Convexity in Free Analysis

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Introduction

Free convexity basics in outline

- LMIs and spectrahedra
- Free spectrahedra
- Free and matrix convex sets
- The Effros-Winkler Theorem
- The Linear Gleichstellensatz

LMIs and spectrahedra

- For selfadjoint matrices $A_1 \dots, A_{\mathrm{g}} \in \mathbb{S}_d$, the expression

$$L_A(x) = I - \left[A_1 x_1 + \cdots + A_{\mathrm{g}} x_{\mathrm{g}}\right]$$

  is a *(monic) linear pencil* of size d.

- $L_A(x) \succeq 0$ ($L_A(x) \succ 0$) is a linear matrix inequality (LMI)
- Its (scalar) solution set

$$\mathcal{D}_A[1] = \left\{x \in \mathbb{R}^{\mathrm{g}} \mid L_A(x) \succeq 0\right\}$$

$$\mathfrak{P}_A[1] = \left\{x \in \mathbb{R}^{\mathrm{g}} \mid L_A(x) \succ 0\right\}$$

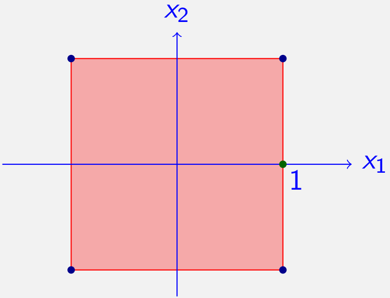
  is an LMI domain or a spectrahedron.

LMIs and spectrhahedra

Polyhedra are spectrahedra

$$L_A(x_1, x_2) = I_4 - \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{bmatrix} x_1 - \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} x_2.$$

$\mathcal{D}_A[1]$ is the square $[-1, 1]^2 \subseteq \mathbb{R}^2$:

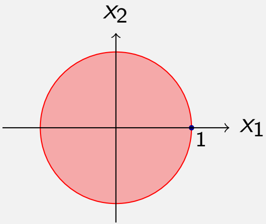


LMIs and spectrahedra

Examples of spectrahedra: Balls

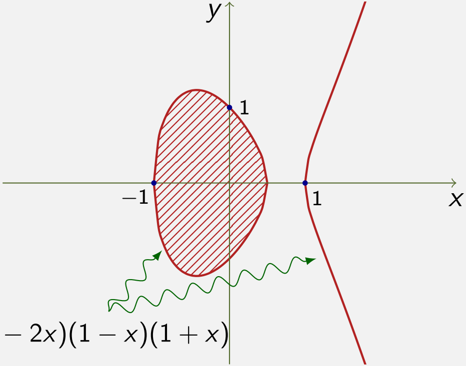
$$L_A(x_1, x_2) \quad = \quad I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_1 - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & -x_1 & -x_2 \\ -x_1 & 1 & 0 \\ -x_2 & 0 & 1 \end{bmatrix}.$$

$\mathcal{D}_A[1] = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ is the closed unit ball in $\mathbb{R}^2$



LMIs and spectrahedra
The interior of an elliptic curve - why real algebraic geometers care

$$L(x,y) = I + x\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{\sqrt{3}}\, y \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$y^2 = (1-2x)(1-x)(1+x)$$

Thank Igor

LMIs and spectrahedra

Spectrahedra in action - a few reasons to care

- Convex optimization and semidefinite programming (SDP), practical with advances in interior point methods.
- real algebraic geometry and determinantal representations.
- Systems engineering and robust control.

Free spectrahedra ...
... relax, go free

For $A_1, \ldots, A_{\mathrm{g}} \in \mathbb{S}_d$ and $X = (X_1, \ldots, X_{\mathrm{g}}) \in \mathbb{S}_n^{\mathrm{g}}$

- The monic linear pencil L_A *evaluates* at X as

$$L_A(X) = I_d \otimes I_n - \sum_{j=1}^{\mathrm{g}} A_j \otimes X_j \in \mathbb{S}_d \otimes \mathbb{S}_n = \mathbb{S}_{dn}.$$

- For each dimension $n \in \mathbb{N}$,

$$\mathcal{D}_A[n] := \left\{ X \in \mathbb{S}_n^{\mathrm{g}} \mid \; L_A(X) \succeq 0 \right\} \subseteq \mathbb{S}^{\mathrm{g}}$$

  are natural relaxations of $\mathcal{D}_A[1]$.

- Completely relaxed: *free spectrahedron* is $\mathcal{D}_A := (\mathcal{D}_A[n])_{n=1}^{\infty}$.
- $\mathcal{D}_A$ is levelwise convex.
- We assume $\mathcal{D}_A$ is bounded. In fact, everything is bounded.

