INVARIANT RANDOM SUBGROUPS OF INDUCTIVE LIMITS OF FINITE ALTERNATING GROUPS

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Abstract. We classify the ergodic invariant random subgroups of inductive limits of finite alternating groups.

1. Introduction

A simple locally finite group G is said to be an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of a strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$. In this paper, we will classify the ergodic invariant random subgroups of the $L(\mathrm{Alt})$-groups, and we will consider the relationship between the existence of "nontrivial" ergodic IRSs, "nontrivial" characters $\chi : G \to \mathbb{C}$ and "nontrivial" 2-sided ideals $I \subseteq \mathbb{C}G$.

Let G be a countably infinite group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. For example, if $N \trianglelefteq G$ is a normal subgroup, then the corresponding Dirac measure δ_N is an IRS of G. Further examples of IRSs arise from from the stabilizer distributions of measure-preserving actions, which are defined as follows. Suppose that G acts via measure-preserving maps on the Borel probability space $(\, Z, \mu\,)$ and let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by

$$z \mapsto G_z = \{\, g \in G \mid g \cdot z = z \,\}.$$

Then the corresponding *stabilizer distribution* $\nu = f_* \mu$ is an IRS of G. In fact, by a result of Abért-Glasner-Virag [1], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, by Creutz-Peterson [2], if ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\, Z, \mu\,)$.

Definition 1.1. A countably infinite group G is said to be *strongly simple* if the only ergodic IRS of G are δ_1 and δ_G.

In other words, a (necessarily simple) group G is strongly simple if G has no nontrivial ergodic IRS.

As we pointed out in Thomas-Tucker-Drob [17], if G is a countably infinite locally finite group and $G \curvearrowright (\, Z, \mu\,)$ is an ergodic action, then an application of the Pointwise Ergodic Theorem for actions of locally finite groups to the *associated character* $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ allows us to regard $G \curvearrowright (\, Z, \mu\,)$ as the "limit" of a suitable sequence of finite permutation groups $G_n \curvearrowright (\, \Omega_n, \mu_n\,)$, where μ_n is the uniform probability measure on Ω_n.

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Definition 1.2. If G is a countable group, then the function $\chi : G \to \mathbb{C}$ is a *character* if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ for all $g, \in G$.
(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$.
(iii) $\chi(1_G) = 1$.

A character χ is said to be *indecomposable* or *extremal* if it is impossible to express $\chi = r\chi_1 + (1-r)\chi_2$, where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters.

The set of characters of G is denoted by $\mathcal{F}(G)$ and the set of indecomposable characters is denoted by $\mathcal{E}(G)$. The set $\mathcal{F}(G)$ always contains the two "trivial" characters χ_{con} and χ_{reg}, where $\chi_{\text{con}}(g) = 1$ for all $g \in G$ and $\chi_{\text{reg}}(g) = 0$ for all $1 \neq g \in G$. It is well-known that χ_{con} is indecomposable, and that χ_{reg} is indecomposable if and only if G is an i.c.c. group, i.e. the conjugacy class g^G of every nonidentity element $g \in G$ is infinite. (For example, see Peterson-Thom [12].) We will say that $\mathcal{F}(G)$ is *trivial* if every $\chi \in \mathcal{F}(G)$ is a convex combination of χ_{con} and χ_{reg}.

Theorem 1.3. *If the countably infinite simple group G is not strongly simple, then $\mathcal{F}(G)$ is nontrivial.*

Proof. Suppose that $\nu \neq \delta_1$, δ_G is a nontrivial ergodic IRS of G. Then, by Creutz-Peterson [2, Proposition 3.3.1], we can suppose that ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\,Z, \mu\,)$. Let $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ be the associated character. Suppose that there exists $0 \leq a \leq 1$ such that $\chi = a\chi_{\text{con}} + (1-a)\chi_{\text{reg}}$. Then, since $\nu \neq \delta_1$, δ_G, it follows that $0 < a < 1$; and so $\inf_{g \in G} \mu(\,\mathrm{Fix}_Z(g)\,) = a > 0$. Applying Ioana-Kechris-Tsankov [6, Theorem 1(i)] in the special case when E is the identity relation, it follows that there exists a G-invariant Borel subset $A \subset Z$ with $\mu(A) > 0$ such that $|A| \leq 1/a$; and, since G acts ergodically on $(\,Z, \mu\,)$, it follows that $\mu(A) = 1$. Let $a \in A$. Then, since G is an infinite simple group and $[G : G_a] \leq |A| < \infty$, it follows that $G_a = G$. Thus $A = \{\,a\,\}$ and $\nu = \delta_G$, which is a contradiction. Consequently, $\chi(g)$ is not a convex combination of χ_{con} and χ_{reg}. □

There exist examples of ergodic actions $G \curvearrowright (\,Z, \mu\,)$ of countably infinite groups such that the associated character χ is *not* indecomposable. For example, if the ergodic action $G \curvearrowright (\,Z, \mu\,)$ is essentially free, then $\chi = \chi_{\text{reg}}$, and so χ is indecomposable if and only if G is an i.c.c. group. There also exist more interesting examples.

Theorem 1.4. *There exists an ergodic action* $\mathrm{Alt}(\mathbb{N}) \curvearrowright (\,Z, \mu\,)$ *such that the associated character is not indecomposable.*

Proof. Suppose that χ is an indecomposable character of the infinite alternating group $\mathrm{Alt}(\mathbb{N})$. Then, by Thoma [16, Satz 6], there exists an indecomposable character θ of the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of the natural numbers such that $\chi = \theta \restriction \mathrm{Alt}(\mathbb{N})$; and hence, by Thoma [16, Satz 1], we have that

$$\chi(\,(\,1\,2\,)(\,3\,4\,)(\,5\,6\,)(\,7\,8\,)\,) = \chi(\,(\,1\,2\,)(\,3\,4\,)\,)\,\chi(\,(\,5\,6\,)(\,7\,8\,)\,). \tag{1.1}$$

Thus it suffice to find an ergodic action $\mathrm{Alt}(\mathbb{N}) \curvearrowright (\,Z, \mu\,)$ such that the associated character $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ fails to satisfy the multiplicative property (1.1).

Let m be the usual uniform product probability measure on $2^{\mathbb{N}}$. Then $\mathrm{Alt}(\mathbb{N})$ acts ergodically on $(\,2^{\mathbb{N}}, m\,)$ via the shift action $(\,g \cdot \xi\,)(n) = \xi(g^{-1}(n))$. For each

$\xi \in 2^{\mathbb{N}}$ and $i = 0,\, 1$, let $B_i^\xi = \{\, n \in \mathbb{N} \mid \xi(n) = i \,\}$. Let $f : 2^{\mathbb{N}} \to \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})}$ be the $\mathrm{Alt}(\mathbb{N})$-equivariant map defined by $\xi \mapsto \mathrm{Alt}(B_0^\xi) \times \mathrm{Alt}(B_1^\xi)$ and let $\nu = f_* m$ be the corresponding ergodic IRS of $\mathrm{Alt}(\mathbb{N})$. Then, by Creutz-Peterson [2], ν is the stabilizer distribution of an ergodic action $\mathrm{Alt}(\mathbb{N}) \curvearrowright (\, Z, \mu\,)$; and the associated character χ is given by

$$\begin{aligned}\chi(g) &= \mu(\, \mathrm{Fix}_Z(g)\,) \\ &= \nu(\, \{\, H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid g \in H \,\}\,) \\ &= m(\, \{\xi \in 2^{\mathbb{N}} \mid g \in \mathrm{Alt}(B_0^\xi) \times \mathrm{Alt}(B_1^\xi)\,\}\,).\end{aligned}$$

Clearly $(\,1\,2\,)\,(\,3\,4\,) \in \mathrm{Alt}(B_0^\xi) \times \mathrm{Alt}(B_1^\xi)$ if and only if $\xi(1) = \xi(2) = \xi(3) = \xi(4)$; and it follows that

$$\chi(\,(\,1\,2\,)\,(\,3\,4\,)\,) = \chi(\,(\,5\,6\,)\,(\,7\,8\,)\,) = 1/2^4 + 1/2^4 = 1/2^3.$$

On the other hand, we have that

$$\chi(\,(\,1\,2\,)\,(\,3\,4\,)\,(\,5\,6\,)\,(\,7\,8\,)\,) = \frac{\binom{4}{0} + \binom{4}{2} + \binom{4}{4}}{2^8} = 1/2^5.$$

Since the multiplicative property (1.1) fails, it follows that χ is not indecomposable. □

Problem 1.5. Find necessary and sufficient conditions for the associated character of an ergodic action $G \curvearrowright (\, Z, \mu\,)$ to be indecomposable.

Vershik [19] has proved a very interesting sufficient condition; namely, that if $G \curvearrowright (\, Z, \mu\,)$ is ergodic and $N_G(G_z) = G_z$ for μ-a.e. $z \in Z$, then the associated character is indecomposable. Using Vershik's criterion, together with our classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \not\cong \mathrm{Alt}(\mathbb{N})$, we will prove the following result.

Theorem 1.6. *If $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-group and $G \curvearrowright (\, Z, \mu\,)$ is an ergodic action, then the associated character is indecomposable.*

The $L(\mathrm{Alt})$-groups with a nontrivial ergodic IRS will be classified as follows. Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, where $|\Delta_1| \geq 5$. For each $i \in \mathbb{N}$, let s_{i+1} be the number of natural orbits of G_i on Δ_{i+1} and let e_{i+1} is the number of points $x \in \Delta_{i+1}$ which lie in a nontrivial non-natural G_i-orbit. Also for each $i < j$, let $s_{ij} = s_{i+1} s_{i+2} \cdots s_j$. Recall that $G = \bigcup_{i \in \mathbb{N}} G_i$ is said to be a *diagonal limit* if $s_{i+1} > 0$ and $e_{i+1} = 0$ for all $i \in \mathbb{N}$; i.e. if for each $i \in \mathbb{N}$, every G_i-orbit on Δ_{i+1} is either natural or trivial.

Definition 1.7. $G = \bigcup_{i \in \mathbb{N}} G_i$ is an *almost diagonal limit* if $s_{i+1} > 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} e_i / s_{0i} < \infty$.

Theorem 1.8. *If G is an $L(\mathrm{Alt})$-group, then G has a nontrivial ergodic IRS if and only if G can be expressed as an almost diagonal limit of finite alternating groups.*

We will present an explicit classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \not\cong \mathrm{Alt}(\mathbb{N})$ in Sections 3 and 4. The classification involves a fundamental dichotomy which was originally introduced by Leinen-Puglisi [10, 11] in the more restrictive setting of diagonal limits of alternating groups, i.e. the linear vs sublinear natural orbit growth condition. This dichotomy arose unexpectedly in the work of Leinen-Puglisi [10, 11] without any natural explanation. By contrast, in this paper, it will appear as a natural consequence of the Pointwise Ergodic Theorem for actions of locally finite groups.

In [20], Vershik showed that the indecomposable characters of the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of the natural numbers were very closely connected with the ergodic IRSs of $\mathrm{Fin}(\mathbb{N})$; and in [19], he suggested that this should also be true of various other locally finite groups. Combining our classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups with the earlier work of Leinen-Puglisi [11], it follows that if $G \not\cong \mathrm{Alt}(\mathbb{N})$ is a diagonal limit of finite alternating groups, then the indecomposable characters of G are precisely the associated characters of the ergodic IRSs of G.

It is clear from Theorems 1.4 and 1.6 that $\mathrm{Alt}(\mathbb{N})$ plays an exceptional role within the class of $L(\mathrm{Alt})$-groups. In Section 9, adapting and slightly correcting Vershik's analysis of the ergodic IRSs of the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of the natural numbers, we will state the classification of the ergodic IRSs of $\mathrm{Alt}(\mathbb{N})$ and we will characterize the ergodic actions $\mathrm{Alt}(\mathbb{N}) \curvearrowright (Z, \mu)$ such that the associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is indecomposable.

If G is a countable group and $\chi \in \mathcal{F}(G)$ is a character, then we can extend χ to a linear function $\chi : \mathbb{C}G \to \mathbb{C}$ and define a corresponding proper 2-sided ideal I_χ of the group ring $\mathbb{C}G$ by

$$I_\chi = \{ x \in \mathbb{C}(G) \mid \chi(g\, x) = 0 \text{ for all } g \in G \}.$$

For example, let $\omega(\mathbb{C}G)$ be the *augmentation ideal*, i.e. the kernel of the homomorphism $\mathbb{C}G \to \mathbb{C}$ defined by $\sum \lambda_i g_i \mapsto \sum \lambda_i$. Then it is easily checked that if χ is a character of G, then $I_\chi = \omega(\mathbb{C}G)$ if and only if $\chi = \chi_{\mathrm{con}}$. It is also easily seen that $I_{\chi_{\mathrm{reg}}} = \{0\}$. In [23], Zalesskiĭ asked whether there exists a simple locally finite group G with an indecomposable character $\chi \neq \chi_{\mathrm{reg}}$ such that $I_\chi = \{0\}$; and he conjectured that if G is a simple locally finite group such that $\omega(\mathbb{C}G)$ is the only nontrivial proper 2-sided ideal of $\mathbb{C}G$, then $\mathcal{F}(G)$ is trivial. In Section 3, we will give an example of a simple locally finite group G such that:

(a) the augmentation ideal $\omega(\mathbb{C}G)$ is the only nontrivial proper 2-sided ideal of $\mathbb{C}G$; and
(b) G has infinitely many indecomposable characters χ such that $I_\chi = \{ 0 \}$.

In this example, the characters of G will be precisely those associated with the ergodic IRSs of G. It should be pointed out that Leinen-Puglisi [10] gave the first examples of simple locally finite groups G with indecomposable characters $\chi \neq \chi_{\mathrm{reg}}$ such that $I_\chi = \{0\}$. However, in their examples, the corresponding group rings $\mathbb{C}G$ had infinitely many nontrivial proper 2-sided ideals.

This paper is organized as follows. In Section 2, we will briefly discuss the pointwise ergodicity and weak mixing properties for ergodic actions of countably infinite locally finite finite groups. In Section 3, we will introduce the notion of an almost diagonal limit of finite alternating groups and the notions of linear/sublinear natural orbit growth; and we will discuss the ergodic IRSs of the $L(\mathrm{Alt})$-groups of linear natural orbit growth. In Section 4, we will discuss the ergodic IRSs of almost diagonal limits with sublinear natural orbit growth. In Section 5, we will present a natural characterization of the almost diagonal limit of finite alternating groups. In Section 6, we will present a series of lemmas concerning upper bounds for the values of the normalized permutation characters of various actions $\mathrm{Alt}(\Delta) \curvearrowright \Omega$ of the finite alternating group $\mathrm{Alt}(\Delta)$. In Sections 7 and 8, we will present our proof of the classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \not\cong \mathrm{Alt}(\mathbb{N})$. Finally, in Section 9, we will discuss the ergodic IRSs of the infinite alternating group $\mathrm{Alt}(\mathbb{N})$.

2. The ergodic theory of locally finite groups

In this section, we will briefly discuss the pointwise ergodicity and weak mixing properties for ergodic actions of countably infinite locally finite finite groups. Throughout, let $G = \bigcup_{i\in\mathbb{N}} G_i$ be the union of the strictly increasing chain of finite subgroups G_i and let $G \curvearrowright (Z, \mu)$ be an ergodic action on a Borel probability space. The following is a special case of more general results of Vershik [18, Theorem 1] and Lindenstrauss [9, Theorem 1.3].

The Pointwise Ergodic Theorem. *With the above hypotheses, if $B \subseteq Z$ is a μ-measurable subset, then for μ-a.e $z \in Z$,*

$$\mu(B) = \lim_{i\to\infty} \frac{1}{|G_i|} |\{ g \in G_i \mid g \cdot z \in B \}|.$$

In particular, the Pointwise Ergodic Theorem applies when B is the μ-measurable subset $\mathrm{Fix}_Z(g) = \{ z \in Z \mid g \cdot z = z \}$ for some $g \in G$. For each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{ g \cdot z \mid g \in G_i \}$ be the corresponding G_i-orbit. Then, as pointed out in Thomas-Tucker-Drob [17, Theorem 2.1], the following result is an easy consequence of the Pointwise Ergodic Theorem.

Theorem 2.1. *With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,*

$$\mu(\mathrm{Fix}_Z(g)) = \lim_{i\to\infty} |\mathrm{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)|.$$

The normalized permutation character $|\mathrm{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)|$ is the probability that an element of $(\Omega_i(z), \mu_i)$ is fixed by $g \in G_i$, where μ_i is the uniform probability measure on $\Omega_i(z)$; and, in this sense, we can regard $G \curvearrowright (Z, \mu)$ as the "limit" of the sequence of finite permutation groups $G_i \curvearrowright (\Omega_i(z), \mu_i)$. Of course, the permutation group $G_i \curvearrowright \Omega_i(z)$ is isomorphic to $G_i \curvearrowright G_i/H_i$, where G_i/H_i is the set of cosets of $H_i = \{ h \in G_i \mid h \cdot z = z \}$ in G_i. The following simple observation will be used repeatedly in our later applications of Theorem 2.1. (For example, see Thomas-Tucker-Drob [17, Proposition 2.2].)

