Optimal approximants and orthogonal polynomials in several variables

Operator Theory With Its Applications

Meredith Sargent [1]
joint with
Alan A. Sola [2]

[1]University of Arkansas

[2]Stockholm University

August 2020

What are optimal approximants?

- $f \in \mathcal{H}$, but $1/f(z) = \sum_{k=0}^{\infty} a_k z^k$ may not be in $\mathcal{H}$

What are optimal approximants?

- $f \in \mathcal{H}$, but $1/f(z) = \sum_{k=0}^{\infty} a_k z^k$ may not be in $\mathcal{H}$
- $f(z) = 1 - z \ \in H^2(\mathbb{D})$ but $\frac{1}{1-z} \ \notin H^2(\mathbb{D})$

What are optimal approximants?

- $f \in \mathcal{H}$, but $1/f(z) = \sum_{k=0}^{\infty} a_k z^k$ may not be in $\mathcal{H}$
- $f(z) = 1 - z \in H^2(\mathbb{D})$ but $\frac{1}{1-z} \notin H^2(\mathbb{D})$
- Partial Taylor sums give *AN* approximation in $\mathcal{H}$, but is it the "best" one?

What are optimal approximants?

- $f \in \mathcal{H}$, but $1/f(z) = \sum_{k=0}^{\infty} a_k z^k$ may not be in $\mathcal{H}$
- $f(z) = 1 - z \in H^2(\mathbb{D})$ but $\frac{1}{1-z} \notin H^2(\mathbb{D})$
- Partial Taylor sums give AN approximation in $\mathcal{H}$, but is it the "best" one?
- Find a degree n polynomial p_n^* that minimizes $\|p \cdot f - 1\|_{\mathcal{H}}$
  - approximating $1/f$ in $\mathcal{H}$
  - approximating 1 in $f\mathcal{P}_n$

What are optimal approximants?

- $f \in \mathcal{H}$, but $1/f(z) = \sum_{k=0}^{\infty} a_k z^k$ may not be in $\mathcal{H}$
- $f(z) = 1 - z \in H^2(\mathbb{D})$ but $\frac{1}{1-z} \notin H^2(\mathbb{D})$
- Partial Taylor sums give *AN* approximation in $\mathcal{H}$, but is it the "best" one?
- Find a degree n polynomial p_n^* that minimizes $\|p \cdot f - 1\|_{\mathcal{H}}$
  - approximating $1/f$ in $\mathcal{H}$
  - approximating 1 in $f\mathcal{P}_n$

Theorem (Bénéteau, Khavinson, Liaw, Seco, and Sola (JLMS 2016))

Let $f \in \mathcal{H}$. Using the orthonormal basis $\{\phi_j\}$ for the weighted space $\mathcal{H}_f$ ($\langle g, h\rangle_{f\mathcal{H}} = \langle gf, hf\rangle_{\mathcal{H}}$),

$$p_n^*(z) = \sum_{k=0}^{n} \langle 1, f\phi_k\rangle_{\mathcal{H}}\, \phi_k(z). \tag{1}$$

This in turn implies that

$$\langle 1, f\phi_n\rangle_{\mathcal{H}}\phi_n(z) = p_n^*(z) - p_{n-1}^*(z), \quad n = 1, 2, 3 \dots. \tag{2}$$

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

Flashback to the 1970s!

Electrical Engineers and “Planar Least Squares Inverses”

- PLSI are secretly just optimal approximants!

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

- PLSI are secretly just optimal approximants!
- Hardy space of the disk

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

- PLSI are secretly just optimal approximants!
- Hardy space of the disk AND the bidisk

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

- PLSI are secretly just optimal approximants!
- Hardy space of the disk AND the bidisk
- Reflections of OG polynomials in weighted spaces, as well as separable functions

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

- PLSI are secretly just optimal approximants!
- Hardy space of the disk AND the bidisk
- Reflections of OG polynomials in weighted spaces, as well as separable functions
- It was about applications to filtering theory:

Flashback to the 1970s!

Electrical Engineers and "Planar Least Squares Inverses"

- PLSI are secretly just optimal approximants!
- Hardy space of the disk AND the bidisk
- Reflections of OG polynomials in weighted spaces, as well as separable functions
- It was about applications to filtering theory:

Filters are stable when the PLSI polynomial has no zeros in the bidisk!