Free spectrahedra ...
... relax, go free

For $A_1, \ldots, A_{\mathrm{g}} \in \mathbb{S}_d$ and $X = (X_1, \ldots, X_{\mathrm{g}}) \in \mathbb{S}_n^{\mathrm{g}}$

- The monic linear pencil L_A evaluates at X as

$$L_A(X) = I_d \otimes I_n - \sum_{j=1}^{\mathrm{g}} A_j \otimes X_j \in \mathbb{S}_d \otimes \mathbb{S}_n = \mathbb{S}_{dn}.$$

- For each dimension $n \in \mathbb{N}$,

$$\mathfrak{P}_A[n] := \left\{X \in \mathbb{S}_n^{\mathrm{g}} \mid \ L_A(X) \succ 0\right\} \subseteq \mathbb{S}^{\mathrm{g}}.$$

are relaxations of $\mathfrak{P}_A[1]$.

- Completely relaxed: *free spectrahedron* is $\mathfrak{P}_A := (\mathfrak{P}_A[n])_{n=1}^{\infty}$.
- $\mathfrak{P}_A$ is levelwise convex.

Free spectrahedra ...

... a few of the reasons to care

$$L_A(X) = I - \sum A_j \otimes X_j, \quad \mathcal{D}_A = \{X : L_A(X) \succeq 0\} \quad \mathfrak{P}_A = \{X : L_A(X) \succ 0\}$$

- Systems engineering.
- The span of $\{I, A_1, \ldots, A_g\}$ is an operator system.
- Connected to cp (completely positive) maps
- Quantum Information Theory.
- Spectrahedral inclusions, the matrix cube problem: $\mathcal{D}_A \subseteq \mathcal{D}_B$ is tractable; $\mathcal{D}_A[1] \subseteq \mathcal{D}_B[1]$ not so much.

Free sets

$$\mathcal{D}_A := (\mathcal{D}_A[n])_{n=1}^{\infty} \subseteq \mathbb{S}^{\mathrm{g}}, \quad \mathfrak{P}_A := (\mathfrak{P}_A[n])_{n=1}^{\infty} \subseteq \mathbb{S}^{\mathrm{g}}.$$

- Let $\mathbb{S}^{\mathrm{g}} = (\mathbb{S}_n^{\mathrm{g}})_n$, the free universe.
- A *free set* $S \subseteq \mathbb{S}^{\mathrm{g}}$ is a sequence $S = (S[n])_n$ satisfying,
  - (a) $S[n] \subseteq \mathbb{S}_n^{\mathrm{g}}$;
  - (b) closed wrt direct sums: If $X \in S[n]$ and and $Y \in S[m]$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} X_{\mathrm{g}} & 0 \\ 0 & Y_{\mathrm{g}} \end{pmatrix} \right) \in S[n+m];$$

  - (c) closed wrt unitary similarity: If $X \in S[n]$ and U is an $n \times n$ unitary, then

$$U^* X U := \left(U^* X_1 U, \quad \cdots, \quad U^* X_{\mathrm{g}} U \right) \in S[n].$$

- It is evident that $\mathcal{D}_A$ (resp $\mathfrak{P}_A$) is a free set.

Matrix convex sets

- A free set $S \subseteq \mathbb{S}^{\mathrm{g}}$ is matrix convex if for each $X \in S[n]$ and isometry $V : \mathbb{C}^m \to \mathbb{C}^n$,

$$V^*XV = (V^*X_1V, \ldots, V^*X_{\mathrm{g}}V) \in S[m];$$

- Each $S[n]$ is convex: For $X, Y \in S[n]$,

$$\begin{pmatrix} \frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} \frac{I}{\sqrt{2}} \\ \frac{I}{\sqrt{2}} \end{pmatrix} = \frac{X+Y}{2};$$

- Free spectrahedra are matrix convex. They are the free analog of a half plane in convex analysis.
  `Proof.`

$$(V \otimes I_n)^* L(X)(V \otimes I_n) = L(V^*XV).$$

The Effros-Winkler Separation Theorem

The E-W Matricial Hahn-Banach Separation Theorem.

If $S \subseteq \mathbb{S}^{\mathrm{g}}$ is closed, matrix convex and contains 0 and if $Y \in \mathbb{S}^{\mathrm{g}}_{\ell} \setminus S[\ell]$, then there is a monic linear pencil $L = I_\ell - \sum A_j \otimes x_j$ of size ℓ such that

$$L(S) \succeq 0, \quad L(Y) \not\succeq 0.$$

Thus $S \subseteq \mathcal{D}_A$, but $Y \notin \mathcal{D}_A$.

$$S = \bigcap \{\mathcal{D}_A : S \subseteq \mathcal{D}_A\}.$$

The Linear Gleichstellensatz

Suppose $\mathcal{D}$ is a free spectrahedron. A tuple $A \in \mathbb{S}_d^{\mathrm{g}}$ is minimal for $\mathcal{D}$ if $\mathcal{D}_A = \mathcal{D}$ and if $B \in \mathbb{S}_e^{\mathrm{g}}$ and $\mathcal{D}_B = \mathcal{D}$, then $d \leq e$.

The Linear Gleichstellensatz [Helton, Klep, M]. If $\mathcal{D}$ is a free spectrahedron, then there is a minimal A such that $\mathcal{D} = \mathcal{D}_A$.

If A and B are both minimal for $\mathcal{D}$, then

$$A = U^* B U.$$

$\mathcal{D}$ determines A; $\mathcal{D}[1]$ does not.