Proposition 2.2. *If $H \leqslant A$ are finite groups and θ is the normalized permutation character corresponding to the action $A \curvearrowright A/H$, then*

$$\theta(g) = \frac{|g^A \cap H|}{|g^A|} = \frac{|\{s \in A \mid sgs^{-1} \in H\}|}{|A|}.$$

The following consequence of Proposition 2.2 implies that when computing upper bounds for the normalized permutation characters of actions $A \curvearrowright A/H$, we can restrict our attention to those coming from maximal subgroups $H < A$.

Corollary 2.3. *If $H \leqslant H' \leqslant A$ are finite groups and θ, θ' are the normalized permutation characters corresponding to the actions $A \curvearrowright A/H$ and $A \curvearrowright A/H'$, then $\theta(g) \leq \theta'(g)$ for all $g \in A$.*

Finally we point out the following straightforward but useful observation.

Theorem 2.4. *If G is a countably infinite simple locally finite group, then every ergodic action $G \curvearrowright (Z, \mu)$ is weakly mixing.*

Proof. Suppose that the ergodic action $G \curvearrowright (Z, \mu)$ is not weakly mixing. Then, by Schmidt [14, Proposition 2.2], it follows that G has a nontrivial finite dimensional unitary representation; and since G is simple, this representation is necessarily faithful. However, this is impossible since Schur [15] has proved that every periodic linear group over the complex field has an abelian subgroup of finite index. (For a more accessible reference, see Curtis-Reiner [3, Theorem 36.14].) □

Corollary 2.5. *If G is a countably infinite simple locally finite group and the action $G \curvearrowright (Z, \mu)$ is ergodic, then the product action $G \curvearrowright (Z^r, \mu^{\otimes r})$ is also ergodic for each $r \geq 1$.*

3. Linear natural orbit growth

In this section, we will begin our analysis of the ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \ncong \mathrm{Alt}(\mathbb{N})$. First we need to introduce some notation. For the remainder of this paper, suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, where $|\Delta_1| \geq 5$. For each $i \in \mathbb{N}$, let

- $n_i = |\Delta_i|$;
- s_{i+1} be the number of natural orbits of G_i on Δ_{i+1};
- f_{i+1} be the number of trivial orbits of G_i on Δ_{i+1};
- $e_{i+1} = n_{i+1} - (s_{i+1} n_i + f_{i+1})$; and
- $t_{i+1} = e_{i+1} + f_{i+1}$.

Thus e_{i+1} is the number of points $x \in \Delta_{i+1}$ which lie in a nontrivial non-natural G_i-orbit and $t_{i+1} = n_{i+1} - s_{i+1} n_i$ is the number of points $x \in \Delta_{i+1}$ which lie in a (possibly trivial) non-natural G_i-orbit. For each $i < j$, let s_{ij} be the number of natural orbits of G_i on Δ_j and let $t_{ij} = n_j - s_{ij} n_i$. Finally let $\tau = \sum_{i=1}^{\infty} e_i / s_{0i}$.

Definition 3.1. $G = \bigcup_{i \in \mathbb{N}} G_i$ is an *almost diagonal limit* if $s_{i+1} > 0$ for all $i \in \mathbb{N}$ and $\tau = \sum_{i=1}^{\infty} e_i / s_{0i} < \infty$.

Remark 3.2. If $s_{i+1} > 0$ and $e_{i+1} = 0$ for all $i \in \mathbb{N}$, then $G = \bigcup_{i \in \mathbb{N}} G_i$ is a *diagonal limit* in the sense of Lavrenyuk-Nekrashevych [8].

Definition 3.3. The $L(\mathrm{Alt})$-group G has *almost diagonal type* if G can be expressed as an almost diagonal limit of finite alternating groups.

We are now in a position to state the first of the main results of this paper.

Theorem 3.4. *If G is an $L(\mathrm{Alt})$-group, then G has a nontrivial ergodic IRS if and only if G has almost diagonal type.*

The classification of the ergodic IRSs of the groups of almost diagonal type involves a fundamental dichotomy which was introduced by Leinen-Puglisi [10, 11] in the more restrictive setting of diagonal limits of alternating groups, i.e. the linear vs sublinear natural orbit growth condition. The following result, which is an immediate consequence of Zalesskiĭ [22, Lemma 10], will allow us to usefully extend the notion of the linear natural orbit condition to the general setting of arbitrary $L(\mathrm{Alt})$-groups.

Lemma 3.5. *Let $\mathrm{Alt}(\Omega_1) \hookrightarrow \mathrm{Alt}(\Omega_2) \hookrightarrow \mathrm{Alt}(\Omega_3)$ be proper embeddings of finite alternating groups with $|\Omega_1| \geq 5$. If Σ is a natural $\mathrm{Alt}(\Omega_1)$-orbit on Ω_3 and Σ' is the $\mathrm{Alt}(\Omega_2)$-orbit on Ω_3 such that $\Sigma' \supseteq \Sigma$, then Σ' is a natural $\mathrm{Alt}(\Omega_2)$-orbit.*

The following result is an immediate consequence of Lemma 3.5.

Lemma 3.6. *If $i < j < k$, then $s_{ik} = s_{ij}s_{jk}$.*

In particular, for each $i > 0$, we have that $s_{0i} = s_1 s_2 \cdots s_i$. The following observation will be used repeatedly throughout this paper.

Proposition 3.7. *Suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. If $(\, j_i \mid i \in \mathbb{N}\,)$ is a strictly increasing sequence of natural numbers and $G'_i = \mathrm{Alt}(\Delta_{j_i})$, then $G = \bigcup_{i\in\mathbb{N}} G'_i$ is also an almost diagonal limit.*

Proof. For each $i < j$, let e_{ij} be the number of points $x \in \Delta_j$ which lie in a nontrivial non-natural G_i-orbit. Then an easy induction on $j \geq i+1$ shows that

$$e_{ij} \leq \sum_{k=i+1}^{j-1} s_{kj}e_k + e_j.$$

Since $s_{0j} = s_{0k}s_{kj}$, we obtain that

$$e_{ij}/s_{0j} \leq \sum_{k=i+1}^{j} e_k/s_{0k},$$

and the result follows. □

Remark 3.8. Suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit of finite alternating groups. If $s_{i+1} = 1$ for all but finitely many $i \in \mathbb{N}$, then $e_{i+1} = 0$ for all but finitely many $i \in \mathbb{N}$, and it follows that $G \cong \mathrm{Alt}(\mathbb{N})$. Hence, applying Proposition 3.7, if $G \ncong \mathrm{Alt}(\mathbb{N})$, then we can suppose that the almost diagonal limit $\bigcup_{i\in\mathbb{N}} G_i$ has been chosen such that $s_{i+1} > 1$ for $i \in \mathbb{N}$.

The statement and proof of the following lemma are identical to Leinen-Puglisi [11, Lemma 2.2].

Lemma 3.9. *For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j\to\infty} s_{ij}/n_j$ exists.*

Proof. If $i < j < k$, then $s_{ik} = s_{ij}s_{jk}$ and clearly $n_j s_{jk} \leq n_k$. Hence

$$\frac{s_{ik}}{n_k} = \frac{s_{ij}}{n_j} \cdot \frac{n_j s_{jk}}{n_k} \leq \frac{s_{ij}}{n_j}$$

and the sequence $(\, s_{ij}/n_j \mid i < j \in \mathbb{N}\,)$ converges to $\inf_{j>i} s_{ij}/n_j$. □

Definition 3.10. G is said to have *linear natural orbit growth* if $a_i > 0$ for some $i \in \mathbb{N}$. Otherwise, G is said to have *sublinear natural orbit growth*.

Remark 3.11. We will soon see that if G has linear natural orbit growth, then G has almost diagonal type.

Note that $a_i = s_{i+1}a_{i+1}$. Thus G has linear natural orbit growth if and only if $a_i > 0$ for all but finitely many $i \in \mathbb{N}$. It is easily checked that if $G = \bigcup_{i\in\mathbb{N}} G'_i$ is another expression of G as the union of a strictly increasing chain of finite alternating groups $G'_i = \mathrm{Alt}(\Delta'_i)$ with corresponding parameters s'_{ij}, n'_i and a'_i, then $a_i > 0$ for all but finitely many $i \in \mathbb{N}$ if and only if $a'_i > 0$ for all but finitely many $i \in \mathbb{N}$. (This is clear in the case when G is expressed as the union of a chain of finite alternating groups $G'_i = \mathrm{Alt}(\Delta_{j_i})$ for some strictly increasing sequence $(\, j_i \mid i \in \mathbb{N}\,)$ of natural numbers; and the general case follows easily.) Thus the notion of linear natural orbit growth is independent of the expression of G as a union of a strictly increasing chain of finite alternating groups.

Lemma 3.12. *If G has linear natural orbit growth, then G has almost diagonal type.*

Proof. Suppose that G has linear natural orbit growth. Then, after passing to after passing to a suitable subsequence if necessary, we can suppose that $s_{i+1} > 0$ and hence $a_i > 0$ for all $i \in \mathbb{N}$. Now an easy induction shows that if $j > 0$, then

$$n_j = s_{0j} n_0 + \sum_{i=1}^{j-1} s_{ij} t_i + t_j$$

and hence

$$1 = \frac{s_{0j}}{n_j} + \frac{s_{0j}}{n_j} \sum_{i=1}^{j} \frac{t_i}{s_{0i}}.$$

Since $a_0 = \inf_{j>0} s_{0j}/n_j > 0$, this implies that $\sum_{i=1}^{\infty} t_i/s_{0i} < \infty$; and since each $e_i \leq t_i$, it follows that $\tau = \sum_{i=1}^{\infty} e_i/s_{0i} < \infty$. □

We will next prove that if G has linear natural orbit growth, then G has a nontrivial ergodic IRS. Note that if G has linear natural orbit growth, then $s_{i+1} > 0$ for all but finitely many $i \in \mathbb{N}$; and hence we can suppose that $s_{i+1} > 0$ for all $i \in \mathbb{N}$. We will initially work with this strictly weaker hypothesis. As we will see, the linear vs sublinear natural orbit growth dichotomy will appear naturally in our analysis via an application of the Pointwise Ergodic Theorem for actions of locally finite groups. Let $t_0 = n_0$ and recall that $t_{i+1} = e_{i+1} + f_{i+1} = n_{i+1} - s_{i+1} n_i$. Clearly we can suppose that:

- $\Delta_0 = \{ \alpha^0_\ell \mid \ell < t_0 \}$; and
- $\Delta_{i+1} = \{ \sigma^\frown k \mid \sigma \in \Delta_i, 0 \leq k < s_{i+1} \} \cup \{ \alpha^{i+1}_\ell \mid 0 \leq \ell < t_{i+1} \}$;

and that the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ satisfies

$$\varphi_i(g)(\sigma^\frown k) = g(\sigma)^\frown k$$

for each $\sigma \in \Delta_i$ and $0 \leq k < s_{i+1}$. Let Δ consist of all sequences of the form $(\alpha^i_\ell, k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$, where $i \in \mathbb{N}$ and k_j is an integer such that $0 \leq k_j < s_j$. For each $i \in \mathbb{N}$ and $\sigma \in \Delta_i$, let $\Delta(\sigma) \subseteq \Delta$ be the subset of sequences of the form $\sigma^\frown (k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$. Then the sets $\Delta(\sigma)$ form a clopen basis for a locally compact topology on Δ. (This is a special case of the "space of paths" of Lavrenyuk-Nekrashevych [8].) Consider the action $G \curvearrowright \Delta$ defined by

$$g \cdot (\alpha^i_\ell, k_{i+1}, \cdots, k_j, k_{j+1} \cdots) = (g(\alpha^i_\ell, k_{i+1}, \cdots, k_j), k_{j+1} \cdots), \quad g \in G_j.$$

Then we will show that there exists a G-invariant ergodic probability measure on Δ if and only if G has linear natural orbit growth; in which case, the action $G \curvearrowright \Delta$ is uniquely ergodic.

Of course, if m is a G-invariant ergodic probability measure on Δ, then m is uniquely determined by $m \restriction \mathcal{A}$, where $\mathcal{A}$ is the algebra of Borel subsets of Δ generated by the basic clopen sets $\{ \Delta(\sigma) \mid \sigma \in \bigcup_{i \in \mathbb{N}} \Delta_i \}$.

Lemma 3.13. *If m is a G-invariant ergodic probability measure on Δ and $\sigma \in \Delta_i$, then $m(\Delta(\sigma)) = a_i$.*

Proof. Applying the Pointwise Ergodic Theorem, choose an element $z \in \Delta$ such that

$$m(\Delta(\sigma)) = \lim_{j \to \infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot z \in \Delta(\sigma) \}|.$$

Suppose that $z = (\, \alpha^r_\ell, k_{r+1}, k_{r+2}, \cdots)$ and for each $j > r$, let

$$z_j = (\, \alpha^r_\ell, k_{r+1}, \cdots , k_j \,) \in \Delta_j.$$

For each $j > \max\{\, i, r \,\}$, let $S_j \subseteq \Delta_j$ be the set of sequences of the form

$$s = \sigma ^\frown (\, t_{i+1}, \cdots , t_j \,).$$

Then $|S_j| = s_{ij}$ and

$$\{\, g \in G_j \mid g \cdot z \in \Delta(\sigma) \,\} = \{\, g \in G_j \mid g \cdot z_j \in S_j \,\};$$

and it follows that

$$m(\Delta(\sigma)) = \lim_{j\to\infty} \frac{1}{|G_j|} |\{\, g \in G_j \mid g \cdot z_j \in S_j \,\}| = \lim_{j\to\infty} |S_j|/|\Delta_j| = a_i.$$

□

Corollary 3.14. *With the above hypotheses, if G has sublinear natural orbit growth, then there does not exist a G-invariant ergodic probability measure on Δ.*

Proof. If m is a G-invariant ergodic probability measure on Δ, then

$$1 = m(\Delta) = \sum_{i\in\mathbb{N}} \sum_{0\leq\ell\leq t_i} m(\Delta(\alpha^i_\ell)) = \sum_{i\in\mathbb{N}} t_i a_i = 0,$$

which is a contradiction. □

Recall that if $i < j$, then $t_{ij} = n_j - s_{ij} n_i$. In order to simplify notation, we will continue to write t_{i+1} instead of t_{ii+1}. Applying Lemma 3.9, it follows that the limit $b_i = \lim_{j\to\infty} t_{ij}/n_j$ exists and that $b_i = 1 - n_i a_i$. Thus we obtain:

Lemma 3.15. *If m is a G-invariant ergodic probability measure on Δ and $i \in \mathbb{N}$, then $m(\bigsqcup\{\, \Delta(\alpha^j_\ell) \mid i < j, \ell < t_j \,\}) = b_i$.*

Note that if $A \in \mathcal{A}$, then there exists $i \in \mathbb{N}$ and $S \subseteq \Delta_i$ such that either

(a) $A = \bigsqcup\{\, \Delta(\sigma) \mid \sigma \in S \,\}$; or
(b) $A = \bigsqcup\{\, \Delta(\sigma) \mid \sigma \in S \,\} \sqcup \bigsqcup\{\, \Delta(\alpha^j_\ell) \mid i < j, \ell < t_j \,\}$.

Furthermore, by Lemmas 3.13 and 3.15, if m is a G-invariant ergodic probability measure on Δ, then $m_0 = m \restriction \mathcal{A}$ must be defined by

$$m_0(A) = \begin{cases} |S|\, a_i, & \text{if (a) holds;} \\ |S|\, a_i + b_i, & \text{if (b) holds.} \end{cases} \tag{3.1}$$

Since $a_{i+1} = a_i/s_{i+1}$ and $b_i = t_{i+1} a_{i+1} + b_{i+1}$, it follows that m_0 is well-defined. It is also clear that $m_0(\Delta) = 1$ and that m_0 is G-invariant. The following lemma will be used to prove that m_0 is σ-additive.

Lemma 3.16. *If G has linear natural orbit growth, then* $\lim_{i\to\infty} b_i = 0$.