The Several Variable Case

Questions of Degree

The Several Variable Case

Questions of Degree

- total degree
  - total degree 1:
    $p_1^* = a_0 + a_1 z_1 + a_2 z_2$
  - total degree 2:
    $p_2^* = b_0 + b_1 z_1 + b_2 z_2 + b_3 z_1^2 + b_4 z_1 z_2 + b_5 z_2^2$

The Several Variable Case

Questions of Degree

- total degree
  - total degree 1:
    $p_1^* = a_0 + a_1 z_1 + a_2 z_2$
  - total degree 2:
    $p_2^* = b_0 + b_1 z_1 + b_2 z_2 + b_3 z_1^2 + b_4 z_1 z_2 + b_5 z_2^2$
- multidegree
  - multidegree (1,1):
    $p_1^* = a_0 + a_1 z_1 + a_2 z_2 + a_3 z_1 z_2$
  - multidegree (2,2):
    $p_2^* = b_0 + b_1 z_1 + b_2 z_2 + b_3 z_1 z_2 + b_4 z_1^2 + b_5 z_2^2 + b_6 z_1^2 z_2 + b_6 z_1 z_2^2 + b_7 z_1^2 z_2^2$

The Several Variable Case

Questions of Degree

- total degree
  - total degree 1:
    $p_1^* = a_0 + a_1 z_1 + a_2 z_2$
  - total degree 2:
    $p_2^* = b_0 + b_1 z_1 + b_2 z_2 + b_3 z_1^2 + b_4 z_1 z_2 + b_5 z_2^2$
- multidegree
  - multidegree (1,1):
    $p_1^* = a_0 + a_1 z_1 + a_2 z_2 + a_3 z_1 z_2$
  - multidegree (2,2):
    $p_2^* = b_0 + b_1 z_1 + b_2 z_2 + b_3 z_1 z_2 + b_4 z_1^2 + b_5 z_2^2 + b_6 z_1^2 z_2 + b_6 z_1 z_2^2 + b_7 z_1^2 z_2^2$
- Both of these are problematic!

Example

$f = 2 - z_1 - z_2$, note that $g(z_1, z_2) = z_1 \in H^2(\mathbb{D}^2)_f$, but it couldn't be built from differences of the above

The Several Variable Case

Monomial Orderings

$$\chi_0 = 1,\ \ \chi_1 = z_1,\ \ \chi_2 = z_2,\ \ \chi_3 = z_1^2,\ \ \chi_4 = z_1 z_2,\ \ \chi_5 = z_2^2,\ \ \chi_6 = z_1^3, \dots$$

$$\mathcal{P}_n = \mathrm{span}\{\chi_j : j = 0, \dots, n\}, \quad n = 0, 1, 2, \dots$$

The Several Variable Case

Monomial Orderings

$$
\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4 z_2 \\
 & & z_1^2 & & z_1^3 z_2 & \\
 & z_1 & & z_1^2 z_2 & & z_1^3 z_2 \\
1 & & z_1 z_2 & & z_1^2 z_2^2 & \\
 & z_2 & & z_1 z_2^2 & & z_1^2 z_2^3 \\
 & & z_2^2 & & z_1 z_2^3 & \\
 & & & z_2^3 & & z_1 z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}
$$

The Several Variable Case

Monomial Orderings

$$\chi_0 = 1,\ \ \chi_1 = z_1,\ \ \chi_2 = z_2,\ \ \chi_3 = z_1^2,\ \ \chi_4 = z_1 z_2,\ \ \chi_5 = z_2^2,\ \ \chi_6 = z_1^3, \ldots$$

$$\mathcal{P}_n = \mathrm{span}\{\chi_j : j = 0, \ldots, n\}, \quad n = 0, 1, 2, \ldots$$

Definition

Let $f \in \mathcal{H}(\Omega)$ be given. The *nth-order optimal polynomial approximant to* $1/f$ with respect to $\mathcal{P}_n$ is defined as

$$p_n^*(z) = \mathrm{Proj}_{f \cdot \mathcal{P}_n}[1](z),$$

where $\mathrm{Proj}_{f \cdot \mathcal{P}_n} \colon \mathcal{H} \to f \cdot \mathcal{P}_n$ denotes the orthogonal projection onto the subspace $f \cdot \mathcal{P}_n$.
In other words, p_n^* is the unique polynomial that minimizes $\|p \cdot f - 1\|_{\mathcal{H}}$ among all $p \in \mathcal{P}_n$.