Global outline

- Free convexity basics
- Free semialgebraic sets - and matrix inequalities
  - Free polynomials
  - Motivation
  - Convex semialgebraic sets
  - Convex polynomials
  - Quasiconvexity and Volcic's Free Bertini Theorem
  - The convex positivstellensatz - a side trip.
- Partial convexity and rational functions
- Extreme Points
- Some Analytic Theory.

Free polynomials ...
... and their evaluations

- $x = (x_1, \ldots, x_g)$ freely noncommuting variables;
- $\alpha = x_{i_1} x_{i_2} \cdots x_{i_m} \in \langle x \rangle$ is a *word*;
- $\mathbb{C}\langle x \rangle$ is the *free algebra* of noncommutative polynomials; e.g.,

$$p(x) = 5 + 2x_1x_2 - 3x_2x_1 + x_1^2x_2x_1, \quad q(x) = x_1x_2 - x_2x_1;$$

- for $X = (X_1, \ldots, X_g) \in \mathbb{S}_n^g$,

$$X^\alpha = X_{i_1} X_{i_2} \cdots X_{i_m}; \quad p(X) = 5I_n + 2X_1X_2 - 3X_2X_1 + X_1^2X_2X_1;$$

- $p = \sum p_\alpha \otimes \alpha \in M_\mu(\mathbb{C}\langle x \rangle)$ is *evaluated* at $X \in M_n(\mathbb{C})^g$ by

$$p(X) = \sum p_\alpha \otimes X^\alpha \in M_\mu(\mathbb{C}) \otimes M_n(\mathbb{C})$$
$$p(X) = \left(p_{j,k}(X)\right)_{j,k=1}^{\mu} \in M_\mu(M_n(\mathbb{C})).$$

Free polynomials

The positivity domain - semialgebraic sets

$$p(X) = \sum p_\alpha \otimes X^\alpha = (p_{j,k}(X)) \in M_\mu(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

- $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is *symmetric* if, for $X \in \mathbb{S}^{\mathrm{g}}$,

$$p(X)^* = p(X);$$

- The *positivity domain* $\mathfrak{P}_p$ *of* p is the sequence $(\mathfrak{P}_p[n])_n$,

$$\mathfrak{P}_p[n] = \{X \in \mathbb{S}_n^{\mathrm{g}} : p(X) \succ 0\};$$

- $\mathfrak{P}_p$ is a free set. It is a *(basic) free semialgebraic set*;
- For $A \in \mathbb{S}_d^{\mathrm{g}}$ and $L_A(x) = I - \sum A_j x_j$,

$$\mathfrak{P}_A = \mathfrak{P}_{L_A} = \{X : L_A(X) \succ 0\}.$$

Motivation
Engineering reality

The system of Matrix Inequalities

$$
\begin{aligned}
AX + XA^T + X(\gamma^2 - C)X &\prec 0 \\
BY + YB^T + Y(\gamma^2 - D)Y &\prec 0 \\
X, Y &\prec 0 \\
X - Y^{-1} &\prec 0
\end{aligned}
$$

arises in linear systems theory - [Doyle,Glover,Karganakar,Francis].

- $X = X^T$, $Y = Y^T$, $C = C^T$, $D = D^T$;
- X, Y are unknowns, A, B, C, D, γ are knowns (system parameters);
- the inequalities depend only on the signal flow diagram.
- the sizes depend upon the particular system;

Motivation

Convexity and the Ricatti inequality

The Ricatti inequality is a simple ubiquitous example of a (scalar) Matrix Inequality,

$$AX + XA^T - XBB^TX + C^TC \succeq 0.$$

Its solution set is convex in X (also separately in A and B), since the Ricatti inequality is equivalent to the Linear Matrix Inequality,

$$\begin{pmatrix} I & B^TX \\ XB & AX + XA + C^TC \end{pmatrix} \succeq 0.$$

Motivation
Matrix inequalities

The take away: Some systems engineering problems are modeled by matrix inequalities:

$$p(A, X) \succ 0.$$

- p is determined by the signal flow diagram;
- The A are the known unknowns - the plant;
- The X are the unknown unknowns (unc uncs to Rumsfeld);
- Convexity in X – for fixed A is desirable;
- More generally, p could be rational.

Convex Semialgebraic sets ...
... are free spectrahedra

Convex Trivialization Theorem [Helton, M 12] [Kriel 19].
Suppose $p \in M_\mu(\mathbb{C}{<}x{>})$ is symmetric and $p(0) \succ 0$.

The basic free semialgebraic set $\mathfrak{P}_p = \{X \in \mathbb{S}^g : p(X) \succ 0\}$ is convex[1] if and only if it is a free spectrahedron, $\mathfrak{P}_A$.

- That $\mathfrak{P}_p$ is a (possibly infinite) intersection of free spectrahedra by EW HB separation Theorem;
- Not so easy to see it is in fact a single free spectrahedron;
- True for p a free rational function [HM14].