Proof. Suppose that G has linear natural orbit growth. Since $t_{ij} = \sum_{k=i+1}^{j-1} s_{kj} t_k + t_j$, it follows that

$$\frac{t_{ij}}{n_j} = \frac{s_{0j}}{n_j} \sum_{k=i+1}^{j} \frac{t_k}{s_{0k}}$$

and hence

$$b_i = a_0 \sum_{k=i+1}^{\infty} \frac{t_k}{s_{0k}}.$$

In the proof of Lemma 3.12, we showed that if G has linear natural orbit growth, then $\sum_{k=1}^{\infty} \frac{t_k}{s_{0k}} < \infty$ and it follows that $b_i = a_0 \sum_{k=i+1}^{\infty} t_k/s_{0k} \to 0$ as $i \to \infty$. □

Proposition 3.17. *If G has linear natural orbit growth, then the action $G \curvearrowright \Delta$ is uniquely ergodic.*

Proof. Since any probability measure μ on Δ is uniquely determined by $\mu \restriction \mathcal{A}$, it is already clear that there exists at most one G-invariant ergodic probability measure on Δ. Hence it is enough to show that the function m_0, defined by (3.1), can be extended to a G-invariant probability measure on Δ. We have already noted that $b_i = t_{i+1}a_{i+1} + b_{i+1}$; and an easy inductive argument shows that if $i < j$, then

$$b_i = t_{i+1}a_{i+1} + t_{i+2}a_{i+2} + \cdots + t_j a_j + b_j;$$

Since $\lim_{j\to\infty} b_j = 0$, it follows that $b_i = \sum_{j>i} t_j a_j$. It is now clear that m_0 is a pre-measure on $\mathcal{A}$. By the Carathéodory Extension Theorem, m_0 can be extended to a probability measure m on Δ; and since m_0 is G-invariant, it follows that m is also G-invariant. □

It is easily checked that the stabilizer distribution of the action $G \curvearrowright (\Delta, m)$ does not depend on the expression of G as the union of a strictly increasing chain of finite alternating groups. (Once again, this is clear in the case when G is expressed as the union of a chain of finite alternating groups $G'_i = \mathrm{Alt}(\Delta_{j_i})$ for some strictly increasing sequence $(j_i \mid i \in \mathbb{N})$ of natural numbers; and the general case follows easily.) From now on, we will refer to $G \curvearrowright (\Delta, m)$ as the *canonical ergodic action*. By Corollary 2.5, the action $G \curvearrowright (\Delta^r, m^{\otimes r})$ is ergodic for all $r \geq 1$, and hence the corresponding stabilizer distributions ν_r are ergodic IRS of G. We are now in a position to state the second of the main results of this paper.

Theorem 3.18. *If G is an $L(\mathrm{Alt})$-group with linear natural orbit growth, then the ergodic IRS of G are $\{\delta_1, \delta_G\} \cup \{\nu_r \mid r \in \mathbb{N}^+\}$.*

We are now ready to present the proof of Theorem 1.6. So suppose that G is an $L(\mathrm{Alt})$-group with $G \ncong \mathrm{Alt}(\mathbb{N})$ and that $G \curvearrowright (Z, \mu)$ is an ergodic action. Let ν be the corresponding stabilizer distribution and let $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ be the associated character. By Theorem 3.4, if G does not have almost diagonal type, then $\nu \in \{\delta_1, \delta_G\}$, and so $\chi \in \{\chi_{\mathrm{reg}}, \chi_{\mathrm{con}}\}$, and it follows that χ is indecomposable. Hence we can suppose that $G \ncong \mathrm{Alt}(\mathbb{N})$ has almost diagonal type; and so Theorem 1.6 is a consequence of the following result.

Corollary 3.19. *If $G \ncong \mathrm{Alt}(\mathbb{N})$ has almost diagonal type and $G \curvearrowright (Z, \mu)$ is an ergodic action, then the associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is indecomposable.*

Proof. Let ν be the stabilizer distribution of the ergodic action $G \curvearrowright (Z, \mu)$. Then, as above, we can suppose that $\nu \neq \delta_1$, δ_G.

First suppose that G has linear natural orbit growth. Then $\nu = \nu_r$ is the stabilizer distribution of the ergodic action $G \curvearrowright (\Delta^r, m^{\otimes r})$ for some $r \geq 1$, where $G \curvearrowright (\Delta, m)$ is the canonical ergodic action. Let $\bar{x} = (x_1, \cdots x_r) \in \Delta^r$ and let

$$G_{\bar{x}} = \{ g \in G \mid g \cdot x_\ell = x_\ell \text{ for } 1 \leq \ell \leq r \}$$

be the corresponding stabilizer. Then it is easily checked that

$$\mathrm{Fix}_\Delta(G_{\bar{x}}) = \{ x_\ell \mid 1 \leq \ell \leq r \}.$$

Suppose that $g \in N_G(G_{\bar{x}}) \smallsetminus H$. Then g permutes the the elements of the set $\mathrm{Fix}_\Delta(G_{\bar{x}})$ nontrivially, and hence there exist $1 \leq \ell < m \leq r$ such that $g \cdot x_\ell = x_m$, and this implies that the sequences x_ℓ, x_m are eventually equal. It follows that $G_{\bar{x}}$ is self-normalizing for $m^{\otimes r}$-a.e. $\bar{x} \in \Delta^r$, and this implies that G_z is self-normalizing for μ-a.e. $z \in Z$. Applying Vershik [19], it follows that χ is indecomposable.

Hence we can suppose that G has sublinear natural orbit growth. Express $G = \bigcup_{i\in\mathbb{N}} G_i$ as an almost diagonal limit of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. Since $G \not\cong \mathrm{Alt}(\mathbb{N})$, we can suppose that $s_{i+1} > 1$ for all $i \in \mathbb{N}$. For each $\ell \in \mathbb{N}$, define the subsets $\Sigma^\ell_j \subseteq \Delta_j$ and subgroups $G(\ell)_j = \mathrm{Alt}(\Sigma^\ell_j)$ for $j \geq \ell$ inductively as follows:

- $\Sigma^\ell_\ell = \Delta_\ell$;
- $\Sigma^\ell_{j+1} = \Delta_{j+1} \smallsetminus \mathrm{Fix}_{\Delta_{j+1}}(G(\ell)_j)$.

Let $G(\ell) = \bigcup_{\ell\leq j\in\mathbb{N}} G(\ell)_j$. Then it is easily checked that if $\ell < m$, then $G(\ell) \leqslant G(m)$ and that $G = \bigcup_{\ell\in\mathbb{N}} G(\ell)$.

Claim 3.20. *$G(\ell)$ has linear natural orbit growth for all $\ell \in \mathbb{N}$.*

Proof. For each $i \geq \ell$, let $n^\ell_i = |\Sigma^\ell_i|$. Note that if $i \geq \ell$, then $G(\ell)_i$ has s_{i+1} natural orbits on Σ^ℓ_{i+1} and that

$$n^\ell_{i+1} \leq s_{i+1} n^\ell_i + e_{i+1}.$$

It follows that if $\ell \leq i < j$, then

$$n^\ell_j \leq s_{ij} n^\ell_i + s_{0j} \sum_{k=i+1}^{j} e_k/s_{0k} \leq s_{ij} n^\ell_i + s_{0j}\gamma_i = s_{ij}(\, n^\ell_i + s_{0i}\gamma_i\,).$$

Thus $n^\ell_j/s_{ij} \leq n^\ell_i + s_{0i}\gamma_i$ and it follows that $\lim_{j\to\infty} s_{ij}/n^\ell_j > 0$. □

In particular, it follows that each $G(\ell)$ is a proper subgroup of G. For each $\ell \in \mathbb{N}$, let $G(\ell) \curvearrowright (\,\Delta_\ell, m_\ell\,)$ be the canonical ergodic action and for each $r \in \mathbb{N}^+$, let $\nu(\ell)_r$ be the stabilizer distribution of $G(\ell) \curvearrowright (\,\Delta^r_\ell, m^{\otimes r}_\ell\,)$. Let $\nu_{G(\ell)}$ be the IRS of $G(\ell)$ arising from the $G(\ell)$-equivariant map $\mathrm{Sub}_G \to \mathrm{Sub}_{G(\ell)}$ defined by $H \mapsto H \cap G(\ell)$. Then Theorem 3.18 implies that there exist $\alpha(\ell)$, $\beta(\ell)$, $\gamma(\ell)_r \in [\,0,1\,]$ with $\alpha(\ell) + \beta(\ell) + \sum_{r\in\mathbb{N}^+} \gamma(\ell)_r = 1$ such that

$$\nu_{G(\ell)} = \alpha(\ell)\delta_1 + \beta(\ell)\delta_{G(\ell)} + \sum_{r\in\mathbb{N}^+} \gamma(\ell)_r \nu(\ell)_r. \tag{3.2}$$

Recall that $\nu \neq \delta_1,\ \delta_G$. Thus (3.2), together with the analysis in the second paragraph of this proof, implies that for ν-a.e. $H \in \mathrm{Sub}_G$, there exists an integer ℓ_H such that $H \cap G(\ell)$ is a (proper) self-normalizing subgroup of G for all $\ell \geq \ell_H$, and this implies that H is also self-normalizing. It follows that G_z is self-normalizing for μ-a.e. $z \in Z$; and by Vershik [19], this implies that χ is indecomposable. □

For later use, we record the following recognition theorem, which will play a role in the proofs of Theorems 3.4 and 3.18.

Theorem 3.21. *Suppose that G is an $L(\mathrm{Alt})$-group with linear natural orbit growth and that ν is an ergodic IRS of G. If there exists a constant $s \geq 1$ and an expression $G = \bigcup_{i\in\mathbb{N}} G'_i$ of G as a union of finite alternating groups $G'_i = \mathrm{Alt}(\Delta'_i)$ such that for ν-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists an integer $1 \leq r_i \leq s$ and a subset $U_i \in [\,\Delta'_i\,]^{r_i}$ such that $H_i = H \cap G'_i = \mathrm{Alt}(\Delta'_i \smallsetminus U_i)$, then $\nu = \nu_r$ for some $1 \leq r \leq s$.*

Proof. In order to simplify notation, we will write Δ_i instead of Δ'_i. (This is harmless, since we have already noted that the stabilizer distribution of the action $G \curvearrowright (\Delta, m)$ does not depend on the expression of G as the union of a strictly increasing chain of finite alternating groups.)

For each $i \in \mathbb{N}$ and $1 \leq t \leq s$, let p_{it} be the ν-probability that there exists $U_i \in [\Delta_i]^t$ such that $H_i = \mathrm{Alt}(\Delta_i \smallsetminus U_i)$. By Lemma 3.16, since G has linear natural orbit growth, we have that $\lim_{j\to\infty} b_j = 0$, where

$$b_j = \lim_{k\to\infty} \frac{(n_k - s_{jk} n_j)}{n_k}.$$

It follows that for all $i \in \mathbb{N}$, if $j > i$ is sufficiently large, then b_j is sufficiently small so that there exists $k > j$ such that

$$\sum_{t=1}^{t=s} p_{kt} \left[1 - \frac{\binom{s_{jk} n_j}{t}}{\binom{n_k}{t}} \right] \leq \left(\frac{1}{2}\right)^{i+1}.$$

Hence we can inductively define a sequence of integers k_i such that

$$\sum_{t=1}^{t=s} p_{k_{i+1} t} \left[1 - \frac{\binom{s_{k_i k_{i+1}} n_{k_i}}{t}}{\binom{n_{k_{i+1}}}{t}} \right] \leq \left(\frac{1}{2}\right)^{i+1}.$$

Let $\Phi_{k_{i+1}}$ be the union of the $s_{k_i k_{i+1}}$ natural G_{k_i}-orbits on $\Delta_{k_{i+1}}$. Then, applying the Borel-Cantelli Lemma, it follows that for ν-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $i \in \mathbb{N}$, there exists a subset $U_{k_{i+1}} \subseteq \Phi_{k_{i+1}}$ of cardinality $r_{k_{i+1}}$ such that $H_{k_{i+1}} = \mathrm{Alt}(\Delta_{k_{i+1}} \smallsetminus U_{k_{i+1}})$. Furthermore, by ergodicity, there exists a constant $1 \leq r \leq s$ such that $r = \liminf r_{k_i}$ for ν-a.e. $H \in \mathrm{Sub}_G$. Suppose that $H \in \mathrm{Sub}_G$ is such a ν-generic subgroup and that $U_{k_{i+1}} \subseteq \Phi_{k_{i+1}}$ is a subset of cardinality $r_{k_{i+1}} = r$ such that $H_{k_{i+1}} = \mathrm{Alt}(\Delta_{k_{i+1}} \smallsetminus U_{k_{i+1}})$. Using the fact that $U_{k_{i+1}} \subseteq \Phi_{k_{i+1}}$, it follows that there exists a subset $U'_{k_i} \subseteq \Delta_{k_i}$ such that $r_{k_i} \leq |U'_{k_i}| \leq |U_{k_{i+1}}| = r$ and $\mathrm{Alt}(\Delta_{k_i} \smallsetminus U'_{k_i}) \leqslant H_{k_i}$. Consequently, it follows that $k_i = r$ for all but finitely many $i \in \mathbb{N}$.

Definition 3.22. Let $\mathcal{S}_r$ be the standard Borel space of subgroups $H \leqslant G$ such that for all but finitely many $i \in \mathbb{N}$, there exists a subset $U_{k_i} \in [\Delta_{k_i}]^r$ such that $H_{k_i} = \mathrm{Alt}(\Delta_{k_i} \smallsetminus U_{k_i})$.

Then we have shown that the ergodic IRS ν concentrates on $\mathcal{S}_r$. Since the stabilizer distribution ν_r of $G \curvearrowright (\Delta^r, m^{\otimes r})$ also concentrates on $\mathcal{S}_r$, the following claim completes the proof of Theorem 3.21.

Claim 3.23. *The action $G \curvearrowright \mathcal{S}_r$ is uniquely ergodic.*

Proof of Claim 3.23. (The following argument is essentially identical to the proof of Thomas-Tucker-Drob [17, Proposition 6.8].) It is enough to show that if μ is an ergodic probability measure on $\mathcal{S}_r$ and $B \subseteq \mathrm{Sub}_G$ is a basic clopen subset, then $\mu(B) = \nu_r(B)$. Let $B = \{ H \in \mathrm{Sub}_G \mid H \cap G_\ell = L \}$, where $\ell \in \mathbb{N}$ and $L \leqslant G_\ell$ is a subgroup. By the Pointwise Ergodic Theorem, there exists $H \in \mathcal{S}_r$ such that

$$\begin{aligned}
\mu(B) &= \lim_{i\to\infty} |\{ g \in G_i \mid gHg^{-1} \in B \}|/|G_i| \\
&= \lim_{i\to\infty} |\{ g \in G_i \mid gH_ig^{-1} \cap G_\ell = L \}|/|G_i| \\
&= \lim_{i\to\infty} |\{ g \in G_{k_i} \mid gH_{k_i}g^{-1} \cap G_\ell = L \}|/|G_{k_i}|.
\end{aligned}$$

Similarly, there exists $H' \in \mathcal{S}_r$ such that

$$\sigma_r(B) = \lim_{i\to\infty} |\{\, g \in G_{k_i} \mid gH'_{k_i}g^{-1} \cap G_\ell = L \,\}|/|G_{k_i}|.$$

Since H, $H' \in \mathcal{S}_r$, there exists $i_0 \in \mathbb{N}$ such that H_{k_i} and H'_{k_i} are conjugate in G_{k_i} for all $i \geq i_0$ and this implies that

$$\lim_{i\to\infty} |\{\, g \in G_{k_i} \mid gH_{k_i}g^{-1} \cap G_\ell = L \,\}|/|G_{k_i}| = \lim_{i\to\infty} |\{\, g \in G_{k_i} \mid gH'_{k_i}g^{-1} \cap G_\ell = L \,\}|/|G_{k_i}|.$$

□

□

Finally recall that if G is a countable group and $\chi \in \mathcal{F}(G)$ is a character, then the corresponding proper 2-sided ideal I_χ of the group $\mathbb{C}G$ is defined by

$$I_\chi = \{\, x \in \mathbb{C}(G) \mid \chi(g\,x) = 0 \text{ for all } g \in G \,\}.$$

As explained in Section 1, the following result exhibits a counterexample to Zalesskiĭ [23, Conjecture 1.24] and also answers Zalesskiĭ [23, Question 5.12].

Proposition 3.24. *There exists an $L(\mathrm{Alt})$-group G such that:*

- (i) *The augmentation ideal $\omega(\mathbb{C}G)$ is the only nontrivial proper 2-sided ideal of $\mathbb{C}G$.*
- (ii) *G has a nontrivial ergodic IRS.*
- (iii) *G has infinitely many indecomposable characters χ such that $I_\chi = \{\, 0 \,\}$.*

Proof. Define $G_i = \mathrm{Alt}(\Delta_i)$ and s_{i+1} inductively as follows.

- $\Delta_0 = \{\, 0, 1, 2, 3, 4 \,\}$;
- $\Delta_{i+1} = \{\, \sigma^\frown k \mid \sigma \in \Delta_i, 0 \leq k < s_{i+1} \,\} \sqcup G_i$, where $s_{i+1} = 2^i\,|G_i|$;

and the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ is defined by

- $\varphi_i(g)(\sigma^\frown k) = g(\sigma)^\frown k$ for each $\sigma \in \Delta_i$ and $0 \leq k < s_{i+1}$;
- $\varphi_i(g)(h) = g\,h$ for each $h \in G_i$.