The Several Variable Case

Matrix method for computing OAs

Let $f \in \mathcal{H} \setminus \{0\}$. Then the coefficients of the n-order optimal approximant $p_n^* = \sum_{j=0}^{n} c_j^* \chi_j$ are given by solution to the linear system

$$M\vec{c}^{\,*} = \vec{b},$$

where M is an $(n+1) \times (n+1)$ Grammian matrix with entries given by

$$M_{ij} = \langle \chi_j f, \chi_i f \rangle$$

and

$$\vec{b} = \begin{pmatrix} \langle 1, \chi_0 f \rangle \\ \vdots \\ \langle 1, \chi_n f \rangle \end{pmatrix}.$$

Reinterpretation of previous results by Bénéteau, Condori, Liaw, Seco, and Sola (2015), and Fricain, Mashreghi, and Seco (2014)

Explicit example: $f(z_1, z_2) = 1 - \frac{1}{2}(z_1 + z_2)$

In $H^2(\mathbb{D}^2)$, the first few optimal approximants to $1/f$ are

$$
\begin{aligned}
p_0 &= \frac{2}{3} \\
p_1 &= \frac{3}{4} + \frac{1}{4} z_1 \\
p_2 &= \frac{14}{17} + \frac{4}{17} z_1 + \frac{4}{17} z_2 \\
p_3 &= \frac{186}{223} + \frac{60}{223} z_1 + \frac{52}{223} z_2 + \frac{20}{223} z_1^2 \\
p_4 &= \frac{1794}{2039} + \frac{684}{2039} z_1 + \frac{620}{2039} z_2 + \frac{160}{2039} z_1^2 + \frac{408}{2039} z_1 z_2 \\
p_5 &= \frac{182}{205} + \frac{68}{205} z_1 + \frac{68}{205} z_2 + \frac{16}{205} z_1^2 + \frac{8}{41} z_1 z_2 + \frac{16}{205} z_2^2
\end{aligned}
$$

Explicit example: $f(z_1, z_2) = 1 - z_1 z_2$

In $H^2(\mathbb{D}^2)$, the first few optimal approximants to $1/f$ are

$$\begin{aligned} p_0 &= \frac{1}{2} \\ p_4 &= \frac{2}{3} + \frac{1}{3} z_1 z_2 \\ p_{12} &= \frac{3}{4} + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_1^2 z_2^2 \end{aligned}$$

??????

Explicit example: $f(z_1, z_2) = 1 - z_1 z_2$

In $H^2(\mathbb{D}^2)$, the first few optimal approximants to $1/f$ are

$$\begin{aligned} p_0 &= \frac{1}{2} \\ p_4 &= \frac{2}{3} + \frac{1}{3} z_1 z_2 \\ p_{12} &= \frac{3}{4} + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_1^2 z_2^2 \end{aligned}$$

??????

In this case, we can't use the optimal approximants to recover the orthogonal polynomials! We won't "get all of them."

Explicit example: $f(z_1, z_2) = 1 - z_1 z_2$

In $H^2(\mathbb{D}^2)$, the first few optimal approximants to $1/f$ are

$$
\begin{aligned}
p_0 &= \frac{1}{2} \\
p_4 &= \frac{2}{3} + \frac{1}{3} z_1 z_2 \\
p_{12} &= \frac{3}{4} + \frac{1}{2} z_1 z_2 + \frac{1}{4} z_1^2 z_2^2
\end{aligned}
$$

??????

In this case, we can't use the optimal approximants to recover the orthogonal polynomials! We won't "get all of them."

For some functions, we won't be able to get ANY!

Weakly inner functions

Definition

We say that $g \in \mathcal{H}(\Omega) \setminus \{0\}$ is *weakly inner* if

$$\langle g, \chi_j g \rangle = 0 \quad \text{for all} \quad j \neq 0.$$

Weakly inner functions

Definition

We say that $g \in \mathcal{H}(\Omega) \setminus \{0\}$ is *weakly inner* if

$$\langle g, \chi_j g \rangle = 0 \quad \text{for all} \quad j \neq 0.$$

Proposition

If $g \in \mathcal{H}(\Omega)$ is weakly inner, then its optimal approximants are all equal to a single constant: $p_n^* = p_0$ for $n = 1, 2, \ldots$.

Weakly inner functions

Definition

We say that $g \in \mathcal{H}(\Omega) \setminus \{0\}$ is *weakly inner* if

$$\langle g, \chi_j g \rangle = 0 \quad \text{for all} \quad j \neq 0.$$

Proposition

If $g \in \mathcal{H}(\Omega)$ is weakly inner, then its optimal approximants are all equal to a single constant: $p_n^* = p_0$ for $n = 1, 2, \ldots$.

Lemma

Suppose $\theta\colon \mathbb{D}^d \to \mathbb{C}$ is inner. Then θ is weakly H^2-inner.

Weakly inner functions

Shapiro-Shields functions

- Explicit examples!
- Constructed using determinants

Example

The Shapiro-Shields function for $H^2(\mathbb{D}^2)$ associated with a point $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ is

$$s_\lambda(z) = \frac{1}{(1-|\lambda_1|^2)(1-|\lambda_2|^2)} \frac{\overline{\lambda}_1(\lambda_1 - z_1) + \overline{\lambda}_2(z_2 - \lambda_2) - \overline{\lambda_1\lambda_2}(\lambda_1\lambda_2 - z_1 z_2)}{(1-\overline{\lambda}_1 z_1)(1-\overline{\lambda}_2 z_2)}.$$

This is weakly inner, but not classically inner.