[1] $\mathfrak{P}_p$ is matrix convex iff each $\mathfrak{P}_p[n]$ is convex as a subset of $\mathbb{S}^g_n$.

Convex polynomials

- A symmetric $f \in \mathbb{C}<x>$ is convex on a free set $S \subseteq \mathbb{S}^{\mathrm{g}}$ if

$$H_f(X, Y) := \frac{f(X) + f(Y)}{2} - f\left(\frac{X+Y}{2}\right) \succeq 0, \quad X, Y \in S[n];$$

- $f(x) = x^4$ is not convex on any open set in $\mathbb{S}_2^1$;
- If f is globally convex, then, $\{-f(X) \succ 0\}$ is convex, in fact

$$C_\tau = \{X \in \mathbb{S}^{\mathrm{g}} : f(X) \prec \tau I\}$$

is matrix convex for each $\tau \in \mathbb{R}$; r is *quasiconvex*.

Convex polynomials ...

... are trivial

- Suppose $f \in \mathbb{C}<x>$ is selfadjoint:

$$f(X)^* = f(X), \quad X \in \mathbb{S}_n^g$$

- f is convex on a free set $S \subseteq \mathbb{S}^g$ if

$$H_f(X,Y) := \frac{f(X)+f(Y)}{2} - f\left(\frac{X+Y}{2}\right) \succeq 0, \quad X, Y \in S[n].$$

Local-Global and SoS [Helton,M] For $f^* = f \in \mathbb{C}<x>$, TFAE:

- f is convex on some nonempty open free set;
- f is globally convex;
- there exists an linear $\ell \in \mathbb{R}\langle x\rangle$ and $\Lambda_j \in \mathbb{C}\langle x\rangle$ such that

$$f(X) = f(0) + \ell(X) + \sum_j \Lambda_j(X)^* \Lambda_j(X).$$

In particular, f has degree (at most) two.

Quasiconvex polynomials
Convex positivity domain

$$f(X) = \lambda(X) + \sum_j \Lambda_j(X)^* \Lambda_j(X), \quad \text{degree two.}$$

If f is globally convex, then, for each $\tau \in \mathbb{R}$, the set C_τ is convex:

$$C_\tau = \{X \in \mathbb{S}^g(\mathbb{R}) : f(X) \prec \tau I\}.$$

Theorem [Helton, Klep, M, Volčič[2]] If (1) $f \in \mathbb{R}\langle x \rangle$ is *irreducible* as an element of $\mathbb{C}\langle x \rangle$, (2) $f(0) \succ 0$ and (3)

$$\mathfrak{P}_f = \{X : f(X) \succ 0\}$$

is convex, then f is convex.

[2] Jurij's variant.

Quasiconvex polynomials
The Free Bertini Theorem

Volčič's Free Bertini Theorem. Suppose $f \in \mathbb{C}\langle x\rangle \setminus \mathbb{C}$.

$f - \tau$ is not irreducible in $\mathbb{C}\langle x\rangle$ for infinitely many $\tau \in \mathbb{C}$
if and only if
there exists a $p \in \mathbb{C}[t]$, $\deg p > 1$, and a $q \in \mathbb{C}\langle x\rangle$ such that

$$f = p \circ q.$$

- $(\Leftarrow)$ $f - \tau = (p - \tau) \circ q$ and thus $f - \tau$ factors for all τ;
- The (entirely nontrivial) converse uses Cohn's theory of the free algebra and free skew fields; and Bergman's centralizer theorem.

Quasiconvexity
Convexity corollary to Free Bertini

The selfadjoint $f \in \mathbb{R}\langle x \rangle$ with $f(0) = 0$ is *locally quasiconvex*

$$C_\tau = \{X \in \mathbb{S}^{\mathrm{g}}(\mathbb{R}) : f(X) \prec \tau\}$$

is convex for all τ in some open interval $(0, \epsilon)$.

`Corollary.` [Volčič] If f is locally quasiconvex, then either

(i) $-f = \sum_{j=1}^{\mathrm{g}} g_j^* g_j$ (a hermitian SoS); or

(ii) $f = p \circ g$, where $g \in \mathbb{R}\langle x \rangle$ is globally convex; and $p \in \mathbb{R}[t]$ and there is an iff version with more information about p.

The convex positivstellensatz

A bit of free real algebraic geometry

- Positivstellensätze are central to real algebraic geometry. They are *algebraic certificates* for a polynomial p to be positive on a semialgebraic set;
- Free analogs are typically much cleaner. E.g.

`The convex positivstellensatz.` [Helton, Klep, M] Consider

$$L_A(x) = I - \sum A_j x_j, \quad \mathcal{D}_A = \{X : L_A(X) \succeq 0\}$$

and suppose $p \in M_\mu(\mathbb{C}\langle x \rangle)$ is a symmetric.

$$p(\mathcal{D}_A) \succeq 0 \iff p(x) = \sum_j s_j(x)^* s_j(x) + \sum f_k(x)^* L_A(x) f_k(x),$$

where s_j, f_k are polynomials of degree at most $\lceil \frac{\deg p}{2} \rceil$.

