymbol{x} - \zeta)}{|\boldsymbol{x} - \zeta|^2}, \boldsymbol{\nu} \right) f(\zeta, \omega) dS(\omega) dS(\zeta),$$

where

$$\boldsymbol{\omega} = \frac{(\widetilde{\boldsymbol{e}_m \zeta + 1})\boldsymbol{u}(\boldsymbol{e}_m \zeta + 1)}{|\boldsymbol{e}_m \zeta + 1|^2} \quad \text{and} \quad \boldsymbol{\nu} = \frac{(\widetilde{\boldsymbol{e}_m \boldsymbol{x} + 1})\boldsymbol{v}(\boldsymbol{e}_m \boldsymbol{x} + 1)}{|\boldsymbol{e}_m \boldsymbol{x} + 1|^2}.$$

Poisson Integral Formula

Let $\boldsymbol{x} = 0$ in the Poisson integral formula above, one has a mean-value property as follows.

Theorem (Mean-value property: sphere version)

Assume $f \in C^2(B(\boldsymbol{a}, r) \times \mathbb{B}^m, \mathcal{H}_k) \cap C(\overline{B(\boldsymbol{a}, r)} \times \overline{\mathbb{B}^m}, \mathcal{H}_k)$ *and* $\mathcal{D}_k f = 0$ *in* $B(\boldsymbol{a}, r) \times \mathbb{B}^m$. *We have*

$$f(\boldsymbol{a}, \boldsymbol{\nu}) = \frac{c_{m,k}}{2} \int_{\mathbb{S}^{m-1}} f(\boldsymbol{a} + r\zeta, \zeta\boldsymbol{\nu}\zeta) dS(\zeta), \quad \forall \boldsymbol{\nu} \in \mathbb{B}^m.$$

Note: A volume-version MVP can also be obtained as well.

Two particular cases:

- When $k = 0$, it is the MVP for harmonic functions.
- When $f(\boldsymbol{x}, \boldsymbol{\nu}) = f(\boldsymbol{\nu}) \in \mathcal{H}_k$, this becomes the Lemma 6 in [Dunkl et. al., 2013].

Poisson Integral Formula

Let $\boldsymbol{x} = 0$ in the Poisson integral formula above, one has a mean-value property as follows.

Theorem (Mean-value property: sphere version)

Assume $f \in C^2(B(\boldsymbol{a}, r) \times \mathbb{B}^m, \mathcal{H}_k) \cap C(\overline{B(\boldsymbol{a}, r)} \times \overline{\mathbb{B}^m}, \mathcal{H}_k)$ *and* $\mathcal{D}_k f = 0$ *in* $B(\boldsymbol{a}, r) \times \mathbb{B}^m$. *We have*

$$f(\boldsymbol{a}, \boldsymbol{\nu}) = \frac{c_{m,k}}{2} \int_{\mathbb{S}^{m-1}} f(\boldsymbol{a} + r\zeta, \zeta\boldsymbol{\nu}\zeta) dS(\zeta), \quad \forall \boldsymbol{\nu} \in \mathbb{B}^m.$$

Note: A volume-version MVP can also be obtained as well.

Two particular cases:

- When $k = 0$, it is the MVP for harmonic functions.
- When $f(\boldsymbol{x}, \boldsymbol{\nu}) = f(\boldsymbol{\nu}) \in \mathcal{H}_k$, this becomes the Lemma 6 in [Dunkl et. al., 2013].

Cauchy's estimates and Liouville's Theorem

From the Poisson integral formula, we can obtain

Theorem (Cauchy's estimates)

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be multi-indices. Assume $f \in C^2(\Omega \times \mathbb{B}^m, \mathcal{H}_k)$ and $\mathcal{D}_k f = 0$ in $\Omega \times \mathbb{B}^m$. Then there exists a constant $c_{\boldsymbol{\alpha},m,k}$ such that

$$|D_{\boldsymbol{\nu}}^{\boldsymbol{\beta}} D_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f(\boldsymbol{a}, \boldsymbol{\nu}_0)| \leq \frac{c_{\boldsymbol{\alpha},m,k} \|f\|_{L^\infty(B(\boldsymbol{a},r_1)\times B(\boldsymbol{\nu}_0,r_2),\mathcal{H}_k)}}{r_1^{|\boldsymbol{\alpha}|} r_2^{|\boldsymbol{\beta}|}},$$

for any $B(\boldsymbol{a}, r_1) \Subset \Omega$ and $B(\boldsymbol{\nu}_0, r_2) \Subset \mathbb{B}^m$.