Weakly inner functions

Shapiro-Shields functions

- Explicit examples!
- Constructed using determinants

Example

The Shapiro-Shields function for $H^2(\mathbb{D}^2)$ associated with a point $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ is

$$s_\lambda(z) = \frac{1}{(1-|\lambda_1|^2)(1-|\lambda_2|^2)} \frac{\overline{\lambda}_1(\lambda_1 - z_1) + \overline{\lambda}_2(z_2 - \lambda_2) - \overline{\lambda_1\lambda_2}(\lambda_1\lambda_2 - z_1 z_2)}{(1-\overline{\lambda}_1 z_1)(1-\overline{\lambda}_2 z_2)}.$$

This is weakly inner, but not classically inner.

Shapiro-Shields functions give examples of weakly inner functions in any RKHS!

Still have time? just a little?

Thank You!

Summary

- Things are harder in several variables!
  - Must choose a monomial ordering (which one is "best"??)
  - Weakly inner doesn't imply inner!
- Other neat stuff
  - We actually found a closed form for the OG polynomials for $1 - az_1z_2$ (for spaces on the bidisk and the 2-ball)
  - And a closed form for the OG polynomials for $1 - a(z_1 + z_2)$ for the 2-ball (bidisk is harder)

OG polynomials in weighted spaces

$f(z_1, z_2) = 1 - az_1z_2$

- Recall: OAs aren't enough to recover all of the OGs

$$p_0 = \frac{1}{2}$$
$$p_4 = \frac{2}{3} + \frac{1}{3}z_1z_2$$
$$p_{12} = \frac{3}{4} + \frac{1}{2}z_1z_2 + \frac{1}{4}z_1^2z_2^2$$

OG polynomials in weighted spaces

$f(z_1, z_2) = 1 - az_1z_2$

- Do Gram-Schmidt on the monomials with the weighted inner product $\langle g, h\rangle_{f\mathcal{H}} = \langle gf, hf\rangle_{\mathcal{H}}$

$$\phi_0 = 1$$
$$\phi_1 = z_1$$
$$\phi_2 = z_2$$
$$\phi_3 = z_1^2$$
$$\phi_4 = \frac{1}{2} + z_1z_2$$
$$\phi_5 = z_2^2$$

$$\phi_6 = z_1^3$$
$$\phi_7 = \frac{1}{2}z_1 + z_1^2z_2$$
$$\phi_8 = \frac{1}{2}z_2 + z_1z_2^2$$
$$\phi_9 = z_2^3$$
$$\vdots$$

OG polynomials in weighted spaces

$f(z_1, z_2) = 1 - az_1z_2$

- Do Gram-Schmidt on the monomials with the weighted inner product $\langle g, h \rangle_{f\mathcal{H}} = \langle gf, hf \rangle_{\mathcal{H}}$

$$\phi_0 = 1$$
$$\phi_1 = z_1$$
$$\phi_2 = z_2$$
$$\phi_3 = z_1^2$$
$$\phi_4 = \frac{1}{2} + z_1z_2$$
$$\phi_5 = z_2^2$$

$$\phi_6 = z_1^3$$
$$\phi_7 = \frac{1}{2}z_1 + z_1^2z_2$$
$$\phi_8 = \frac{1}{2}z_2 + z_1z_2^2$$
$$\phi_9 = z_2^3$$
$$\vdots$$

$$\phi_{12} = \frac{1}{3} + \frac{2}{3}z_1z_2 + z_1^2z_2^2$$
$$\phi_{17} = \frac{1}{3}z_1 + \frac{2}{3}z_1^2z_2 + z_1^3z_2^2$$
$$\phi_{18} = \frac{1}{3}z_2 + \frac{2}{3}z_1z_2^2 + z_1^2z_2^3$$

$f(z_1, z_2) = 1 - az_1z_2$

$$\phi_4 = \frac{1}{2} + z_1 z_2$$

$$\phi_7 = \frac{1}{2} z_1 + z_1^2 z_2$$

$$\phi_8 = \frac{1}{2} z_2 + z_1 z_2^2$$

$f(z_1, z_2) = 1 - az_1z_2$

$$\phi_4 = \frac{1}{2} + z_1z_2$$
$$\phi_7 = \frac{1}{2}z_1 + z_1^2z_2$$
$$\phi_8 = \frac{1}{2}z_2 + z_1z_2^2$$

$$\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4z_2 \\
 & & z_1^2 & & z_1^3z_2 & \\
 & z_1 & & z_1^2z_2 & & z_1^3z_2 \\
1 & & z_1z_2 & & z_1^2z_2^2 & \\
 & z_2 & & z_1z_2^2 & & z_1^2z_2^3 \\
 & & z_2^2 & & z_1z_2^3 & \\
 & & & z_2^3 & & z_1z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}$$