Let $G = \bigcup_{i\in\mathbb{N}} G_i$. By construction, if $i < j$, then G_i has a regular orbit on Δ_j. Hence, by Zalesskiĭ [22, Lemma 10], the augmentation ideal is the only nontrivial proper 2-sided ideal of $\mathbb{C}G$. Also if $i < j$, then s_{ij} is clearly the number of natural orbits of G_i on Δ_j. Furthermore, an easy induction shows that if $i < j$, then

$$|\Delta_j| = s_{ij}|\Delta_i| + \sum_{k=i}^{j-2} s_{k+1j}|G_k| + |G_{j-1}|$$

and hence

$$\begin{aligned} \frac{n_j}{s_{ij}} &= |\Delta_i| + \sum_{k=i}^{j-2} \frac{s_{k+1j}|G_k|}{s_{ij}} + \frac{|G_{j-1}|}{s_{ij}} \\ &\leq |\Delta_i| + \sum_{k=i}^{j-1} \frac{|G_k|}{s_{k+1}} \\ &= |\Delta_i| + \sum_{k=i}^{j-1} \frac{1}{2^k} < |\Delta_i| + 2. \end{aligned}$$

It follows that $a_i = \lim_{j\to\infty} s_{ij}/n_j > 0$ and thus G has linear natural orbit growth. Let $G \curvearrowright (\Delta, m)$ be the canonical ergodic action. Then for each $r \geq 1$,

$$\chi_r(g) = m^{\otimes r}(\,\mathrm{Fix}_{\Delta^r}(g)\,)$$

is an indecomposable character of G; and it is easily checked that if $r \neq s$, then $\chi_r \neq \chi_s$. Since $\chi_r \neq \chi_{\mathrm{con}}$, it follows that $I_{\chi_r} \neq \omega(\mathbb{C}G)$ and so $I_{\chi_r} = \{\,0\,\}$. □

4. Sublinear natural orbit growth

In this section, we will discuss the ergodic IRSs of the $L(\mathrm{Alt})$-groups G of almost diagonal type such that G has sublinear natural orbit growth. Examining the list of ergodic IRSs in the statement of Theorem 3.18, we see that $G = \bigcup_{i\in\mathbb{N}} G_i$ is an $L(\mathrm{Alt})$-group with linear natural orbit growth and $\nu \neq \delta_1,\ \delta_G$ is an ergodic IRS, then ν concentrates on the subspace of subgroups $H \in \mathrm{Sub}_G$ such that there exists a *fixed* integer $r \geq 1$ such that for all but finitely many $i \in \mathbb{N}$, there exists a subset $\Sigma_i \subseteq \Delta_i$ of cardinality r such that:

- $H \cap G_i = \mathrm{Alt}(\Delta_i \smallsetminus \Sigma_i)$; and
- Σ_{i+1} is contained in the union of the natural G_i-orbits on Δ_i.

As is suggested by the proof of Corollary 3.19, a similar result holds if $G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit with sublinear natural orbit growth, except that in this case:

- $d_i = |\Sigma_i| \to \infty$ as $i \to \infty$; and
- Σ_{i+1} is contained in the union of the natural and trivial G_i-orbits on Δ_{i+1}.

In order to simplify notation, we will work with the G-invariant probability measures on the space of corresponding sequences of subsets $(\,\Sigma_i\,)$ rather than directly with the IRSs on Sub_G. Of course, such a measure can be identified with a corresponding IRS via the map

$$(\,\Sigma_i\,) \mapsto H = \bigcup \mathrm{Alt}(\Delta_i \smallsetminus \Sigma_i).$$

Throughout this section, we will suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. Initially we will *not* assume that G has sublinear natural orbit growth. Let Σ consist of the infinite sequences of sets $(\,\Sigma_i\,)_{i\geq i_0}$ for some $i_0 \in \mathbb{N}$ such that the following conditions are satisfied for all $i \geq i_0$

- $\Sigma_i \subseteq \Delta_i$;
- $\mathrm{Alt}(\Delta_{i+1} \smallsetminus \Sigma_{i+1}) \cap G_i = \mathrm{Alt}(\Delta_i \smallsetminus \Sigma_i)$;
- Σ_{i+1} is contained in the union of the natural and trivial G_i-orbits on Δ_{i+1};
- if $i_0 > 0$, then $\mathrm{Alt}(\Delta_{i_0} \smallsetminus \Sigma_{i_0}) \cap G_{i_0-1}$ does not have the form $\mathrm{Alt}(\Delta_{i_0-1} \smallsetminus U)$ for any subset $U \subseteq \Delta_{i_0-1}$.

Then the natural action of G on Σ corresponds to the conjugacy action of G on the subspace of subgroups $\{\,\bigcup_{i\geq i_0} \mathrm{Alt}(\Delta_i \smallsetminus \Sigma_i) \mid (\,\Sigma_i\,)_{i\geq i_0} \in \Sigma\,\}$.

Remark 4.1. For later use, note that if $(\,\Sigma_i\,)_{i\geq i_0} \in \Sigma$ and $i_0 \leq i < j$, then $|\Sigma_i| \leq |\Sigma_j|$; and if $|\Sigma_i| = |\Sigma_j|$, then Σ_j is contained in the union of the natural G_i-orbits on Δ_j.

Fix some $\beta_0 \in (\,0,\infty\,)$ and let $\gamma_0 = \beta_0\,\tau = \beta_0 \sum_{i=1}^{\infty} e_i/s_{0i}$. For each $i \in \mathbb{N}$, let

- $\beta_{i+1} = \beta_i/s_{i+1} = \beta_0/s_{0i+1}$; and
- $\gamma_{i+1} = \gamma_i - \beta_i e_{i+1}/s_{i+1} = \beta_0 \sum_{j=i+2}^{\infty} e_j/s_{0j}$.

For each $i \in \mathbb{N}$ and $X \subseteq \Delta_i$, let $\Sigma(X)$ be the set of sequences $(\Sigma_j)_{j \geq j_0} \in \Sigma$ for some $j_0 \leq i$ such that $\Sigma_i = X$. Then the sets $\Sigma(X)$ form a clopen basis for a locally compact topology on Σ. First define μ_{β_0} on the basic clopen sets by

$$\mu_{\beta_0}(\Sigma(X)) = \frac{1}{e^{\beta_i n_i + \gamma_i}}(e^{\beta_i} - 1)^{|X|}. \tag{4.1}$$

Note that (4.1) can be rewritten as:

$$\mu_{\beta_0}(\Sigma(X)) = \frac{1}{e^{\gamma_i}}\left(1 - \frac{1}{e^{\beta_i}}\right)^{|X|}\left(\frac{1}{e^{\beta_i}}\right)^{n_i - |X|}. \tag{4.2}$$

Thus, modulo the "correction factor" $1/e^{\gamma_i}$, the probability that $\Sigma_i = X$ is simply that given by the binomial distribution when the probability of selecting a point $x \in \Delta_i$ is $p_i = 1 - (1/e^{\beta_i})$.

Let $\mathcal{A}$ be the algebra of Borel subsets of Σ generated by the basic clopen sets $\Sigma(X)$. Note that if $A \in \mathcal{A}$, then there exists $i \in \mathbb{N}$ and $S \subseteq \mathcal{P}(\Delta_i)$ such that either:

(a) $A = \bigsqcup\{\Sigma(X) \mid X \in S\}$, or
(b) $A = \bigsqcup\{\Sigma(X) \mid X \in S\} \sqcup (\Sigma \smallsetminus B_i)$, where $B_i = \bigsqcup\{\Sigma(X) \mid X \in \mathcal{P}(\Delta_i)\}$.

We next extend μ_{β_0} to the algebra $\mathcal{A}$ by defining

$$\mu_{\beta_0}(A) = \begin{cases} \sum_{X \in S} \mu_{\beta_0}(\Sigma(X)), & \text{if (a) holds;} \\ \sum_{X \in S} \mu_{\beta_0}(\Sigma(X)) + (1 - (1/e)^{\gamma_i}), & \text{if (b) holds.} \end{cases}$$

We claim that μ_{β_0} is a pre-measure on $\mathcal{A}$. Of course, we must first check that μ_{β_0} is well-defined. To see this, fix some $i \in \mathbb{N}$ and for each $X \subseteq \Delta_i$, let E_X be the collection of subsets $Y \subseteq \Delta_{i+1}$ such that $\mathrm{Alt}(\Delta_{i+1} \smallsetminus Y) \cap G_{i+1} = \mathrm{Alt}(\Delta_i \smallsetminus X)$ and Y is contained in the union of the natural and trivial G_i-orbits on Δ_{i+1}. We will prove by induction on $\ell = |X|$ that $\mu_{\beta_0}(\Sigma(X)) = \sum_{Y \in E_X} \mu_{\beta_0}(\Sigma(Y))$. First suppose that $\ell = 0$. Then

$$\mu_{\beta_0}(\Sigma(\emptyset)) = \frac{1}{e^{\beta_i n_i + \gamma_i}}.$$

Also $Y \in E_\emptyset$ if and only if Y is a subset of the trivial G_i-orbits on Δ_{i+1}. Thus

$$\begin{aligned} \mu_{\beta_0}(E_\emptyset) &= \frac{1}{e^{\beta_{i+1} n_{i+1} + \gamma_{i+1}}} \sum_{t=0}^{f_{i+1}} \binom{f_{i+1}}{t} (e^{\beta_{i+1}} - 1)^t \\ &= \frac{1}{e^{\beta_{i+1} n_{i+1} + \gamma_{i+1}}} e^{\beta_{i+1} f_{i+1}} \\ &= \frac{1}{e^{\beta_{i+1}(n_{i+1} - f_{i+1}) + \gamma_{i+1}}}. \end{aligned}$$

By definition, we have that

$$\beta_{i+1}(n_{i+1} - f_{i+1}) + \gamma_{i+1} = \frac{\beta_i}{s_{i+1}}(s_{i+1} n_i + e_{i+1}) + \gamma_i - \frac{\beta_i e_{i+1}}{s_{i+1}} = \beta_i n_i + \gamma_i.$$

Hence the result holds when $\ell = 0$. Suppose inductively that the result holds for $\ell \geq 0$ and let $X \subseteq \Delta_i$ with $|X| = \ell + 1$. Then

$$\mu_{\beta_0}(\Sigma(X)) = \frac{1}{e^{\beta_i n_i + \gamma_i}}(e^{\beta_i} - 1)^{\ell+1}.$$

Write $X = X_0 \cup \{ x \}$, where $|X_0| = \ell$. Then $Y \in E_X$ if and only if $Y = Y_0 \sqcup Z$, where $Y_0 \in E_{X_0}$ and $Z \in E_{\{ x \}}$. Thus

$$\begin{aligned}\mu_{\beta_0}(E_X) &= \sum_{Y_0 \in E_{X_0}} \mu_{\beta_0}(\Sigma(Y_0)) \sum_{t=1}^{s_{i+1}} \binom{s_{i+1}}{t} (e^{\beta_{i+1}} - 1)^t \\ &= \frac{1}{e^{\beta_i n_i + \gamma_i}} (e^{\beta_i} - 1)^{\ell} (e^{\beta_{i+1} s_{i+1}} - 1) \\ &= \frac{1}{e^{\beta_i n_i + \gamma_i}} (e^{\beta_i} - 1)^{\ell+1}.\end{aligned}$$

Thus $\mu_{\beta_0}(A)$ is well-defined if $A = \bigsqcup \{ \Sigma(X) \mid X \in S \}$ for some $S \subseteq \mathcal{P}(\Delta_i)$. Also, since

$$\mu_{\beta_0}(B_i) = \frac{1}{e^{\beta_i n_i + \gamma_i}} \sum_{\ell=0}^{n_i} \binom{n_i}{\ell} (e^{\beta_i} - 1)^{\ell} = \frac{1}{e^{\gamma_i}},$$

it follows that $\mu_{\beta_0}(A)$ is well-defined if $A = \bigsqcup \{ \Sigma(X) \mid X \in S \} \sqcup (\Sigma \smallsetminus B_i)$; and it also follows that $\mu_{\beta_0}(\Sigma) = 1$. Finally to check that μ_{β_0} is σ-additive, it is enough to show that for all $i \in \mathbb{N}$,

$$\sum_{j=i}^{\infty} \mu_{\beta_0}(B_{j+1} \smallsetminus B_j) = \mu(\Sigma \smallsetminus B_i) = 1 - (1/e)^{\gamma_i}.$$

To see this, note that if $k > i$, then

$$\sum_{j=i}^{k} \mu_{\beta_0}(B_{j+1} \smallsetminus B_j) = \sum_{j=i}^{k} [(1/e)^{\gamma_{j+1}} - (1/e)^{\gamma_j}] = (1/e)^{\gamma_{k+1}} - (1/e)^{\gamma_i};$$

and since $\gamma_{k+1} = \beta_0 \sum_{j=k+1}^{\infty} e_{j+1}/s_{0j+1} \to 0$ as $k \to \infty$, it follows that

$$\sum_{j=i}^{k} \mu_{\beta_0}(B_{j+1} \smallsetminus B_j) \to 1 - (1/e)^{\gamma_i}$$

as $k \to \infty$. This completes the proof that μ_{β_0} is a pre-measure on $\mathcal{A}$. Clearly μ is G-invariant. Hence, by the Carathéodory Extension Theorem, μ_{β_0} can be extended to a G-invariant probability measure μ_{β_0} on Σ.

Theorem 4.2. *If $G = \bigcup_{i \in \mathbb{N}} G_i$ is an almost diagonal limit of finite alternating groups and $\beta_0 \in (0, \infty)$, then the action $G \curvearrowright (\Sigma, \mu_{\beta_0})$ is ergodic if and only if G has sublinear natural orbit growth.*

We will begin with the easy direction in Theorem 4.2.

Proposition 4.3. *If G has linear natural orbit growth, then $G \curvearrowright (\Sigma, \mu_{\beta_0})$ is not ergodic.*

Proof. If G has linear natural orbit growth, then

$$\lim_{i \to \infty} \beta_i n_i + \gamma_i = \lim_{i \to \infty} \beta_0 \frac{n_i}{s_{0i}} + \gamma_i = \frac{\beta_0}{a_0} > 0.$$

Hence if $\sigma \in \Sigma$ is the sequence with constant value $\emptyset$ and $X_0 = \{ \sigma \}$, then

$$\mu_{\beta_0}(X_0) = \lim_{i \to \infty} \frac{1}{e^{\beta_i n_i + \gamma_i}} = \frac{1}{e^{\beta_0 / a_0}}.$$

Since X is a G-invariant Borel subset with $0 < \mu_{\beta_0}(X) < 1$, it follows that the action $G \curvearrowright (\Sigma, \mu_{\beta_0})$ is not ergodic. $\square$

Remark 4.4. If G has linear natural orbit growth, then we can calculate the ergodic decomposition of the action $G \curvearrowright (\Sigma, \mu_{\beta_0})$ as follows. Let $\lambda = \beta_0/a_0$. For each $r \geq 0$, if $X_r \subseteq \Sigma$ is the Borel subset consisting of the sequences $(\Sigma_j)_{j \geq j_0}$ such that $|\Sigma_j| = r$ for all but finitely many $j \geq j_0$, then

$$\mu_{\beta_0}(X_r) = \frac{1}{e^{\lambda}} \frac{\lambda^r}{r!}.$$

To see this, note that

$$\mu_{\beta_0}(X_1) = \lim_{j\to\infty} \frac{1}{\beta_j n_j + \gamma_j} \cdot n_j (e^{\beta_0 \frac{n_j}{s_{0j}} \frac{1}{n_j}} - 1) = \frac{1}{e^{\lambda}} \cdot \lambda,$$

and that if $r \geq 2$, then

$$\begin{aligned}\mu_{\beta_0}(X_r) &= \lim_{j\to\infty} \frac{1}{\beta_j n_j + \gamma_j} \binom{n_j}{r} (e^{\beta_0 \frac{n_j}{s_{0j}} \frac{1}{n_j}} - 1)^r \\ &= \lim_{j\to\infty} \frac{1}{\beta_j n_j + \gamma_j} \frac{n_j^r}{r!} (e^{\beta_0 \frac{n_j}{s_{0j}} \frac{1}{n_j}} - 1)^r \\ &= \frac{1}{e^{\lambda}} \frac{\lambda^r}{r!}.\end{aligned}$$

If we identify μ_{β_0} with the corresponding IRS of G, then X_r corresponds to the IRS ν_r of Theorem 3.18. Thus, writing $\delta_G = \nu_0$, we obtain the ergodic decomposition:

$$\mu_{\beta_0} = \frac{1}{e^{\lambda}} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \nu_r.$$

For the remainder of this section, we will suppose that G has sublinear natural orbit growth. Here the analysis splits into two cases depending on whether or not $G \cong \mathrm{Alt}(\mathbb{N})$; equivalently, on whether or not $s_{i+1} = 1$ and $e_{i+1} = 0$ for all but finitely many many $i \in \mathbb{N}$. First suppose that $G \cong \mathrm{Alt}(\mathbb{N})$. In order to simplify notation, we will suppose that $s_{i+1} = 1$ and $e_{i+1} = 0$ for all $i \in \mathbb{N}$. And we can also suppose that $G = \mathrm{Alt}(\mathbb{N})$ and that each $\Delta_i = \{0, 1, \cdots, n_i - 1\}$. Let $\alpha_0 = 1 - (1/e^{\beta_0})$ and $\alpha_1 = 1/e^{\beta_0}$. Let p_α be the probability measure on $\{0, 1\}$ defined by $p_\alpha(\{\ell\}) = \alpha_\ell$ and let μ_α be the corresponding product probability measure on $2^{\mathbb{N}}$. Then $\mathrm{Alt}(\mathbb{N})$ acts ergodically on $(2^{\mathbb{N}}, \mu_\alpha)$ via the shift action $(g \cdot \xi)(n) = \xi(g^{-1}(n))$. Let $\xi \overset{f_\alpha}{\mapsto} (\Sigma_i^\xi)_{i \geq 0}$ be the $\mathrm{Alt}(\mathbb{N})$-equivariant map from $2^{\mathbb{N}}$ to Σ defined by $\Sigma_i^\xi = \{k \in \Delta_i \mid \xi(k) = 0\}$. Then $\mu_{\beta_0} = (f_\alpha)_* \mu_\alpha$ and it follows that the action $\mathrm{Alt}(\mathbb{N}) \curvearrowright (\Sigma, \mu_{\beta_0})$ is ergodic. (Using the notation of Section 9, the stabilizer distribution corresponding to μ_{β_0} is the ergodic IRS $\nu_\alpha^{E_\alpha}$ of $\mathrm{Alt}(\mathbb{N})$.)