The convex positivstellensatz
Bianalytic maps between free spectrahedra

$$p(\mathcal{D}_A) \succeq 0 \Longleftrightarrow p = \sum_j s_j^* s_j + \sum f_k^* L_A f_k,$$

The convex positivstellensatz is a point of departure for studying free bianalytic maps between free spectrahedra:

- $f : \mathcal{D}_A \to \mathcal{D}_B$ if and only if $p(x) = L_B(f(x)) \succeq 0$ on $\mathcal{D}_A$;
- Polynomial approximation uniformly on compact subsets[3] [Agler, McCarthy];
- See Nicole Tuovila's talk at 3:30 today.

[3] Disclaimer: In the free free setting

Partial Convexity and Rational Functions
Outline

- Free convexity basics
- Free semialgebraic sets
- Partial convexity and rational functions
  - Rational functions and realizations
  - The domain of a rational function
  - Partially convexity
- Extreme points
- Some Analytic Theory.

Rational functions
Realizations

- A symmetric (free) rational function $r \in M_\mu(\mathbb{C}\langle\!\langle x \rangle\!\rangle)$ that is regular at 0 has a symmetric *descriptor realization*

$$r = c^* \left(J - \sum_{k=1}^{g} T_k x_k \right)^{-1} c,$$

where, for some positive integer e, $J, T_k \in \mathbb{S}_e$, $c \in M_{e,\mu}$ and $J = J^* = J^{-1}$ is a signature matrix;

- We view r as a function: r evaluates at a tuple $X \in \mathbb{S}_n^g$ as

$$r(X) = (c^* \otimes I_n) \left(J \otimes I_n - \sum_{k=1}^{g} T_k \otimes X_k \right)^{-1} (c \otimes I_n);$$

- $r(X)^* = r(X) \in \mathbb{S}$ (symmetric);
- $r[n] : \mathbb{S}_n^g \dashrightarrow \mathbb{S}_{n\mu}$.

Rational functions

Rational Expressions - an example

A rational function is an equivalence class of *rational expressions*; e.g.,

$$
\begin{aligned}
r = r(x_1, x_2) = &\left(1 - x_2 - x_1(1 - x_2)^{-1}x_1\right)^{-1} \\
&= x_1^{-1}(1 - x_2)\left[(1 - x_2)x_1^{-1}(1 - x_2) - x_1\right]^{-1}.
\end{aligned}
$$

- The expressions agree on tuples X where they are both defined;
- The first, but not the second, is defined at $(1, 1)$;
- The second, but not the first, is defined at $(0, 0)$;
- The realization

$$
r = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I_2 - x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - x_2 I_2 \right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

  is defined at both $(1, 1)$ and $(0, 0)$.

Rational functions
The domain of a rational function

$$r = c^* \left(J - \sum_{k=1}^{g} T_k x_k \right)^{-1} c, \quad J, T_k \in \mathbb{S}_e; \quad c \in M_{e,\mu}; J = J^* = J^{-1}.$$

- If e is the smallest over all realizations, then the realization is *minimal*;
- The *resolvent* is

$$R(X) = \left(J \otimes I_n - \sum_{k=1}^{g} T_k \otimes X_k \right)^{-1}.$$

- minimal realizations are essentially unique - in particular, invertibility of $R(X)$ does not depend upon the choice of minimal realization.

Rational functions

The domain of a rat function - singularities can't hide

$$r = c^* \left(J - \sum_{k=1}^{g} T_k x_k \right)^{-1} c, \quad J, T_k \in \mathbb{S}_e; \quad c \in M_{e,\mu}; J = J^* = J^{-1}.$$

- If e is the smallest over all realizations, then the realization is minimal.
- $R(x) = (J - \sum T_k x_k)^{-1}$ is the resolvent;

The Rational Domain Theorem [Volčič] [KVerbovetskyi-Vinnikov] justifies calling $\operatorname{dom} r = (\operatorname{dom} r[n])_n$ *the domain* of r, where

$$\operatorname{dom} r[n] = \{X \in \mathbb{S}_n^g : R(X) \text{ exists } \} \subseteq \mathbb{S}_n^g.$$

Rational functions

The domain of a rat function - singularities can't hide

The Rational Domain Theorem [Volčič] [KVerbovetskyi-Vinnikov] justifies calling $\operatorname{dom} r = (\operatorname{dom} r[n])_n$ *the domain* of r, where

$$\operatorname{dom} r[n] = \{X \in \mathbb{S}_n^{\mathrm{g}} : R(X) \text{ exists } \} \subseteq \mathbb{S}_n^{\mathrm{g}}.$$

- ★ $\operatorname{dom} r$ is the largest free set contained in the *ordinary* domains of the $r[n]$.
- In the one variable case, the domain of the resolvent is the domain of r.