$f(z_1, z_2) = 1 - az_1z_2$

$$\phi_4 = \frac{1}{2} + z_1z_2$$

$$\phi_7 = \frac{1}{2}z_1 + z_1^2z_2$$

$$\phi_8 = \frac{1}{2}z_2 + z_1z_2^2$$

$$\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4z_2 \\
 & & z_1^2 & & z_1^3z_2 & \\
 & z_1 & & z_1^2z_2 & & z_1^3z_2 \\
1 & & z_1z_2 & & z_1^2z_2^2 & \\
 & z_2 & & z_1z_2^2 & & z_1^2z_2^3 \\
 & & z_2^2 & & z_1z_2^3 & \\
 & & & z_2^3 & & z_1z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}$$

$f(z_1, z_2) = 1 - az_1 z_2$

$$\phi_4 = \frac{1}{2} + z_1 z_2$$
$$\phi_7 = \frac{1}{2} z_1 + z_1^2 z_2$$
$$\phi_8 = \frac{1}{2} z_2 + z_1 z_2^2$$

$$\phi_{12} = \frac{1}{3} + \frac{2}{3} z_1 z_2 + z_1^2 z_2^2$$
$$\phi_{17} = \frac{1}{3} z_1 + \frac{2}{3} z_1^2 z_2 + z_1^3 z_2^2$$
$$\phi_{18} = \frac{1}{3} z_2 + \frac{2}{3} z_1 z_2^2 + z_1^2 z_2^3$$

$$\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4 z_2 \\
 & & z_1^2 & & z_1^3 z_2 & \\
 & z_1 & & z_1^2 z_2 & & z_1^3 z_2 \\
1 & & z_1 z_2 & & z_1^2 z_2^2 & \\
 & z_2 & & z_1 z_2^2 & & z_1^2 z_2^3 \\
 & & z_2^2 & & z_1 z_2^3 & \\
 & & & z_2^3 & & z_1 z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}$$

$f(z_1, z_2) = 1 - az_1 z_2$

$$\phi_4 = \frac{1}{2} + z_1 z_2$$
$$\phi_7 = \frac{1}{2} z_1 + z_1^2 z_2$$
$$\phi_8 = \frac{1}{2} z_2 + z_1 z_2^2$$

$$\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4 z_2 \\
 & & z_1^2 & & z_1^3 z_2 & \\
 & z_1 & & z_1^2 z_2 & & z_1^3 z_2 \\
1 & & z_1 z_2 & & z_1^2 z_2^2 & \\
 & z_2 & & z_1 z_2^2 & & z_1^2 z_2^3 \\
 & & z_2^2 & & z_1 z_2^3 & \\
 & & & z_2^3 & & z_1 z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}$$

$$\phi_{12} = \frac{1}{3} + \frac{2}{3} z_1 z_2 + z_1^2 z_2^2$$
$$\phi_{17} = \frac{1}{3} z_1 + \frac{2}{3} z_1^2 z_2 + z_1^3 z_2^2$$
$$\phi_{18} = \frac{1}{3} z_2 + \frac{2}{3} z_1 z_2^2 + z_1^2 z_2^3$$

OG polynomials in weighted spaces

$f(z_1, z_2) = 1 - az_1z_2$

Theorem (MS, Sola (2020))

For $n = 0, 1, \ldots,$ let

$$r_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} (k+1)z^k.$$

Then the polynomials

$$\varphi^{(1)}_{M,m}(z_1, z_2) = z_1^M r_m(z_1z_2) \quad \textit{and} \quad \varphi^{(2)}_{N,n}(z_1, z_2) = z_2^N r_n(z_1z_2),$$

with $M, m, N, n \in \mathbb{N}_0$, form an orthogonal basis for $H^2_{1-z_1z_2}(\mathbb{D}^2)$.

STILL have time????

Thank You!