Hence we can suppose that $G \ncong \mathrm{Alt}(\mathbb{N})$ and that $\lim_{i\to\infty} \beta_i = 0$. In order to prove that $G \curvearrowright (\Sigma, \mu_{\beta_0})$ is ergodic, it is enough to find a G-invariant Borel subset $\Sigma_{\beta_0} \subseteq \Sigma$ such that $\mu_{\beta_0}(\Sigma_{\beta_0}) = 1$ and such that if m is an ergodic probability measure on Σ_{β_0}, then

$$m(\Sigma(X) \cap \Sigma_{\beta_0}) = \mu_{\beta_0}(\Sigma(X)).$$

for all $X \in \bigcup_{i\in\mathbb{N}} \mathcal{P}(\Delta_i)$. The definition of Σ_{β_0} will involve the following sequence of random variables.

Definition 4.5. For each $i \in \mathbb{N}$, let d_i be the random variable on Σ defined by

$$d_i((\Sigma_j)_{j \geq j_0}) = \begin{cases} |\Sigma_i|, & \text{if } i \geq j_0; \\ 0, & \text{otherwise.} \end{cases}$$

In preparation for an application of Chebyshev's inequality, we will next compute the expectation $\mathbb{E}[d_i]$ and the variance $\mathrm{Var}(d_i)$ of the random variable d_i. Here we will make use of the observation that modulo the "correction factor" $1/e^{\gamma_i}$, the probability that $\Sigma_i = X$ is that given by the binomial distribution when the probability of selecting a point $x \in \Delta_i$ is $p_i = 1 - (1/e^{\beta_i})$.

Lemma 4.6. $\mathbb{E}[d_i] = e^{-\gamma_i}(1 - e^{-\beta_i})\, n_i$.

Proof. Using equation (4.2), we see that

$$\mathbb{E}[d_i] = e^{-\gamma_i} n_i p_i = e^{-\gamma_i}(1 - e^{-\beta_i})\, n_i.$$

□

Lemma 4.7. $\mathrm{Var}(d_i) = (e^{\gamma_i} - 1)\, \mathbb{E}[d_i]^2 + e^{-\beta_i}\mathbb{E}[d_i]$.

Proof. Again using equation (4.2), we see that

$$\mathbb{E}[d_i^2] = e^{-\gamma_i}[\, n_i p_i + n_i(n_i - 1)p_i^2\,]$$

and a routine computation shows that

$$\begin{aligned}\mathrm{Var}(d_i) &= \mathbb{E}[d_i^2] - \mathbb{E}[d_i]^2 \\ &= (e^{\gamma_i} - 1)\, \mathbb{E}[d_i]^2 + e^{-\beta_i}\mathbb{E}[d_i].\end{aligned}$$

□

Proposition 4.8. *There exists an increasing sequence $I = (\, i_k \mid k \in \mathbb{N}\,)$ such that $\lim_{k\to\infty} d_{i_k}/\beta_{i_k} n_{i_k} = 1$ for μ_{β_0}-a.e. $(\Sigma_i)_{i \geq i_0} \in \Sigma$.*

Proof. Since $\beta_i = \beta_0/s_{0i} \to 0$, it follows that $(1 - e^{-\beta_i})/\beta_i \to 1$. Since we also have that $\gamma_i \to 0$, it follows from Lemma 4.6 that

$$\lim_{i\to\infty} \mathbb{E}[d_i]/\beta_i n_i = 1. \tag{4.3}$$

In particular, since G has sublinear natural orbit growth and

$$\mathbb{E}[d_i] \approx \beta_i n_i = \beta_0 \frac{n_i}{s_{0i}},$$

it follows that $\mathbb{E}[d_i] \to \infty$. Hence, letting $\sigma(d_i) = \sqrt{\mathrm{Var}(d_i)}$ denote the standard deviation, applying Lemma 4.7, we see that

$$\lim_{i\to\infty} \sigma(d_i)/\mathbb{E}[d_i] = 0. \tag{4.4}$$

Combining (4.3) and (4.4), there exists an increasing sequence $I = (\, i_k \mid k \in \mathbb{N}\,)$ such that for all $k \in \mathbb{N}$,

(a) $(1 - 1/2^k)\beta_{i_k} n_{i_k} \leq \mathbb{E}[d_i] \leq (1 + 1/2^k)\beta_{i_k} n_{i_k}$ and
(b) $\sigma(d_{i_k}) \leq \mathbb{E}[d_{i_k}]/4^k$.

Let E_k be the event that $|d_{i_k} - \mathbb{E}[d_{i_k}]| \geq \mathbb{E}[d_{i_k}]/2^k$. Applying Chebyshev's inequality, since $\mathbb{E}[d_{i_k}]/2^k \geq 2^k\sigma(d_{i_k})$, it follows that $\mathbb{P}[E_k] \leq 1/4^k$. Applying the Borel-Cantelli Lemma, for μ_{β_0}-a.e. $(\Sigma_i)_{i\geq i_0} \in \Sigma$, for all but finitely many $k \in \mathbb{N}$,

$$(1 - 1/2^k)\, \mathbb{E}[d_{i_k}] \leq d_{i_k} \leq (1 + 1/2^k)\, \mathbb{E}[d_{i_k}]$$

and hence

$$(1 - 1/2^k)^2\, \beta_{i_k} n_{i_k} \leq d_{i_k} \leq (1 + 1/2^k)^2\, \beta_{i_k} n_{i_k}.$$

It follows that $\lim_{k\to\infty} d_{i_k}/\beta_{i_k} n_{i_k} = 1$ for μ_{β_0}-a.e. $(\Sigma_i)_{i\geq i_0} \in \Sigma$. □

Definition 4.9. Σ_{β_0} is the set of $(\Sigma_i)_{i\geq i_0} \in \Sigma$ such that $\lim_{k\to\infty} d_{i_k}/\beta_{i_k} n_{i_k} = 1$

Since $\mu_{\beta_0}(\Sigma_{\beta_0}) = 1$, in order to show that $G \curvearrowright (\Sigma, \mu_{\beta_0})$ is ergodic, it is enough to prove the following result.

Proposition 4.10. *If m is an ergodic probability measure on Σ_{β_0}, then*

$$m(\Sigma(X) \cap \Sigma_{\beta_0}) = \mu_{\beta_0}(\Sigma(X))$$

for all $X \in \bigcup_{i \in \mathbb{N}} \mathcal{P}(\Delta_i)$.

So suppose that m is an ergodic probability measure on Σ_{β_0}. Then by the Pointwise Ergodic Theorem, there exists an element $(\Sigma_k)_{k \geq k_0} \in \Sigma_{\beta_0}$ such that

$$m(\Sigma(X) \cap \Sigma_{\beta_0}) = \lim_{j \to \infty} \frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot (\Sigma_k)_{k \geq k_0} \in \Sigma(X) \}| \tag{4.5}$$

for all $X \in \bigcup_{i \in \mathbb{N}} \mathcal{P}(\Delta_i)$. Fix some $X \subset \Delta_i$. For each $j > \max\{ i, k_0 \}$, let $d_j = |\Sigma_j|$ and let

$$\begin{aligned} m_{ij} &= s_{ij} n_i + \sum_{k=i+1}^{j-1} s_{kj} e_k + e_j \\ &= s_{ij} n_i + s_{0j} \sum_{k=i+1}^{j} e_k / s_{0k}. \end{aligned}$$

Then an easy induction on $\ell = |X|$ shows that

$$\frac{1}{|G_j|} |\{ g \in G_j \mid g \cdot (\Sigma_k)_{k \geq k_0} \in \Sigma(X) \}| = \sum_{t=0}^{\ell} (-1)^{\ell - t} \binom{\ell}{t} \frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}};$$

and a second induction using (4.5) shows that for all $0 \leq t \leq n_i$, the limit

$$\lim_{j \to \infty} \frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}} \tag{4.6}$$

exists. We will make repeated use of the following lemma in the remaining sections of this paper.

Lemma 4.11. *Suppose that $(n_j)_{j \in \mathbb{N}}$, $(m_j)_{j \in \mathbb{N}}$ and $(d_j)_{j \in \mathbb{N}}$ are sequences of natural numbers such that the following conditions are satisfied:*

(a) m_j, $d_j \leq n_j$.
(b) $m_j / n_j \to 0$ *and* $d_j / n_j \to 0$ *as* $j \to \infty$.
(c) $\lim_{j \to \infty} \binom{n_j - m_j}{d_j} / \binom{n_j}{d_j}$ *exists.*

Then $\lim_{j \to \infty} d_j m_j / n_j$ *exists and*

$$\lim_{j \to \infty} \frac{\binom{n_j - m_j}{d_j}}{\binom{n_j}{d_j}} = \left(\frac{1}{e} \right)^{\lim_{j \to \infty} d_j m_j / n_j}.$$

Proof. In order to simply notation, we will write n, d, m instead of n_j, d_j, m_j. Note that, since

$$\frac{\binom{n-m}{d}}{\binom{n}{d}} = \frac{(n-m)}{n} \frac{(n-m-1)}{n-1} \cdots \frac{(n-m-d+1)}{(n-d+1)},$$

it follows that

$$\left(\frac{n-m-d+1}{n-d+1} \right)^d \leq \frac{\binom{n-m}{d}}{\binom{n}{d}} \leq \left(\frac{n-m}{n} \right)^d,$$

and hence that

$$\left(1-\frac{m}{n-d+1}\right)^{\frac{n-d+1}{m}\frac{dm}{n-d+1}} \leq \frac{\binom{n-m}{d}}{\binom{n}{d}} \leq \left(1-\frac{m}{n}\right)^{\frac{n}{m}\frac{dm}{n}}. \tag{4.7}$$

Since $\frac{m}{n} \to 0$ and $\frac{m}{n-d+1} \to 0$, it follows that

$$\left(1-\frac{m}{n}\right)^{\frac{n}{m}} \to \left(\frac{1}{e}\right) \quad \text{and} \quad \left(1-\frac{m}{n-d+1}\right)^{\frac{n-d+1}{m}} \to \left(\frac{1}{e}\right). \tag{4.8}$$

Let $\varepsilon > 0$. Since $\frac{d}{n} \to 0$, for all but finitely many j, we have that

$$n-d+1 \geq n-\varepsilon n = (\,1-\varepsilon\,)n$$

and so

$$\frac{dm}{n} \leq \frac{dm}{n-d+1} \leq \frac{1}{(\,1-\varepsilon\,)}\frac{dm}{n}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), together with the fact that $\lim_{j\to\infty}\binom{n-m}{d}/\binom{n}{d}$ exists, it follows that $\lim_{j\to\infty} dm/n$ exists and that

$$\lim_{j\to\infty}\frac{\binom{n-m}{d}}{\binom{n}{d}} = \left(\frac{1}{e}\right)^{\lim_{j\to\infty} dm/n}.$$

□

We next check that Lemma 4.11 can be applied to each of the limits (4.6). First note that if $m_j = m_{ij} - ts_{ij}$, then

$$\begin{aligned}\frac{m_j}{n_j} &= \frac{s_{ij}}{n_j}(n_i - t) + \frac{s_{0j}}{n_j}\sum_{k=i+1}^{j} e_k/s_{0k} \\ &\leq \frac{s_{ij}}{n_j}(n_i - t) + \frac{s_{0j}}{n_j}\gamma_j;\end{aligned}$$

and since G has sublinear natural orbit growth, this implies that $m_j/n_j \to 0$. Also note that

$$\lim_{k\to\infty}\frac{d_{j_k}}{n_{j_k}} = \lim_{k\to\infty}\frac{d_{j_k}}{\beta_{j_k}n_{j_k}}\beta_{j_k} = \lim_{k\to\infty}\beta_{j_k} = 0.$$

Hence, applying Lemma 4.11, we obtain that

$$\lim_{j\to\infty}\frac{\binom{n_j-m_{ij}+ts_{ij}}{d_j}}{\binom{n_j}{d_j}} = \left(\frac{1}{e}\right)^{\lim_{k\to\infty} d_{j_k}(m_{ij_k}-ts_{ij_k})/n_{j_k}}. \tag{4.10}$$

Lemma 4.12. *For all $i \in \mathbb{N}$ and $0 \leq t \leq n_i$,*

$$\lim_{k\to\infty} d_{j_k}(m_{ij_k} - ts_{ij_k})/n_{j_k} = \beta_i(n_i - t) + \gamma_i.$$

Proof. First note that since $\beta_j ts_{ij} = t\beta_i$ and

$$\beta_j m_{ij} = \beta_j s_{ij} n_i + \beta_j s_{0j}\sum_{k=i+1}^{j} e_k/s_{0k} = \beta_i n_i + \beta_0 \sum_{k=i+1}^{j} e_k/s_{0k},$$

it follows that $\lim_{j\to\infty} \beta_j(m_{ij} - ts_{ij}) = \beta_i(n_i - t) + \gamma_i$. Hence, using the fact that $\lim_{k\to\infty} d_{j_k}/\beta_{j_k} n_{j_k} = 1$, we obtain that

$$\begin{aligned}\lim_{k\to\infty} d_{j_k}(m_{ij_k} - ts_{ij_k})/n_{j_k} &= \lim_{k\to\infty} \frac{d_{j_k}}{\beta_{j_k} n_{j_k}} \beta_{j_k}(m_{ij_k} - ts_{ij_k}) \\ &= \beta_i(n_i - t) + \gamma_i.\end{aligned}$$

□

Summing up, we have shown that if $X \subsetneq \Delta_i$ with $|X| = \ell$, then

$$\begin{aligned}m(\Sigma(X) \cap \Sigma_{\beta_0}) &= \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} \lim_{j\to\infty} \frac{\binom{n_j - m_{ij} + ts_{ij}}{d_j}}{\binom{n_j}{d_j}} \\ &= \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} \left(\frac{1}{e}\right)^{\beta_i(n_i - t) + \gamma_i} \\ &= \left(\frac{1}{e}\right)^{\beta_i n_i + \gamma_i} \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} e^{\beta_i t} (-1)^{\ell-t} \\ &= \left(\frac{1}{e}\right)^{\beta_i n_i + \gamma_i} (\, e^{\beta_i} - 1\,)^{\ell} \\ &= \mu_{\beta_0}(\Sigma(X)),\end{aligned}$$

as desired. This completes the proof that the action $G \curvearrowright (\,\Sigma, \mu_{\beta_0}\,)$ is ergodic.

Definition 4.13. Let ν_β be the stabilizer distribution of the action $G \curvearrowright (\,\Sigma, \mu_{\beta_0}\,)$.

We will next prove that the ergodic IRS ν_β is independent of the particular expression of G as an almost diagonal limit of finite alternating groups. Suppose that $K \leqslant F$ are finite subgroups of G and consider the basic clopen subset $B = \{H \in \mathrm{Sub}_G \mid H \cap F = K\} \subseteq \mathrm{Sub}_G$. Suppose that $F \leqslant G_{i_0}$ and for every $i \geq i_0$, define $S_i = S_i(K, F)$ by

$$S_i = \{\, X \subsetneq \Delta_i \mid \mathrm{Alt}(\Delta_i \smallsetminus X) \cap F = K \,\}.$$

Then clearly

$$\begin{aligned}\nu_{\beta_0}(B) &= \lim_{i\to\infty} \frac{1}{e^{\beta_i n_i + \gamma_i}} \sum_{X \in S_i} (\, e^{\beta_i} - 1\,)^{|X|} \\ &= \lim_{i\to\infty} \frac{1}{e^{\beta_i n_i}} \sum_{X \in S_i} (\, e^{\beta_i} - 1\,)^{|X|}\end{aligned}$$

In particular, if G is expressed as the union of a subchain of $G'_i = \mathrm{Alt}(\Delta_{j_i})$ for some strictly increasing sequence $(\, j_i \mid i \in \mathbb{N}\,)$ of natural numbers, then we obtain the same stabilizer distribution ν_{β_0}. Now the result follows easily from the following result.

Lemma 4.14. *Suppose that $G = \bigcup_{i\in\mathbb{N}} G'_i$ is a second expression of G as an almost diagonal limit of finite alternating groups $G'_i = \mathrm{Alt}(\Delta'_i)$ and that*

$$\Delta_0 \subsetneq \Delta'_0 \subsetneq \Delta_1 \subsetneq \Delta'_1 \subsetneq \cdots \subsetneq \Delta_i \subsetneq \Delta'_i \subsetneq \cdots \tag{4.11}$$

Then the chain (4.11) is also an almost diagonal limit.