- Given positive integers $\mathtt{h}$ and $\mathtt{g}$, write

$$(A,X) = (A_1,\ldots,A_{\mathtt{h}},X_1,\ldots,X_{\mathtt{g}}) \in \mathbb{S}_n^{\mathtt{h}} \times \mathbb{S}_n^{\mathtt{g}} = \mathbb{S}_n^{\mathtt{h}+\mathtt{g}};$$

- A subset $S \subseteq \mathbb{S}_n^{\mathtt{h}} \times \mathbb{S}_n^{\mathtt{g}}$ is *convex (resp. open) in x*, or **partially convex** if for each $A \in \mathbb{S}_n^{\mathtt{h}}$ the *slice*

$$S[A] = \{X \in \mathbb{S}_n^{\mathtt{g}} : (A,X) \in S\} \subseteq \mathbb{S}_n^{\mathtt{g}}$$

is convex (resp open).

Partial convexity
Free rational functions in a and x

- Let $a = (a_1, \ldots, a_{\mathrm{h}})$ and $x = (x_1, \ldots, x_{\mathrm{g}})$ be collections of freely noncommuting variables.
- A symmetric (free) rational function $r \in M_\mu(\mathbb{C}\langle\!\!\langle a, x \rangle\!\!\rangle)$ that is regular at 0 has a symmetric descriptor realization

$$r = c^* \left(J - \sum_{j=1}^{\mathrm{h}} S_j a_j - \sum_{k=1}^{\mathrm{g}} T_k x_k \right)^{-1} c,$$

  where, for some positive integer e, $J, S_j, T_k \in \mathbb{S}_e$ and $J = J^* = J^{-1}$ is a signature matrix and $c \in M_{e,\mu}(\mathbb{C})$;
- $\operatorname{dom} r \subseteq \mathbb{S}^{\mathrm{h}} \times \mathbb{S}^{\mathrm{g}} := \mathbb{S}^{\mathrm{h}+\mathrm{g}}$.

Partial convexity

Partially convex functions

$$r = c^* \left(J - \sum_{j=1}^{\mathtt{h}} S_j a_j - \sum_{k=1}^{\mathtt{g}} T_k x_k \right)^{-1} c \;=\; c^* R(a, x) c.$$

Suppose $S \subseteq \operatorname{dom} r$ is convex in x. The function r is *convex in x*, or *partially convex* on S if, for each $A \in \mathbb{S}^{\mathtt{h}}$, the function

$$S[A] \ni X \mapsto r(A, X)$$

is convex: that is, for each A and $X, Y \in S[A]$,

$$H_r(A; X, Y) = \frac{r(A, X) + r(A, Y)}{2} - r\left(A, \frac{X + Y}{2}\right) \succeq 0.$$

Partial convexity

Partially convex rational functions

$$r = c^* \left(J - \sum_{j=1}^{\mathtt{h}} S_j a_j - \sum_{k=1}^{\mathtt{g}} T_k x_k \right)^{-1} c \;=\; c^* R(a,x) c.$$

- Let V_T : range $T \to \mathbb{C}^e$ denote the inclusion;
- Let $R_T(a,x) = V_T^*(J - \sum S_j a_j - \sum T_k x_k)^{-1} V_T$;

$$R_T(A,X) = (V_T \otimes I_n)^* R(A,X)(V_T \otimes I_n);$$

- Let

$$\operatorname{dom}^+ r[n] = \{(A,X) \in \operatorname{dom} r : R_T(A,X) \succeq 0\}.$$

Partial convexity
The domain of partial convexity

$$r = c^* \left(J - \sum_{j=1}^{\mathrm{h}} S_j a_j - \sum_{k=1}^{\mathrm{g}} T_k x_k \right)^{-1} c \; = \; c^* R(a,x) c,$$

$$R_T(a,x) = V_T^* R(a,x) V, \quad \Lambda_T[H] = \sum T_j \otimes H_j,$$

$$r_{xx}(a,x)[h] = c^* R(A,X) \Lambda_T[H] R_T(A,X) \Lambda_T[H] R(A,X)(c \otimes I_n),$$

$$\operatorname{dom}^+ r = \{(A,X) \in \operatorname{dom} r : R_T(A,X) \succeq 0\}.$$

$\operatorname{dom}^+ r$ is the *domain of partial convexity* [JKMMP].

- $\operatorname{dom}^+ r$ is both open in x and convex in x;
- r is convex in x on $\operatorname{dom}^+ r$;
- Conversely, if r is convex in x on the free set $S \subseteq \operatorname{dom} r$, then $S \subseteq \operatorname{dom}^+ r$.

Partial convexity

Partially convex rational functions – the fine print

$$\frac{r(A,X)+r(A,Y)}{2} - r\Big(A, \frac{X+Y}{2}\Big) \succeq 0, \quad R_T(a,x) = V_T^* R(a,x) V_T,$$
$$r_{xx}(a,x)[h] = c^* R(A,X)\Lambda_T[H] R_T(A,X) \Lambda_T[H] R(A,X)(c \otimes I_n),$$

$\operatorname{dom}^+ r = \{R_T(A,X) \succeq 0\}$.

Theorem [Jury, Klep, Mancuso, M, Pascoe].