Summary

- Things are harder in several variables!
  - Must choose a monomial ordering (which one is "best"??)
  - Weakly inner doesn't imply inner!
- Other neat stuff
  - We actually found a closed form for the OG polynomials for $1 - az_1z_2$ (for spaces on the bidisk and the 2-ball)
  - And a closed form for the OG polynomials for $1 - a(z_1 + z_2)$ for the 2-ball (bidisk is harder)

OG polynomials in the weighted Drury-Arveson space

$f(z_1, z_2) = 1 - \frac{\sqrt{2}}{2}(z_1 + z_2)$

$$
\begin{aligned}
p_0^* &= \frac{1}{2} \\
p_1^* &= \frac{1}{12}\left(7 + 2\sqrt{2}z_1\right) \\
p_2^* &= \frac{1}{6}\left(4 + \sqrt{2}z_1 + \sqrt{2}z_2\right) \\
p_3^* &= \frac{1}{48}\left(33 + 10\sqrt{2}z_1 + 8\sqrt{2}z_2 + 6z_1^2\right) \\
p_4^* &= \frac{1}{48}\left(35 + 12\sqrt{2}z_1 + 10\sqrt{2}z_2 + 6z_1^2 + 12z_1z_2\right) \\
p_5^* &= \frac{1}{8}\left(6 + 2\sqrt{2}z_1 + 2\sqrt{2}z_2 + z_1^2 + 2z_1z_2 + z_2^2\right)
\end{aligned}
$$

OG polynomials in the weighted Drury-Arveson space

$f(z_1, z_2) = 1 - \frac{\sqrt{2}}{2}(z_1 + z_2)$

$$\phi_0 = 1$$

$$\phi_1 = \frac{1}{12}\left(1 + 2\sqrt{2}z_1\right)$$

$$\phi_2 = \frac{1}{12}\left(1 + 2\sqrt{2}z_2\right)$$

$$\phi_3 = \frac{1}{48}\left(1 + 2\sqrt{2}z_1 + 3z_1^2\right)$$

$$\phi_4 = \frac{1}{24}\left(1 + \sqrt{2}z_1 + \sqrt{2}z_2 + 6z_1z_2\right)$$

$$\phi_5 = \frac{1}{48}\left(1 + 2\sqrt{2}z_2 + 3z_2^2\right)$$

$$\begin{array}{cccccc}
 & & & & & z_1^5 \\
 & & & & z_1^4 & \\
 & & & z_1^3 & & z_1^4 z_2 \\
 & & z_1^2 & & z_1^3 z_2 & \\
 & z_1 & & z_1^2 z_2 & & z_1^3 z_2 \\
1 & & z_1 z_2 & & z_1^2 z_2^2 & \\
 & z_2 & & z_1 z_2^2 & & z_1^2 z_2^3 \\
 & & z_2^2 & & z_1 z_2^3 & \\
 & & & z_2^3 & & z_1 z_2^4 \\
 & & & & z_2^4 & \\
 & & & & & z_2^5
\end{array}$$

OG polynomials in the weighted Drury-Arveson space

$f(z_1, z_2) = 1 - \frac{\sqrt{2}}{2}(z_1 + z_2)$

$$\phi_0 = 1$$
$$\phi_1 = \frac{1}{12}\left(1 + 2\sqrt{2}z_1\right)$$
$$\phi_2 = \frac{1}{12}\left(1 + 2\sqrt{2}z_2\right)$$
$$\phi_3 = \frac{1}{48}\left(1 + 2\sqrt{2}z_1 + 3z_1^2\right)$$
$$\phi_4 = \frac{1}{24}\left(1 + \sqrt{2}z_1 + \sqrt{2}z_2 + 6z_1z_2\right)$$
$$\phi_5 = \frac{1}{48}\left(1 + 2\sqrt{2}z_2 + 3z_2^2\right)$$

					z_1^5
				z_1^4	
			z_1^3		$z_1^4 z_2$
		z_1^2		$z_1^3 z_2$	
	z_1		$z_1^2 z_2$		$z_1^3 z_2$
1		$z_1 z_2$		$z_1^2 z_2^2$	
	z_2		$z_1 z_2^2$		$z_1^2 z_2^3$
		z_2^2		$z_1 z_2^3$	
			z_2^3		$z_1 z_2^4$
				z_2^4	
					z_2^5

OG polynomials in the weighted Drury-Arveson space

Theorem

In Drury-Arveson space of the 2-ball, weighted by $f(z_1, z_2) = 1 - \frac{\sqrt{2}}{2}(z_1 + z_2)$*, let* $\phi_{j,k}(z_1, z_2)$ *be the first orthogonal polynomial containing* $z_1^j z_2^k$*. Then*

$$\phi_{j,k}(z_1, z_2) = \sum_{m=0}^{j} \sum_{n=0}^{k} \hat{\phi}_{j,k}(m, n) z_1^m z_2^n \tag{3}$$

where the coefficients $\hat{\phi}_{j,k}(z_1, z_2)$ *are given by*

$$\hat{\phi}_{j,k}(m, n) = \left(\frac{\sqrt{2}}{2}\right)^{j+k-m-n} \frac{(m+n+1)!}{(j+k+1)!} \left(\frac{j!k!}{m!n!} \frac{(j+k-m-n)!}{(j-m)!\,(k-n)!}\right). \tag{4}$$

We also have that

$$\|\phi_{j,k}\|_f^2 = \frac{j+k+2}{j+k+1} \frac{j!k!}{(j+k)!}. \tag{5}$$

Maybe just stop talking

Thank You!