Proof. Suppose that the chain $G = \bigcup_{i\in\mathbb{N}} G'_i$ has parameters n'_i, s'_{ij}, e'_i, etc. Then $\tau' = \sum_{i=1}^{\infty} e'_i/s'_{0i} < \infty$. Let $\Delta''_{2i} = \Delta_i$, let $\Delta''_{2i+1} = \Delta'_i$ and let $G''_i = \mathrm{Alt}(\Delta''_i)$. Let the chain $G = \bigcup_{i\in\mathbb{N}} G''_i$ have parameters n''_i, s''_{ij}, e''_i, etc. Then clearly we have that $s''_{02i+2} = s_{0i+1}$ and $s''_{02i+3} = s''_{01}s'_{0i+1}$. Also by considering the inclusions

$$\Delta_i \subsetneq \Delta'_i \subsetneq \Delta_{i+1},$$

we see that $e_{i+1} \geq s''_{2i+2}e''_{2i+1}$; and similarly $e'_{i+1} \geq s''_{2i+3}e''_{2i+2}$. It follows that

$$\begin{aligned}
\sum_{i=1}^{\infty} e''_i/s''_{0i} &= \sum_{i=0}^{\infty} e''_{2i+1}/s''_{02i+1} + \sum_{i=0}^{\infty} e''_{2i+2}/s''_{02i+2} \\
&\leq \sum_{i=0}^{\infty} e_{i+1}/s''_{02i+2} + \sum_{i=0}^{\infty} e'_{i+1}/s''_{02i+3} \\
&= \sum_{i=0}^{\infty} e_{i+1}/s_{0i+1} + \frac{1}{s''_{01}} \sum_{i=0}^{\infty} e'_{i+1}/s'_{0i+1} \\
&= \tau + \frac{1}{s''_{01}}\tau'.
\end{aligned}$$

□

We can now state the third of the main results of this paper.

Theorem 4.15. *If $G \not\cong \mathrm{Alt}(\mathbb{N})$ has almost diagonal type and sublinear natural orbit growth, then the ergodic IRSs of G are $\{\, \delta_1, \delta_G \,\} \cup \{\, \nu_{\beta_0} \mid \beta_0 \in (\, 0, \infty \,) \,\}$.*

In Section 9, we will see that Theorem 4.15 is false if $G \cong \mathrm{Alt}(\mathbb{N})$.

5. Groups of almost diagonal type

In Section 3, we proved that an $L(\mathrm{Alt})$-group G has linear natural orbit growth if and only if there exists a G-invariant ergodic probability measure on Δ. In this section, we will prove a corresponding characterization of the almost diagonal limits of finite alternating groups. (This characterization will play a key role in our proof of the classification of the ergodic IRSs of the $L(\mathrm{Alt})$-group.) Throughout this section, suppose that $G = \bigcup_{i\in\mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that $s_{i+1} > 1$ for all $i \in \mathbb{N}$. Let Σ be the spaces of sequences defined in Section 4.

Theorem 5.1. *With the above hypotheses, the following statements are equivalent:*

(i) *$G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit.*
(ii) *There exists a nonatomic G-invariant ergodic probability measure on Σ.*

In Section 4, we saw that if $G = \bigcup_{i\in\mathbb{N}} G_i$ is an almost diagonal limit, then there exists a nonatomic G-invariant ergodic on Σ. Conversely, let ν be a nonatomic G-invariant ergodic probability measure on Σ. Suppose that G is not an almost diagonal limit; i.e. that $\tau = \sum_{i=1}^{\infty} e_i/s_{0i} = \infty$. Then, applying Lemma 3.12, it follows that G has sublinear natural orbit growth.

Applying the Pointwise Ergodic Theorem, there exists $(\, \Sigma_j \,)_{j\geq j_0} \in \Sigma$ such that for all $i \in \mathbb{N}$ and $X \subseteq \Delta_i$,

$$\nu(\, \Sigma(X) \,) = \lim_{j\to\infty} \frac{1}{|G_j|}\, |\{\, g \in G_j \mid g \cdot (\, \Sigma_i \,)_{j\geq j_0} \in \Sigma(X) \,\}|. \tag{5.1}$$

Let $|\Sigma_j| = d_j$.

Claim 5.2. $\lim_{j\to\infty} d_j = \infty$.

Proof. Suppose not. Then, by Remark 4.1, there exist integers $d \geq 0$ and $j_1 \geq j_0$ such that $d_j = d$ for all $j \geq j_1$. Suppose that $X \subseteq \Delta_i$ with $|X| = \ell \geq 1$ and that $\nu(\,\Sigma(X)\,) \neq 0$. Let $j \geq \max\{\,i, j_1\,\}$ and let Φ_{ij} be the union of the natural orbits of G_i on Δ_j. If $g \in G_j$ satisfies $g \cdot (\,\Sigma_i\,)_{j \geq j_0} \in \Sigma(X)$, then we must have that $|g(\Sigma_j) \cap \Phi_{ij}| \geq \ell$. Hence (5.1) implies that $\ell \leq d$ and that

$$\begin{aligned}\nu(\,\Sigma(X)\,) &\leq \lim_{j\to\infty} \sum_{t=\ell}^{d} \frac{\binom{s_{ij}n_i}{t}\binom{n_j-m_j}{d-t}}{\binom{n_j}{d}} \\ &= \lim_{j\to\infty} \sum_{t=\ell}^{d} \frac{1}{\binom{d}{t}} \frac{\binom{s_{ij}n_i}{t}}{\binom{n_j-(d-t)}{t}} \frac{\binom{n_j-m_j}{d-t}}{\binom{n_j}{d-t}} \\ &\leq \lim_{j\to\infty} \sum_{t=\ell}^{d} \frac{\binom{s_{ij}n_i}{t}}{\binom{n_j-(d-t)}{t}}.\end{aligned}$$

Since G has sublinear natural orbit growth, it follows that if $\ell \leq t \leq d$, then

$$\lim_{j\to\infty} \frac{\binom{s_{ij}n_i}{t}}{\binom{n_j-(d-t)}{t}} = 0.$$

But this means that $\nu(\,\Sigma(X)\,) = 0$, which is a contradiction. Thus no such $X \subseteq \Delta_i$ exists and it follows that ν concentrates on the G-invariant sequence $\sigma \in \Sigma$ with constant value $\emptyset$, which is a contradiction. □

Arguing as in Section 4, we see that if $X \subseteq \Delta_i$ with $|X| = \ell$, then

$$\nu(\,\Sigma(X)\,) = \lim_{j\to\infty} \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} \frac{\binom{n_j-m_{ij}+ts_{ij}}{d_j}}{\binom{n_j}{d_j}}; \tag{5.2}$$

and that the limit

$$\lim_{j\to\infty} \frac{\binom{n_j-m_{ij}+ts_{ij}}{d_j}}{\binom{n_j}{d_j}}$$

exists for all $0 \leq t \leq n_i$. We will now work towards verifying that the hypotheses of Lemma 4.11 are satisfied. For each $0 \leq t \leq n_i$, let $m_{itj} = m_{ij} - ts_{ij}$. Then

$$\lim_{j\to\infty} m_{itj} = \lim_{j\to\infty} [\,(n_i - t)s_{ij} + s_{0j} \sum_{k=i+1}^{j} e_k/s_{0k}\,] = \infty.$$

Claim 5.3. *If $i \in \mathbb{N}$ and $0 \leq t \leq n_i$, then* $\lim_{j\to\infty} m_{itj}/n_j = 0$.

Proof. Suppose that there exist i, t with $0 \leq t \leq n_i$ such that $\lim_{j\to\infty} m_{itj}/n_j \neq 0$. Since

$$\frac{m_{itj}}{n_j} = (n_i - t)\frac{s_{ij}}{n_j} + \frac{s_{0j}}{n_j} \sum_{k=i+1}^{j} e_k/s_{0k}$$

and $\lim_{j\to\infty} s_{ij}/n_j = 0$, it follows that

$$\limsup_{j\to\infty} \frac{m_{itj}}{n_j} = \limsup_{j\to\infty} \frac{s_{0j}}{n_j} \sum_{k=i+1}^{j} e_k/s_{0k};$$

and hence there exists a constant $0 < c \leq 1$ such that $\limsup_{j\to\infty} m_{itj}/n_j = c$ for all $0 \leq t \leq n_i$. Note that if $\ell < m$, then

$$\frac{s_{0j}}{n_j}\sum_{k=\ell+1}^{j} e_k/s_{0k} = \frac{s_{0j}}{n_j}\sum_{k=\ell+1}^{m} e_k/s_{0k} + \frac{s_{0j}}{n_j}\sum_{k=m+1}^{j} e_k/s_{0k}$$

and that $lim_{j\to\infty} \frac{s_{0j}}{n_j}\sum_{k=\ell+1}^{m} e_k/s_{0k} = 0$. It follows that $\limsup_{j\to\infty} m_{itj}/n_j = c$ for all i, t with $0 \leq t \leq n_i$. Since

$$\frac{\binom{n_j - m_{itj}}{d_j}}{\binom{n_j}{d_j}} \leq \left(\frac{n_j - m_{itj}}{n_j}\right)^{d_j} = \left(1 - \frac{m_{itj}}{n_j}\right)^{d_j}$$

and $\lim_{j\to\infty} d_j = \infty$, it follows that

$$\lim_{j\to\infty} \frac{\binom{n_j - m_{itj}}{d_j}}{\binom{n_j}{d_j}} = \lim_{j\to\infty} \frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}} = 0$$

for all i, t with $0 \leq t \leq n_i$. But then (5.2) implies that $\nu(\,\Sigma(X)\,) = 0$ for all $X \in \bigcup_{i\in\mathbb{N}} \mathcal{P}(\Delta_i)$, which is a contradiction. □

Claim 5.4. $\lim_{j\to\infty} d_j/n_j = 0$.

Proof. Suppose not. Then there exists a constant $0 < c \leq 1$ and an infinite subset $J \subseteq \mathbb{N}$ such that $d_j/n_j \geq c$ for all $j \in J$. Let i, t with $0 \leq t \leq n_i$. Since $\lim_{j\to\infty} m_{itj}/n_j = 0$, there exists an cofinite subset $J_{it} \subseteq J$ such that

$$\frac{\binom{n_j - m_{itj}}{d_j}}{\binom{n_j}{d_j}} \leq \left(1 - \frac{m_{itj}}{n_j}\right)^{d_j} \leq \left(1 - \frac{m_{itj}}{n_j}\right)^{\frac{n_j}{m_{itj}}\frac{d_j}{n_j} m_{itj}} \leq \left(\frac{1}{2}\right)^{c\, m_{itj}}$$

for all $j \in J_{it}$. Since $\lim_{j\to\infty} m_{itj} = \infty$, it follows that

$$\lim_{j\to\infty} \frac{\binom{n_j - m_{itj}}{d_j}}{\binom{n_j}{d_j}} = \lim_{j\to\infty} \frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}} = 0$$

for all i, t with $0 \leq t \leq n_i$; and, as in the proof of Claim 5.3, this is impossible. □

Thus the hypotheses of Lemma 4.11 are satisfied; and so for all integers i, t with $0 \leq t \leq n_i$, we have that

$$\lim_{j\to\infty} \frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}} = \left(\frac{1}{e}\right)^{\lambda_{ti}},$$

where $\lambda_{ti} = \lim_{j\to\infty} d_j(m_{ij} - t s_{ij})/n_j$. Since $\tau = \sum_{i=1}^{\infty} e_i/s_{0i} = \infty$ and

$$\begin{aligned} d_j m_{ij}/n_j &= \frac{n_i d_j s_{ij}}{n_j} + \frac{d_j s_{0j}}{n_j}\sum_{k=i+1}^{j} e_k/s_{0k} \\ &= \left[\frac{n_i}{s_{0i}} + \sum_{k=i+1}^{j} e_k/s_{0k}\right]\frac{d_j s_{0j}}{n_j}, \end{aligned}$$

it follows that $\lim_{j\to\infty} d_j s_{0j}/n_j = 0$ and hence

$$\lambda_{0i} = \lim_{j\to\infty} d_j m_{ij}/n_j = \lim_{j\to\infty} \frac{d_j s_{0j}}{n_j}\sum_{k=i+1}^{j} e_k/s_{0k}.$$

Also notice that

$$\begin{aligned}\lambda_{0i} &= \lim_{j\to\infty} \frac{d_j s_{0j}}{n_j}\left[e_{i+1}/s_{i+1} + \sum_{k=i+2}^{j} e_k/s_{0k}\right] \\ &= \lim_{j\to\infty} \frac{d_j s_{0j}}{n_j} \sum_{k=i+2}^{j} e_k/s_{0k} \\ &= \lambda_{0i+1}.\end{aligned}$$

Thus there exists a constant λ such that $\lambda_{0i} = \lambda$ for all $i \in \mathbb{N}$. Next notice that if $0 \le t \le n_i$, then

$$\begin{aligned}\lambda_{\ell i} &= \lim_{j\to\infty}\left[d_j m_{ij}/n_j - \frac{t}{s_{0i}} d_j s_{0j}/n_j\right] \\ &= \lim_{j\to\infty} d_j m_{ij}/n_j \\ &= \lambda.\end{aligned}$$

Hence for all $i \in \mathbb{N}$ and $X \subseteq \Delta_i$, if $|X| = \ell$, then

$$\begin{aligned}\nu(\,\Sigma(X)\,) &= \sum_{t=0}^{\ell}(-1)^{\ell-t}\binom{\ell}{t}\left(\frac{1}{e}\right)^{\lambda} \\ &= \begin{cases}\left(\frac{1}{e}\right)^{\lambda}, & \text{if } \ell = 0; \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

It follows that $\lambda = 0$ and that ν concentrates on the sequence $\sigma \in \Sigma$ with constant value $\emptyset$, which contradicts the assumption that ν is nonatomic. This completes the proof of Theorem 5.1.

Finally we record the following recognition theorem, which will be used in the proof of Theorem 4.15.

Theorem 5.5. *If G has sublinear natural orbit growth and μ is a nonatomic G-invariant ergodic probability measure on Σ, then there exists $\beta_0 \in (\,0, \infty\,)$ such that the corresponding stabilizer distribution is ν_{β_0}.*

Proof. Applying the Pointwise Ergodic Theorem, there exists $(\,\Sigma_j\,)_{j\ge j_0} \in \Sigma$ such that for all $i \in \mathbb{N}$ and $X \subseteq \Delta_i$,

$$\nu(\,\Sigma(X)\,) = \lim_{j\to\infty}\frac{1}{|G_j|}\,|\{\,g \in G_j \mid g\cdot(\,\Sigma_i\,)_{j\ge j_0} \in \Sigma(X)\,\}|.$$

Let $|\Sigma_j| = d_j$. Then, arguing as in the proof of Theorem 5.1, we see that if if $X \subseteq \Delta_i$ with $|X| = \ell$, then

$$\begin{aligned}\nu(\,\Sigma(X)\,) &= \lim_{j\to\infty}\sum_{t=0}^{\ell}(-1)^{\ell-t}\binom{\ell}{t}\frac{\binom{n_j - m_{ij} + t s_{ij}}{d_j}}{\binom{n_j}{d_j}} \\ &= \lim_{j\to\infty}\sum_{t=0}^{\ell}(-1)^{\ell-t}\binom{\ell}{t}\left(\frac{1}{e}\right)^{\lambda_{ti}},\end{aligned}$$

where $\lambda_{ti} = \lim_{j\to\infty} d_j(m_{ij} - t s_{ij})/n_j$; and we also see that

$$\beta_0 = \lim_{j\to\infty} d_j s_{0j}/n_j \neq 0.$$

Note that

$$\begin{aligned}\lambda_{ti} &= \lim_{j\to\infty} d_j[\,(n_i-t)s_{ij} + s_{0j}\sum_{k=i+1}^{j} e_k/s_{0k}\,]/n_j \\ &= \lim_{j\to\infty}[\,\frac{1}{s_{0i}}\frac{d_j s_{0j}}{n_j}(n_i-t) + \frac{d_j s_{0j}}{n_j}\sum_{k=i+1}^{j} e_k/s_{0k}\,] \\ &= \frac{1}{s_{0i}}\beta_0(n_i-t) + \beta_0\sum_{k=i+1}^{\infty} e_k/s_{0k} \\ &= \beta_i(n_i-t)+\gamma_i,\end{aligned}$$

where $\beta_i = \beta_0/s_{0i}$ and $\gamma_i = \beta_0\sum_{k=i+1}^{\infty} e_k/s_{0k}$. It follows that $\mu = \mu_{\beta_0}$. □

6. Normalized permutation characters of finite alternating groups

In this section, we will present a series of lemmas concerning upper bounds for the values of the normalized permutation characters of various actions $\mathrm{Alt}(\Delta) \curvearrowright \Omega$ of the finite alternating group $\mathrm{Alt}(\Delta)$. No attempt will be made to prove the best possible results: we will be content to prove easy results which are good enough to serve our purposes in the proofs of Theorems 3.4 and 3.18.