Then, a Liouville-type theorem can be obtained

Theorem (Liouville-type theorem)

Suppose $f \in C^2(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_k) \cap L^\infty(\mathbb{R}^m \times \mathbb{B}^m, \mathcal{H}_k)$ and $\mathcal{D}_k f = 0$ on $\mathbb{R}^m \times \mathbb{B}^m$. Then $f = f(\boldsymbol{v}) \in \mathcal{H}_k$.

Remark

There are some other properties for harmonic functions can not be generalized in this context. For instance, maximum principle, Harnack inequality, positive harmonic functions, etc. Two main difficulties are the following.

- The constant in the mean-value property is greater than 1

$$f(\boldsymbol{a},\boldsymbol{\nu}) = \frac{c_{m,k}}{2}\int_{\mathbb{S}^{m-1}} f(\boldsymbol{a}+r\zeta,\zeta\boldsymbol{\nu}\zeta)dS(\zeta), \quad \forall \boldsymbol{\nu}\in\mathbb{B}^m.$$

- Since f is already a homogeneous harmonic polynomial w.r.t. $\boldsymbol{\nu}\in\mathbb{B}^m$, so we can not assume that $f>0$ for $(\boldsymbol{x},\boldsymbol{\nu})\in\mathbb{B}^m\times\mathbb{B}^m$.

Reference I

- Hendrik De Bie, David Eelbode, Matthias Roels, **The higher spin Laplace operator**, Potential Analysis, Vol. 47, Issue 2, 2017, pp. 123-149.
- F. Brackx, R. Delanghe, F. Sommen, **Clifford Analysis**, Pitman, London, 1982.
- J. Bureš, F. Sommen, V. Souček, P. Van Lancker, **Rarita-Schwinger Type Operators in Clifford Analysis**, J. Funct. Anal. 185 (2001), No. 2, pp. 425-455.
- R. Delanghe, F. Sommen, V. Souček, **Clifford Algebra and Spinor-Valued Functions: A Function Theory for the Dirac Operator**, Kluwer, Dordrecht, 1992.
- Chao Ding, **"Integral formulas for higher order conformally invariant fermionic operators"**, Advances in Applied Clifford Algebras, (2019) 29: 37. https://doi.org/10.1007/s00006-019-0953-4

Reference II

- Chao Ding, Sebastian Bock, Klaus Gürlebeck, **"Special bases for solutions of a generalized Maxwell system in 3-dimensional space"**, Mathematical Methods in the Applied Sciences, Vol. 41, Issue 13, 2018, 5308-5326.
- Chao Ding, Sebastian Bock, Klaus Gürlebeck, **"Construction of special solutions for the Maxwell equations"**, Advances in Applied Clifford Algebras, 2019, https://doi.org/10.1007/s00006-019-1013-9.
- Chao Ding, John Ryan, **"Some Properties of the Higher Spin Laplace Operator"**, Transactions of the American Mathematical Society, DOI: https://doi.org/10.1090/tran/7404.
- C. Ding, R. Walter, J. Ryan, **Arbitrary Order Conformally Invariant Operators in Higher Spin Spaces**, The Journal of Geometric Analysis, Vol. 27, Issue 3, 2017, pp. 2418-2452.

Reference III

- C.F. Dunkl, Junxia Li, John Ryan and Peter Van Lancker, **Some Rarita-Schwinger type operators**, Computational Methods and Function Theory, 2013, Vol. 13, Issue 3, pp. 397-424.
- David Eelbode, Matthias Roels, **Generalised Maxwell equations in higher dimensions**, Complex Analysis and Operator Theory, Vol. 10, Issue 2, 2016, pp. 267-293.
- R. Narasimhan, **Analysis on Real and Complex Manifolds**, North-Holland Mathematical Library, Vol. 35, 1985.
- W. Rarita, J. Schwinger, **On a Theory of Particles with Half-integral Spin**, Phys. Rev., Vol. 60, Issue 1, 1941, pp. 60-61.

Thank you for your Kind attention!