Summary

- Things are harder in several variables!
  - Must choose a monomial ordering (which one is "best"??)
  - Weakly inner doesn't imply inner!
- Other neat stuff
  - We actually found a closed form for the OG polynomials for $1 - az_1z_2$ (for spaces on the bidisk and the 2-ball)
  - And a closed form for the OG polynomials for $1 - a(z_1 + z_2)$ for the 2-ball (bidisk is harder)

The Drury-Arveson space (and friends) are special

- The bidisk analog of $f(z,w) = 1 - \frac{\sqrt{2}}{2}(z_1 + z_2)$ is $f(z,w) = 1 - \frac{1}{2}(z_1 + z_2)$

$$
\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= \frac{1}{3} + z_1 \\
\phi_2 &= \frac{5}{16} - \frac{1}{16} z_1 + z_2 \\
\phi_3 &= \frac{2}{17} + \frac{32}{85} z_1 - \frac{2}{85} z_2 + z_1^2 \\
\phi_4 &= \frac{51}{223} + \frac{74}{223} z_1 + \frac{79}{223} z_2 - \frac{25}{446} z_1^2 + z_1 z_2 \\
\phi_5 &= \frac{208}{2039} - \frac{98}{2039} z_1 + \frac{722}{2039} z_2 - \frac{11}{2039} z_1^2 - \frac{130}{2039} z_1 z_2 + z_2^2 \\
&\vdots
\end{aligned}
$$

this one is just for funsies

Thank You!

Summary

- Things are harder in several variables!
  - Must choose a monomial ordering (which one is "best"??)
  - Weakly inner doesn't imply inner!
- Other neat stuff
  - We actually found a closed form for the OG polynomials for $1 - az_1z_2$ (for spaces on the bidisk and the 2-ball)
  - And a closed form for the OG polynomials for $1 - a(z_1 + z_2)$ for the 2-ball (bidisk is harder)

Zero sets of Optimal Approximants

The Shanks Conjecture

Theorem (Bénéteau, Khavinson, Liaw, Seco, and Sola (JLMS 2016))

Let $f \in D_\alpha$ have $f(0) \neq 0$ and let (p_n) be the optimal approximants to $1/f$.

(i) *For $\alpha \geq 0$, $Z(p_n) \cap \overline{\mathbb{D}} = \varnothing$ for all n.*

(ii) *for $\alpha < 0$, $Z(p_n) \cap \overline{D(0, 2^{\alpha/2})} = \varnothing$ for all n.*[a]

[a] improved by Beneteau, Khavinson, Liaw, Seco, and Simanek in 2019

Zero sets of Optimal Approximants

The Shanks Conjecture

Theorem (Bénéteau, Khavinson, Liaw, Seco, and Sola (JLMS 2016))

Let $f \in D_\alpha$ have $f(0) \neq 0$ and let (p_n) be the optimal approximants to $1/f$.

(i) *For $\alpha \geq 0$, $Z(p_n) \cap \overline{\mathbb{D}} = \varnothing$ for all n.*

(ii) *for $\alpha < 0$, $Z(p_n) \cap \overline{D(0, 2^{\alpha/2})} = \varnothing$ for all n.*[a]

[a] improved by Beneteau, Khavinson, Liaw, Seco, and Simanek in 2019

The Shanks Conjecture (Shanks, Treitel, and Justice 1972)

Given a polynomial f, the optimal approximants to $1/f$ in $H^2(\mathbb{D}^2)$ will be zero-free in the bidisk $\mathbb{D}^2$.

Zero sets of Optimal Approximants

Theorem (Bénéteau, Khavinson, Liaw, Seco, and Sola (JLMS 2016))

Let $f \in D_\alpha$ have $f(0) \neq 0$ and let (p_n) be the optimal approximants to $1/f$.

(i) *For $\alpha \geq 0$, $Z(p_n) \cap \overline{\mathbb{D}} = \varnothing$ for all n.*

(ii) *for $\alpha < 0$, $Z(p_n) \cap \overline{D(0, 2^{\alpha/2})} = \varnothing$ for all n.*[a]

[a] improved by Beneteau, Khavinson, Liaw, Seco, and Simanek in 2019

The Shanks Conjecture (Shanks, Treitel, and Justice 1972)

Given a polynomial f, the optimal approximants to $1/f$ in $H^2(\mathbb{D}^2)$ will be zero-free in the bidisk $\mathbb{D}^2$.