Suppose that $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-group; say, $G = \bigcup_{i\in\mathbb{N}} G_i$ is the union of the increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. Let $\nu \neq \delta_1$, δ_G is an ergodic IRS of G. By Creutz-Peterson [2, Proposition 3.3.1], we can suppose that ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\,Z,\mu\,)$. Let $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ be the corresponding character. For each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{\, g\cdot z \mid g \in G_i \,\}$. Then, by Theorem 2.1, for μ-a.e. $z \in Z$, for all $g \in G$, we have that

$$\mu(\,\mathrm{Fix}_Z(g)\,) = \lim_{i\to\infty} |\,\mathrm{Fix}_{\Omega_i(z)}(g)\,|/|\,\Omega_i(z)\,|.$$

Fix such an element $z \in Z$ and let $H = \{\, h \in G \mid h\cdot z = z \,\}$ be the corresponding point stabilizer. Clearly we can suppose that z has been chosen so that if $g \in H$, then $\chi(g) > 0$.

For each $i \in \mathbb{N}$, let $H_i = H \cap G_i$. Then, examining the list of ergodic IRSs in the statement of Theorem 3.4, we see that it is necessary to show that there exists a *fixed* integer $r \geq 1$ such that for all but finitely many $i \in \mathbb{N}$, there is a subset $U_i \subseteq \Delta_i$ of cardinality r such that $H_i = \mathrm{Alt}(\Delta_i \smallsetminus U_i)$. We will eventually show that if this is not the case, then there exists an element $g \in H$ such that

$$\mu(\,\mathrm{Fix}_Z(g)\,) = \lim_{i\to\infty} |\,g^{G_i} \cap H_i\,|/|\,g^{G_i}\,| = |\,\{s \in G_i \mid sgs^{-1} \in H_i\,\}|/|G_n| = 0,$$

which is a contradiction. For example, Lemma 6.1 will play a key role in the proof that there do not exist infinitely many $i \in \mathbb{N}$ such that H_i acts primitively on Δ_i; and Lemmas 6.3 and 6.5 will play key roles in the proof that there do not exist infinitely many $i \in \mathbb{N}$ such that H_i acts imprimitively on Δ_i.

For the remainder of this section, let $\Delta = \{\,1, 2, \cdots, n\,\}$.

Lemma 6.1. *For each prime p and real number $a > 0$, there exists $n_{p,a} \in \mathbb{N}$ such that if $n \geq n_{p,a}$ and*

(i) *$g \in \mathrm{Alt}(\Delta)$ is a product of $b \geq an$ p-cycles;*
(ii) *$K < \mathrm{Alt}(\Delta)$ is a proper primitive subgroup;*

then the normalized permutation character of the action $\mathrm{Alt}(\Delta) \curvearrowright \Omega = \mathrm{Alt}(\Delta)/K$ *satisfies* $|\,\mathrm{Fix}_\Omega(g)|/|\Omega| < \frac{1}{n}$.

Proof. Clearly we can suppose that n has been chosen so that $b \geq an \geq 2$. In particular, since g contains at least two p-cycles, this implies that the conjugacy classes of g in $\mathrm{Alt}(\Delta)$ and $\mathrm{Sym}(\Delta)$ coincide and hence

$$|g^{\mathrm{Alt}(\Delta)}| = \frac{n!}{p^b b!(n-bp)!}.$$

Applying Stirling's Approximation and the fact that $b \geq a\,n$, it follows that there exist constants r, s such that

$$|g^{\mathrm{Alt}(\Delta)}| > r\,s^n \frac{n^n}{b^b(n-bp)^{n-bp}} > r\,s^n \frac{n^n}{n^b n^{n-bp}} \geq r\,s^n (n^n)^{(p-1)a}.$$

By Praeger-Saxl [13], since K is a proper primitive subgroup of $\mathrm{Alt}(\Delta)$, it follows that $|K| < 4^n$. By Proposition 2.2, this implies that

$$|\,\mathrm{Fix}_\Omega(g)|/|\Omega| = |\,g^{\mathrm{Alt}(\Delta)} \cap K\,|/|\,g^{\mathrm{Alt}(\Delta)}\,| \leq |\,K\,|/|\,g^{\mathrm{Alt}(\Delta)}\,| \leq \frac{4^n}{r\,s^n(n^n)^{(p-1)a}}.$$

The result follows easily. □

Lemma 6.2. *Let* $\Omega = [\Delta]^\ell$ *be the set of* ℓ*-subsets of* Δ *for some* $2 \leq \ell \leq n/2$. *Suppose that* $g \in \mathrm{Alt}(\Delta)$ *has prime order* $p > 2$ *and that* $c = |\,\mathrm{Fix}_\Delta(g)| \leq n/4$. *Then the normalized permutation character of the action* $\mathrm{Alt}(\Delta) \curvearrowright \Omega$ *satisfies:*

(i) $|\,\mathrm{Fix}_\Omega(g)|/|\Omega| < \frac{1}{2}|\,\mathrm{Fix}_\Delta(g)|/|\Delta|$ *if* $c \geq 16$;
(ii) $|\,\mathrm{Fix}_\Omega(g)|/|\Omega| < \frac{5}{n}$ *if* $c < 16$.

Proof. First suppose that $\ell < p$. Then $\mathrm{Fix}_\Omega(g) = [\mathrm{Fix}_\Delta(g)]^\ell$. Clearly we can suppose that $c \geq \ell$ and since $c \leq n/4$, it follows that

$$\frac{|\,\mathrm{Fix}_\Omega(g)|}{|\Omega|} = \frac{\binom{c}{\ell}}{\binom{n}{\ell}} = \frac{c(c-1)\cdots(c-\ell-1)}{n(n-1)\cdots(n-\ell-1)} \leq \frac{c(c-1)}{n(n-1)} < \frac{c}{4n} < \frac{|\,\mathrm{Fix}_\Delta(g)|}{2|\Delta|}.$$

Next suppose that $\ell \geq p > 2$. Let $\mathcal{A} = \{\, S \in \mathrm{Fix}_\Omega(g) \mid S \subseteq \mathrm{Fix}_\Delta(g)\,\}$ and let $\mathcal{B} = \mathrm{Fix}_\Omega(g) \smallsetminus \mathcal{A}$. If $\mathcal{A} \neq \emptyset$, then

$$\frac{|\mathcal{A}|}{|\Omega|} = \frac{\binom{c}{\ell}}{\binom{n}{\ell}} \leq \frac{c(c-1)(c-2)}{n(n-1)(n-2)} < \frac{c}{16n}.$$

For each $S \in \mathcal{B}$, let $\alpha(S) = \min\{\, s \in S \mid g \cdot s \neq s\,\}$. Then, since $\ell > 2$, it follows that the sets

$$\mathcal{B} \cup \{\, (S \smallsetminus \{\,\alpha(S)\,\}) \cup \{\,t\,\} \mid S \in \mathcal{B}, t \in \Delta \smallsetminus (S \cup \mathrm{Fix}_\Delta(g))\,\}$$

are distinct. Note that if $S \in \mathcal{B}$, then $|S \cup \mathrm{Fix}_\Delta(g)| \leq 3n/4$; and it follows that $(1 + \frac{n}{4})|\mathcal{B}| \leq |\Omega|$ and so $|\mathcal{B}|/|\Omega| < 4/n$. If $c \geq 16$, then

$$\frac{|\,\mathrm{Fix}_\Omega(g)|}{|\Omega|} < \frac{c}{16n} + \frac{4}{n} \leq \frac{c}{16n} + \frac{c}{4n} < \frac{|\,\mathrm{Fix}_\Delta(g)|}{2|\Delta|};$$

while if $c < 16$, then

$$\frac{|\,\mathrm{Fix}_\Omega(g)|}{|\Omega|} < \frac{c}{16n} + \frac{4}{n} < \frac{5}{n}.$$

□

If $\mathcal{P}$ is a partition of Δ, then the subsets $B \in \mathcal{P}$ will be called the *blocks* of $\mathcal{P}$; and if $s \in \Delta$, then $[\, s\,]_{\mathcal{P}}$ will denote the block of $\mathcal{P}$ which contains s.

Lemma 6.3. *Let Ω be the set of partitions $\mathcal{P}$ of Δ into ℓ-sets for some* fixed *divisor ℓ of n such that $2 \leq \ell \leq n/2$. If $g \in \mathrm{Alt}(\Delta)$ has prime order $p > 2$, then the normalized permutation character of the action $\mathrm{Alt}(n) \curvearrowright \Omega$ satisfies $|\,\mathrm{Fix}_\Omega(g)|/|\Omega| < 2/n$.*

Proof. Let $\mathcal{P} \in \mathrm{Fix}_\Omega(g)$. Then we define the integer $\alpha(\mathcal{P})$ as follows.

(a) If $\mathcal{P}$ contains a g-invariant block B such that $g \restriction B \neq \mathrm{id}_B$, then $\alpha(\mathcal{P})$ is the least $s \in \Delta$ such that $[\, s\,]_{\mathcal{P}}$ is g-invariant and $g \cdot s \neq s$.

(b) Otherwise, $\alpha(\mathcal{P})$ is the least $s \in \Delta$ such that $g \cdot s \neq s$.

For each $t \in \Delta \smallsetminus [\,\alpha(\mathcal{P})\,]_{\mathcal{P}}$, we define $\mathcal{P}(t) \in \Omega$ to be the partition obtained from $\mathcal{P}$ by replacing the block $[\,\alpha(\mathcal{P})\,]_{\mathcal{P}}$ by $([\,\alpha(\mathcal{P})\,]_{\mathcal{P}} \smallsetminus \{\,\alpha(\mathcal{P})\,\}) \cup \{\,t\,\}$ and the block $[\,t\,]_{\mathcal{P}}$ by $([\,t\,]_{\mathcal{P}} \smallsetminus \{\,t\,\}) \cup \{\,\alpha(\mathcal{P})\,\}$.

Claim 6.4. $\mathcal{P}(t) \notin \mathrm{Fix}_\Omega(g)$.

Proof of Claim 6.4. First suppose that $\mathcal{P}$ contains a g-invariant block. Then clearly $g \cdot [\,t\,]_{\mathcal{P}(t)} \neq [\,t\,]_{\mathcal{P}(t)}$. Also, since $\ell \geq p > 2$, it follows that $g \cdot [\,t\,]_{\mathcal{P}(t)} \cap [\,t\,]_{\mathcal{P}(t)} \neq \emptyset$. Hence $\mathcal{P}(t) \notin \mathrm{Fix}_\Omega(g)$.

Thus we can suppose that $\mathcal{P}$ does not contain a g-invariant block. For each $0 \leq i < p$, let $S_i = g^i \cdot [\,\alpha(\mathcal{P})\,]_{\mathcal{P}}$. Since $p > 2$, there exists $0 < i < p$ such that $S_i \in \mathcal{P}(t)$. Since $S_0 = g^{p-i} \cdot S_i \notin \mathcal{P}(t)$, it follows that $\mathcal{P}(t) \notin \mathrm{Fix}_\Omega(g)$. □

If $\mathcal{P}$, $\mathcal{P}' \in \mathrm{Fix}_\Omega(g)$ and $\mathcal{P}(t) = \mathcal{P}'(t')$, then it is easily checked that $\mathcal{P} = \mathcal{P}'$ and $t = t'$. Thus $(1 + n - \ell)|\,\mathrm{Fix}_\Omega(g)| \leq |\Omega|$ and so $|\,\mathrm{Fix}_\Omega(g)|/|\Omega| < 2/n$. □

The following two results are routine generalizations of Lemmas 5.2 and 5.3 of Thomas-Tucker-Drob [17].

Lemma 6.5. *For any $\varepsilon > 0$ and $0 < a \leq 1$ and $r \geq 0$, there exists an integer $d_{a,r,\varepsilon}$ such that if $d_{a,r,\varepsilon} \leq d \leq (n-r)/2$ and $H < \mathrm{Alt}(\Delta)$ is any subgroup such that*

(i) *there exists an H-invariant subset $U \subseteq \Delta$ of cardinality r, and*

(ii) *H acts imprimitively on $\Delta \smallsetminus U$ with a proper system of imprimitivity $\mathcal{B}$ of blocksize d,*

then for any element $g \in \mathrm{Alt}(\Delta)$ satisfying $|\,\mathrm{supp}(g)| \geq an$,

$$\frac{|\{\, s \in \mathrm{Alt}(\Delta) \mid sgs^{-1} \in H\,\}|}{|\,\mathrm{Alt}(\Delta)|} < \varepsilon.$$

Lemma 6.6. *For any $\varepsilon > 0$ and $0 < a \leq 1$, there exists an integer $r_{a,\varepsilon}$ such that if $r_{a,\varepsilon} \leq r \leq n/2$ and $H < \mathrm{Alt}(\Delta)$ is a subgroup with an H-invariant set $U \subseteq \Delta$ of cardinality $|U| = r$, then for any element $g \in \mathrm{Alt}(\Delta)$ satisfying $|\,\mathrm{supp}(g)| \geq an$,*

$$\frac{|\{s \in \mathrm{Alt}(\Delta) \mid sgs^{-1} \in H\}|}{|\,\mathrm{Alt}(\Delta)|} < \varepsilon.$$

For the sake of completeness, we will sketch the main points of the proofs of Lemmas 6.5 and 6.6. As in Thomas-Tucker-Drob [17, Section 5], our approach will be probabilistic; i.e. we will regard the normalized permutation character

$$\frac{|\{s \in \mathrm{Alt}(\Delta) \mid sgs^{-1} \in H\}|}{|\,\mathrm{Alt}(\Delta)|}$$

as the probability that a uniformly random permutation $s \in \mathrm{Alt}(\Delta)$ satisfies $sgs^{-1} \in H$. Our probability theoretic notation is standard. In particular, if E is an event, then $\mathbb{P}[E]$ denotes the corresponding probability; and if N is a random variable, then $\mathbb{E}[N]$ denotes the expectation, $\mathrm{Var}[N]$ denotes the variance and $\sigma = (\mathrm{Var}[N])^{1/2}$ denotes the standard deviation. The proofs of Lemmas 6.5 and 6.6 make use of the following consequence of Chebyshev's inequality. (See Thomas-Tucker-Drob [17, Lemma 5.1].)

Lemma 6.7. *Suppose that (N_k) is a sequence of non-negative random variables such that $\mathbb{E}[N_k] = \mu_k > 0$ and $\mathrm{Var}[N_k] = \sigma_k^2 > 0$. If $\lim_{k\to\infty} \mu_k/\sigma_k = \infty$, then $\mathbb{P}[N_k > 0] \to 1$ as $k \to \infty$.*

In our arguments, it will be convenient to make use of big O notation. Recall that if (a_m) and (x_m) are sequences of real numbers, then $a_m = O(x_m)$ means that there exists a constant $C > 0$ and an integer $m_0 \in \mathbb{N}$ such that $|a_m| \leq C|x_m|$ for all $m \geq m_0$. Also if (c_m) is another sequence of real numbers, then we write $a_m = c_m + O(x_m)$ to mean that $a_m - c_m = O(x_m)$.

Sketch proof of Lemma 6.5. Suppose that $m = r + d\ell$, where $\ell \geq 2$, and that $H < \mathrm{Alt}(\Delta)$ has an H-invariant set $U \subseteq \Delta$ of cardinality $|U| = r$ such that H acts imprimitively on $T = \Delta \smallsetminus U$ with a proper system of imprimitivity $\mathcal{B}$ of blocksize d. Let $b = a/3$ and suppose that $g \in \mathrm{Alt}(\Delta)$ satisfies $|\,\mathrm{supp}(g)| \geq an = 3bn$. Then there exists a subset $Z \subseteq \mathrm{supp}(g)$ such that $g(Z) \cap Z = \emptyset$ and $|Z| = cn$ for some $b \leq c \leq 1/2$. Fix an element $z_0 \in Z$ and let $y_0 = g(z_0)$. Let $s \in S$ be a uniformly random permutation. If $s(z_0)$,$s(y_0) \in T$, let B_0, $C_0 \in \mathcal{B}$ be the blocks in $\mathcal{B}$ containing $s(z_0)$ and $s(y_0)$ respectively; otherwise, let $B_0 = C_0 = \emptyset$. Let

$$J(s) = \{z \in Z \smallsetminus \{z_0\} \mid s(z) \in B_0 \text{ and } s(g(z)) \notin C_0\}.$$

Note that if $J(s) \neq \emptyset$, then $sgs^{-1}(B_0)$ intersects at least two of the blocks of $\mathcal{B}$ and thus $sgs^{-1} \notin H$. Hence it is enough to show that $\mathbb{P}[\,|J(s)| > 0\,] > 1 - \varepsilon$ for all sufficiently large d (depending only on ε, a and r).