- $\operatorname{dom}^+ r$ is both open in x and convex in x;
- If r is convex in x on some free open set, then r is convex in x on $\operatorname{dom}^+ r$;
- Conversely, if (1) $S \subseteq \operatorname{dom} r$ is a free set that is convex in x; (2) if S contains a nonempty free open set; and (3) if r is convex in x on S, then $S \subseteq \operatorname{dom}^+ r$.

Partial convexity
An algebraic certificate of partial convexity

`The root butterfly realization` [Jury, Klep, Mancuso, M, Pascoe].

$r \in \mathbb{C}\langle a, x \rangle$ is convex in x in a neighborhood of 0 if and only if

$$r(a,x) = \ell(a,x) + \Sigma(a,x)^* \sqrt{w(a)} \left(I - \sum [\sqrt{w(a)}\, \widehat{T}_j \sqrt{w(a)}]\, x_j \right)^{-1} \sqrt{w(a)} \Sigma(a,x).$$

- [Pascoe, Tully-Doyle] [Helton, M, Vinnikov] Free functions, rational functions, no a variables.
- [Helton, Hay, Lim, M] Polynomials.

Partial convexity

The root butterfly realization: an algebraic certificate of x-convexity

Theorem [Same suspects]. A symmetric $r \in \mathbb{C}\langle a, x\rangle$ is convex in x in a neighborhood of 0 if and only if there exists $k \in \mathbb{N}$,

(i) $\widehat{T} \in \mathbb{S}_k^g$;

(ii) a symmetric $w \in \mathbb{C}\langle a\rangle^{k\times k}$;

(iii) $\ell \in \mathbb{C}\langle a, x\rangle$ and $\Sigma \in \mathbb{C}\langle a, x\rangle^{k\times 1}$ each of degree at most one in x and ℓ is symmetric;

such that $w(A) \succeq 0$ and $I - \sum[\sqrt{w(a)}\,\widehat{T}_j\sqrt{w(a)}] \otimes X_j \succ 0$ near 0 and

$$r(a,x) = \ell(a,x) + \Sigma(a,x)^*\sqrt{w(a)}\left(I - \sum[\sqrt{w(a)}\,\widehat{T}_j\sqrt{w(a)}]\,x_j\right)^{-1}\sqrt{w(a)}\Sigma(a,x).$$

Extreme points

- Free convexity basics
- Free semialgebraic sets
- Partial convexity and rational functions
- Extreme Points
- Some Analytic Theory.

Extreme points
The Arveson boundary

- The *matrix convex hull* of $\mathscr{E} \subseteq \mathbb{S}^{\mathrm{g}}$ is the matrix convex set

$$\operatorname{matcohull} \mathscr{E} = \{V^* X V : X \in \mathscr{E}, \quad V^* V = I\};$$

- $V^* X V$ is the free analog of a convex combination;
- A *good* notion of extreme point for a free spectrahedron $\mathcal{D}_A$ produces a *small* collection $\mathscr{E} \subseteq \mathcal{D}_A$ such that $\operatorname{matcohull} \mathscr{E} = \mathcal{D}_A$;
- An *Arveson boundary point* for a free spectrahedron $\mathcal{D}_A$ is a tuple $X \in \mathcal{D}_A$ such that if $Y \in \mathcal{D}_A$ has the form

$$Y_j = \begin{pmatrix} X_j & \alpha_j \\ \alpha_j^* & \beta_j \end{pmatrix} \in \mathcal{D}_A,$$

  then $\alpha_j = 0$.

Extreme points
Arveson boundary points span

$$\mathcal{D}_A \ni Y = \begin{pmatrix} X & \alpha \\ \alpha^* & \beta \end{pmatrix} \implies \alpha = 0.$$

- Thus X is an Arveson boundary point if the only dilations of X are trivial. The analog of a boundary representation. The nc analog of a peak point (in the Shilov boundary).
- If $\mathcal{D}_A$ is the matrix convex hull of $\mathscr{E}$, then $\mathscr{E}$ contains the Arveson boundary points.

Theorem. [Evert, Helton] If $\mathcal{D}_A$ is closed wrt $\mathbb{C}$- conjugation, then

$$\mathcal{D}_A = \text{matcohull}\, \partial\mathcal{D}_A^{\text{Arv}}.$$

Extreme points
Other notions of extreme points

Theorem [Evert, Helton] $\mathcal{D}_A$ is the matco hull of its Arv points.

- Typically, off the shelf techniques produce *operator* Arveson boundary points. In particular:
  - False for general compact matrix convex sets $K \subseteq \mathbb{S}^g$ [Evert];
  - False for $\mathcal{D}_A$ in *free free variables*;
- The tension: pass to operators or liberalize the notion of extreme point;
- There is a highly developed theory involving other notions of extreme points tailored to matco sets; e.g., matrix extreme points.