Weakest Shanks' Conjecture

An irreducible polynomial, b, with no zeros in the bidisk, yields OA polynomials for $1/b$ that are zero free in the bidisk.

Counterexamples to the Weakest Shanks' Conjecture

In the Bergman Space

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1z_2 + z_1^2z_2 - z_2^2 + z_1z_2^2.$$

Counterexamples to the Weakest Shanks' Conjecture

In the Bergman Space

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1 z_2 + z_1^2 z_2 - z_2^2 + z_1 z_2^2.$$

$$p_2 = \frac{4}{835}\left(-\frac{1267}{27} - 24z_1 - 24z_2\right)$$

Counterexamples to the Weakest Shanks' Conjecture

In the Bergman Space

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1z_2 + z_1^2z_2 - z_2^2 + z_1z_2^2.$$

$$p_2 = \frac{4}{835}\left(-\frac{1267}{27} - 24z_1 - 24z_2\right)$$

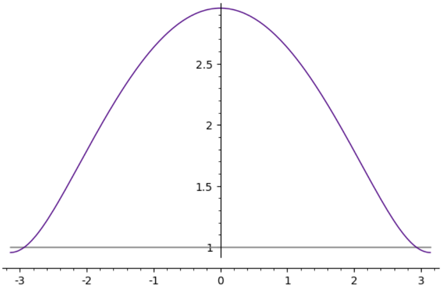
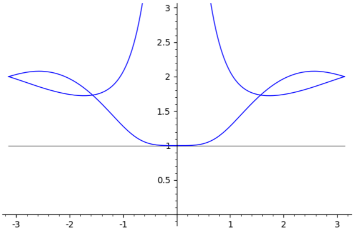


Figure: Solving $p_2(z_1, e^{it}) = 0$ for z_1 and plotting against $t \in (0, 2\pi)$. Note that p_2 is symmetric in z_1 and z_2

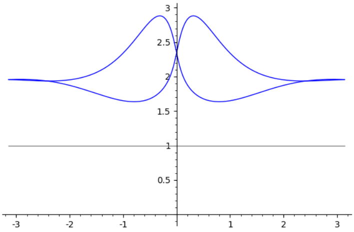
Counterexamples to the Weakest Shanks' Conjecture

In the Bergman Space

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1z_2 + z_1^2z_2 - z_2^2 + z_1z_2^2.$$



(a) Solving $b(z_1, e^{it}) = 0$ for z_1 and plotting against $t \in (0, 2\pi)$



(b) Solving $b(e^{it}, z_2) = 0$ for z_2 and plotting against $t \in (0, 2\pi)$

Figure: Zero sets of b

Do these counterexamples work in the Hardy space?

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1z_2 + z_1^2z_2 - z_2^2 + z_1z_2^2.$$

$$p_2 = -\frac{14}{75} - \frac{2}{25}z_1 - \frac{2}{25}z_2$$

Do these counterexamples work in the Hardy space?

$$b(z_1, z_2) = -4 + 3z_1 - z_1^2 + 3z_2 - 2z_1z_2 + z_1^2z_2 - z_2^2 + z_1z_2^2.$$

$$p_2 = -\frac{14}{75} - \frac{2}{25}z_1 - \frac{2}{25}z_2$$

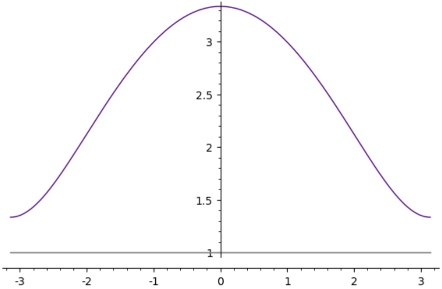


Figure: Solving $p_2(z_1, e^{it}) = 0$ for z_1 and plotting against $t \in (0, 2\pi)$. Note that p_2 is symmetric in z_1 and z_2

Thank You!

Summary

- Things are harder in several variables!
  - Must choose a monomial ordering (which one is "best"??)
  - Weakly inner doesn't imply inner!
- Counter examples in the Bergman space; Hardy?? (Is modified Shanks' true here?)
- Other neat stuff
  - We actually found a closed form for the OG polynomials for $1 - az_1z_2$ (for spaces on the bidisk and the 2-ball)
  - And a closed form for the OG polynomials for $1 - a(z_1 + z_2)$ for the 2-ball (bidisk is harder)