Some Analytic Theory
Outline

- Free convexity basics
- Free semialgebraic sets
- Partial convexity and rational functions
- Extreme Points
- Some Analytic Theory
  - Augat's Free Grothendieck Theorem
  - Pseudoconvex sets and free plurisubharmonic functions

Augat's Free Grothendieck Theorem

free free variables; aka, free complex analysis

- $M(\mathbb{C})^{\mathrm{g}} = (M_n(\mathbb{C})^{\mathrm{g}})_n$, the free free universe;
- $p \in \mathbb{C}<x>$ *evaluates* at $X \in M(\mathbb{C})^{\mathrm{g}}$ in the canonical way;
- E.g.; for $X = (X_1, X_2) \in M_n(\mathbb{C})^2$ and

$$p(x) = 5 + 2x_1x_2 - 3x_2x_1 + x_1^2x_2x_1,$$
$$p(X) = 5I_n + 2X_1X_2 - 3X_2X_1 + X_1^2X_2X_1.$$

Grothendieck's Theorem

Automorphisms of $\mathbb{C}[t_1, \ldots, t_g]$.

Grothendieck's Theorem. Suppose $p : \mathbb{C}^g \to \mathbb{C}^g$ is a polynomial mapping; that is, for some $p^j \in \mathbb{C}[t_1, \ldots, t_g]$.

$$p = \begin{pmatrix} p^1 & \ldots & p^g \end{pmatrix}.$$

If p is injective, then p is bijective and moreover the inverse of p is a polynomial.

For instance, $p : \mathbb{C}^2 \to \mathbb{C}^2$,

$$\begin{aligned} p(t_1, t_2) &= \left(t_1, t_2 - t_1^2\right), \\ p^{-1}(t_1, t_2) &= \left(t_1, t_2 + t_1^2\right). \end{aligned}$$

Augat's Free Grothendieck Theorem

Automorphisms of the free algebra

Augat's Free Grothendieck Theorem.

Suppose $p : M(\mathbb{C})^{\mathrm{g}} \to M(\mathbb{C})^{\mathrm{g}}$ is a free polynomial mapping.
The following are equivalent.

(i) p is injective;

(ii) p is bijective;

(iii) p has a (free) polynomial inverse.

- Grothendieck's Theorem implies p has a free inverse. Showing that this inverse is in fact a polynomial is the challenge.
- The proof involves a good deal of algebra, but also free analysis including Pascoe's free inverse function theorem and some (new) realization theory.
- Meric will discuss potential generalizations and related conjectures as part of Wednesday's 2-3 pm problem session.

Plurisubaharmonic functions - plush
Symmetric rational functions

- $x = (x_1, \ldots, x_{\mathrm{g}})$ with adjoint variables $(x_1^*, \ldots, x_{\mathrm{g}}^*)$;
- A symmetric rational function $r \in \mathbb{C}\langle x, x^* \rangle$ has a *descriptor realization:*

$$r = c^* \left(J - \Lambda_A(x) - \Lambda_A(x)^* \right)^{-1} c, \quad \Lambda_A(x) = \sum_{j=1}^{\mathrm{g}} A_j x_j;$$

- r is *symmetric:* $r(X)^* = r(X)$ for $X \in \operatorname{dom} r[n] \subseteq M_n(\mathbb{C})^g$.

Plush
The complex Hessian

$$r = c^* \left(J - \Lambda_A(x) - \Lambda_A(x)^*\right)^{-1} c, \quad \Lambda_A(x) = \sum_{j=1}^{g} A_j x_j$$

- The *complex Hessian* of r at X in the direction $H \in M_n(\mathbb{C})^g$ is

$$\frac{\partial^2 r}{\partial_{x^*} \partial_x}(X)[H, H];$$

- For $r(x) = x^{*2} x^2$,

$$\frac{\partial^2 r}{\partial_{x^*} \partial_x}(X)[H] = (XH + HX)^* (XH + HX) \succeq 0;$$

- More generally, for $g \in \mathbb{C}<x>$ and $r(x) = g(x) g(x)^*$,

$$\frac{\partial^2 r}{\partial x^* \partial x}(X)[H] = Dg(X)[H] \left(Dg(X)[H]\right)^* \succeq 0;$$

- r is *plush* on a set S if $\frac{\partial^2 r}{\partial_{x^*} \partial_x}(X)[H] \succeq 0$ all $X \in S$, all H.

Plush

Rational Functions

Theorem. [Greene] [Greene, Helton, Vinnikov] A symmetric polynomial $r \in \mathbb{C}\langle x, x^* \rangle$ is (1) plush on a free open set; iff (2) it is globally plush; iff (3)

$$r(x) = s(x) + s(x)^* + \sum p_j(x)^* p_j(x) + \sum q_k(x) q_k(x)^*,$$

with $s, p_j, q_k \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$.

Theorem. [Dym, Helton, Klep, M, Volčič] [Pascoe] A symmetric rational function r is plush in a free neighborhood of 0 iff there exists a convex symmetric rational function $f \in \mathbb{C}\langle\!\langle y, y^* \rangle\!\rangle$ (in $\mathtt{h}$ variables), and $q_j \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$ for $1 \leq j \leq \mathtt{h}$, such that

$$r(x) = f(q_1(x), \ldots, q_{\mathtt{h}}(x)) = f \circ q(x)$$

Thus r is plush if and only if r is convex composed with analytic.