Since we wish to apply Lemma 6.7, we need to compute the asymptotics of the expectation and variance of the random variable $|J(s)|$. Arguing as in the proof of Thomas-Tucker-Drob [17, Lemma 5.2], it can be shown that

$$\mathbb{E}[\,|J(s)|\,] = cd(1 - \tfrac{d}{m}) + O(1); \tag{6.1}$$

and that

$$\mathbb{E}[\,|J(s)|\,]^2 = [cd(1 - \tfrac{d}{m})]^2 + O(d); \tag{6.2}$$

and that

$$\mathbb{E}[\,|J(s)|^2\,] = \left[cd(1 - \tfrac{d}{m})\right]^2 + O(d), \tag{6.3}$$

where the implied constants needed to witness the big-O inequalities are only dependent on the parameter r. Combining (6.2) and (6.3) we obtain that

$$\mathrm{Var}(|J(s)|) = \mathbb{E}[\,|J(s)|^2\,] - \mathbb{E}[\,|J(s)|\,]^2 = O(d),$$

and hence $\mathrm{Var}(|J(s)|)^{1/2} = O(\sqrt{d})$. Of course, (6.1) implies that $d = O(\mathbb{E}[\,|J(s)|\,])$. Thus there exists a constant $C > 0$ such that $\sigma = \mathrm{Var}(|J(s)|)^{1/2} \leq C\sqrt{d}$ and $d \leq C\,\mathbb{E}[\,|J(s)|\,]) = C\mu$ for all sufficiently large d. It follows that

$$\mu/\sigma \geq C^{-1}d/C\sqrt{d} = C^{-2}\sqrt{d} \to \infty \qquad \text{as } d \to \infty.$$

Applying Lemma 6.7, we conclude that $\mathbb{P}[\,|J(s)| > 0\,] \to 1$ as $d \to \infty$. This completes the proof of Lemma 6.5. □

Sketch proof of Lemma 6.6. Let $b = a/3$. Suppose that $H < \mathrm{Alt}(\Delta)$ has an H-invariant set $U \subseteq \Delta$ of cardinality $|U| = r \leq n/2$ and that $g \in \mathrm{Alt}(\Delta)$ satisfies $|\operatorname{supp}(g)| \geq an = 3bn$. Then, once again, there exists a subset $Z \subseteq \operatorname{supp}(g)$ such that $g(Z) \cap Z = \emptyset$ and $|Z| = cn$ for some $b \leq c \leq 1/2$. Let $s \in \mathrm{Alt}(\Delta)$ be a uniformly random permutation and let

$$I(s) = \{\, z \in Z \mid s(z) \in U \text{ and } s(g(z)) \notin U \,\}.$$

If $I(s) \neq \emptyset$, then U is not sgs^{-1}-invariant and thus $sg\, s^{-1} \notin H$. Hence it is enough to show that $\mathbb{P}[\,|I(s)| > 0\,] > 1 - \varepsilon$ for all sufficiently large r (depending only on ε and a).

Arguing as in the proof of Thomas-Tucker-Drob [17, Lemma 5.2], it can be shown that

$$\mathbb{E}[\,|I(s)|\,] = cr(1 - \tfrac{r}{m}) + O(1); \tag{6.4}$$

and that

$$\mathbb{E}[\,|I(s)|\,]^2 = [cr(1 - \tfrac{r}{m})]^2 + O(r); \tag{6.5}$$

and that

$$\mathbb{E}[\,|I(s)|^2\,] = \big[cr(1 - \tfrac{r}{m})\big]^2 + O(r), \tag{6.6}$$

where the implied constants needed to witness the big-O inequalities are absolute. It follows that $\mathrm{Var}(|I(s)|)^{1/2} = O(\sqrt{r})$ and $r = O(\mathbb{E}[\,|I(s)|\,])$; and another application of Lemma 6.7 shows that $\mathbb{P}[\,|I(s)| > 0\,] \to 1$ as $r \to \infty$. □

7. Full limits of finite alternating groups

In this section, we will classify the ergodic IRSs of full limits of finite alternating groups.

Definition 7.1. Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.

(i) The embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is said to be *full* if $\mathrm{Alt}(\Delta_i)$ has no trivial orbits on Δ_{i+1}.

(ii) G is the *full limit* of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if each embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full.

Warning 7.2. A composition of two full embeddings is not necessarily full. Consequently, if $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit and $(\, k_i \mid i \in \mathbb{N}\,)$ is a strictly increasing sequence of natural numbers, then $G = \bigcup_{i \in \mathbb{N}} G_{k_i}$ is not necessarily a full limit. The notion of a full limit is a purely technical one, introduced in order to formulate the following special cases of Theorems 3.4 and 3.18, which will be proved in this section.

Proposition 7.3. *If G is a full limit of finite alternating groups, then G has a nontrivial ergodic IRS if and only if G has linear natural orbit growth.*

Proposition 7.4. *Suppose that G is a full limit of finite alternating groups and that G has linear natural orbit growth. Let $G \curvearrowright (\,\Delta, m\,)$ be the canonical ergodic action; and for each $r \geq 1$, let ν_r be the stabilizer distribution of $G \curvearrowright (\,\Delta^r, m^{\otimes r}\,)$. Then the ergodic IRS of G are $\{\,\delta_1, \delta_G\,\} \cup \{\,\nu_r \mid r \in \mathbb{N}^+\,\}$.*

For the rest of this section, let $G = \bigcup_{i \in \mathbb{N}} G_i$ be the full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.

Lemma 7.5. *Let $p > 2$ be an odd prime, let $a = 1/(p+1)$ and let $n_{p,a}$ be the integer given by Lemma 6.1. Suppose that $|\Delta_{i_0}| \geq \max\{ n_{p,a}, 5(p+1) \}$ and that $g \in \mathrm{Alt}(\Delta_{i_0})$ is an element of order p such that $|\operatorname{Fix}_{\Delta_{i_0}}(g)| \leq |\Delta_{i_0}|/(p+1)$. Then $|\operatorname{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/(p+1)$ for all $i \geq i_0$.*

Proof. Let $i \geq i_0$ and suppose that $|\operatorname{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/(p+1)$. It is enough to show that if Ω is an orbit of $\mathrm{Alt}(\Delta_i)$ on Δ_{i+1}, then $|\operatorname{Fix}_\Omega(g)|/|\Omega| \leq 1/(p+1)$. Let $\omega \in \Omega$ and let $H = \{ h \in \mathrm{Alt}(\Delta_i) \mid h \cdot \omega = \omega \}$ be the corresponding stablizer. Let K be a maximal proper subgroup of $\mathrm{Alt}(\Delta_i)$ such that $H \leqslant K$ and let θ_K be the normalized permutation character of the action $\mathrm{Alt}(\Delta_i) \curvearrowright \mathrm{Alt}(\Delta_i)/K$. Then, applying Corollary 2.3, we have that $|\operatorname{Fix}_\Omega(g)|/|\Omega| \leq \theta_K(g)$.

First suppose that K acts primitively on Δ_i. Let g be a product of a_i p-cycles when regarded as an element of $\mathrm{Alt}(\Delta_i)$. Since $|\operatorname{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/(p+1)$, it follows that $a_i \geq |\Delta_i|/(p+1)$. Hence, by Lemma 6.1, we have that

$$\theta_K(g) < 1/|\Delta_i| < 1/(p+1).$$

Next suppose that K acts imprimitively on Δ_i, preserving a system of imprimitivity $\mathcal{P}$ of blocksize $2 \leq \ell \leq n/2$. Then $\mathrm{Alt}(\Delta_i) \curvearrowright \mathrm{Alt}(\Delta_i)/K$ is isomorphic to the action of $\mathrm{Alt}(\Delta_i)$ on the set $\mathcal{P}$ of partitions of Δ_i into ℓ-sets. Applying Lemma 6.3, we obtain that

$$\theta_K(g) < 2/|\Delta_i| < 1/(p+1).$$

Finally suppose that K acts intransitively on Δ_i, fixing set-wise a subset $S \subseteq \Delta_i$ of size $1 \leq \ell \leq n/2$. Then $\mathrm{Alt}(\Delta_i) \curvearrowright \mathrm{Alt}(\Delta_i)/K$ is isomorphic to the action of $\mathrm{Alt}(\Delta_i)$ on $[\Delta_i]^\ell$. If $\ell = 1$, then $\theta_K(g) = |\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i| \leq 1/(p+1)$. Hence we can suppose that $\ell \geq 2$. Applying Lemma 6.2, either

$$\theta_K(g) < 5/|\Delta_i| \leq 1/(p+1),$$

or else

$$\theta_K(g) < \frac{1}{2}|\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i| \leq 1/2(p+1).$$

□

Corollary 7.6. $\limsup |\operatorname{Fix}_{\Delta_j}(g)|/|\Delta_j| < 1$ *for all* $1 \neq g \in G$.

Proof. Applying Lemma 7.5, it follows that there exists an element $g \in G$ of order 3 such that $\limsup |\operatorname{Fix}_{\Delta_j}(g)|/|\Delta_j| \leq 1/4$. On the other hand, it is easily checked that if $(k_j \mid j \in \mathbb{N})$ is a strictly increasing sequence of natural numbers, then $N = \{ g \in G \mid \lim_{j \to \infty} |\operatorname{Fix}_{\Delta_{k_j}}(g)|/|\Delta_{k_j}| = 1 \}$ is a normal subgroup of G. Since G is simple, the result follows. □

For the rest of this section, suppose that $\nu \neq \delta_1$, δ_G is an ergodic IRS of G. Applying Creutz-Peterson [2, Proposition 3.3.1], we can suppose that ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$. Let $\chi(g) = \mu(\operatorname{Fix}_Z(g))$ be the corresponding character. For each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{ g \cdot z \mid g \in G_i \}$. Then, by Theorem 2.1, for μ-a.e. $z \in Z$, for all $g \in G$, we have that

$$\mu(\operatorname{Fix}_Z(g)) = \lim_{i \to \infty} |\operatorname{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)|.$$

Fix such an element $z \in Z$ and let $H = \{ h \in G \mid h \cdot z = z \}$ be the corresponding point stabilizer. Clearly we can suppose that the element $z \in Z$ has been chosen so that $H \neq 1$, G and so that $\chi(g) > 0$ for all $g \in H$. For each $i \in \mathbb{N}$, let $H_i = H \cap G_i$ and let $n_i = |\Delta_i|$. Clearly $G_i \curvearrowright \Omega_i(z)$ is isomorphic to $G_i \curvearrowright G_i/H_i$.

Lemma 7.7. *There exist only finitely many $i \in \mathbb{N}$ such that the action $H_i \curvearrowright \Delta_i$ is primitive.*

Proof. Suppose that $I = \{ i \in \mathbb{N} \mid H_i \curvearrowright \Delta_i \text{ is primitive } \}$ is infinite. Since $H \neq 1$, there exists an element $g \in H$ of some prime order p. Let $g \in G_{i_0}$ and for each $i \geq i_0$, let g be a product of a_i p-cycles when regarded as an element of G_i. Then, by Corollary 7.6, there exists a constant $a > 0$ such that $a_i \geq a\, n_i$ for all $i \geq i_0$. Let $n_{p,a}$ be the integer given by Lemma 6.1. Then $|\operatorname{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)| < 1/n_i$ for all $i \in I$ such that $n_i \geq n_{p,a}$ and it follows that

$$\chi(g) = \lim_{i\to\infty} |\operatorname{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)| = 0,$$

which is a contradiction. □

Lemma 7.8. *For each integer $d > 1$, there exist only finitely many $i \in \mathbb{N}$ such that H_i acts imprimitively on Δ_i preserving a* maximal *system $\mathcal{B}_i$ of imprimitivity of blocksize d.*

Proof. Suppose that there exists a fixed $d > 1$ and an infinite subset $I \subseteq \mathbb{N}$ such that for all $i \in I$, the subgroup H_i acts imprimitively on Δ_i preserving a maximal system $\mathcal{B}_i$ of imprimitivity of blocksize d. Then H_i is isomorphic to a subgroup of $\operatorname{Sym}(d) \operatorname{wr} \operatorname{Sym}(n_i/d)$ for each $i \in I$. Applying Stirling's Approximation, it follows that there exist constants c, k such that for all n,

$$|\operatorname{Sym}(d) \operatorname{wr} \operatorname{Sym}(n/d)| < c\, k^n n^{n/d}.$$

Claim 7.9. *For all but finitely many $i \in I$, the induced action of H_i on $\mathcal{B}_i$ contains* $\operatorname{Alt}(\mathcal{B}_i)$.

Proof of Claim 7.9. Suppose not and let $g \in H$ be an element of some prime order p. Let $g \in G_{i_0}$ and for each $i \geq i_0$, let g be a product of a_i p-cycles when regarded as an element of G_i. Applying Corollary 7.6, there exists a constant $a > 0$ such that $a_i \geq a\, n_i$ for all $i \geq i_0$. And arguing as in the proof of Lemma 6.1, it follows that there are constants r, s such that

$$|g^{G_i}| > r\, s^{n_i} (n_i^{n_i})^{(p-1)a}.$$

Let $i \in I$ and let $\Gamma_i \leqslant \operatorname{Sym}(\mathcal{B}_i)$ be the group induced by the action of H_i on $\mathcal{B}_i$. Since $\mathcal{B}_i$ is a maximal system of imprimitivity, it follows that Γ_i is a primitive subgroup of $\operatorname{Sym}(\mathcal{B}_i)$. Hence, by Praeger-Saxl [13], if Γ_i does not contain $\operatorname{Alt}(\mathcal{B}_i)$, then $|\Gamma_i| < 4^{n_i}$. Since H_i is isomorphic to a subgroup of $\operatorname{Sym}(d) \operatorname{wr} \Gamma_i$, it follows that

$$|H_i| < (d!)^{n_i/d} 4^{n_i/d} = t^{n_i},$$

where $t = (d!\, 4)^{1/d}$, and so

$$\frac{|g^{G_i} \cap H_i|}{|g^{G_i}|} < \frac{|H_i|}{|g^{G_i}|} < \frac{t^{n_i}}{r\, s^{n_i} (n_i^{n_i})^{(p-1)a}}$$

It follows that

$$\chi(g) = \lim_{i\to\infty} |\operatorname{Fix}_{\Omega_i(z)}(g)|/|\Omega_i(z)| = \lim_{i\to\infty} |g^{G_i} \cap H_i|/|g^{G_i}| = 0,$$

which is a contradiction. □

Let $a = 1/6$ and let $n_{5,a}$ be the integer given by Lemma 6.1. Let $i_0 \in I$ be such that $|\Delta_{i_0}| \geq \max\{ n_{5,a}, 24d \}$ and such that the induced action of H_{i_0} on $\mathcal{B}_{i_0}$ contains $\mathrm{Alt}(\mathcal{B}_{i_0})$. Then there exists an element $g \in H_{i_0}$ of order 5 such that g fixes setwise at most 4 blocks of $\mathcal{B}_{i_0}$ and so $|\,\mathrm{Fix}_{\Delta_{i_0}}(g)| \leq 4\,d \leq |\Delta_{i_0}|/6$. Applying Lemma 7.5, it follows that $|\,\mathrm{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/6$ for all $i \geq i_0$. For each $i \geq i_0$, let g be a product of a_i p-cycles when regarded as an element of G_i. Then it is easily checked that $a_i \geq n_i/6$. Hence, arguing as above, there exist constants r, s such that

$$|g^{G_i}| > r\, s^{n_i} (n_i^{n_i})^{4/6}.$$

Hence, if $i_0 \leq i \in I$, we have that

$$\frac{|\, g^{G_i} \cap H_i\,|}{|\, g^{G_i}\,|} < \frac{|\, H_i\,|}{|\, g^{G_i}\,|} < \frac{c\, k^{n_i} n_i^{n_i/d}}{r\, s^{n_i} (n_i^{n_i})^{4/6}}.$$

Since $4/6 > 1/2 \geq 1/d$, it follows that $\chi(g) = 0$, which is a contradiction. □

Lemma 7.10. *There exist only finitely many $i \in \mathbb{N}$ such that the action $H_i \curvearrowright \Delta_i$ is transitive.*

Proof. Suppose not. Then, by Lemma 7.7, for all but finitely many $i \in \mathbb{N}$, the subgroup H_i acts imprimitively on Δ_i with a maximal system of imprimitivity $\mathcal{B}_i$ of blocksize d_i. Furthermore, by Lemma 7.8, we have that $d_i \to \infty$ as $i \to \infty$. Let $1 \neq h \in H$; say, $h \in H_i$. Then, by Corollary 7.6, there exist a constant $a > 0$ such that $|\,\mathrm{supp}_{\Delta_j}(g)| \geq a|\Delta_j|$ for all $j \geq i$. But then Lemma 6.5 (in the case when $r = 0$) implies that

$$\chi(g) = \lim_{j \to \infty} \frac{|\,\{ s \in G_j \mid sgs^{-1} \in H_i \}|}{|G_j|} = 0,$$

which is a contradiction. □

Lemma 7.11. *There exists a constant s such that for all but finitely many $i \in \mathbb{N}$, there exists a unique H_i-invariant subset $U_i \subseteq \Delta_i$ of cardinality $1 \leq r_i \leq s$ such that H_i induces at least $\mathrm{Alt}(\Sigma_i)$ on $\Sigma_i = \Delta_i \smallsetminus U_i$.*

Proof. Combining Lemmas 7.7, 7.8 and 7.10, we see that there exists $i_0 \in \mathbb{N}$ such that H_i acts intransitively on Δ_i for all $i \geq i_0$. For each such i, let

$$r_i = \max\{\, |U| : U \subseteq \Delta_i \text{ is } H_i\text{-invariant and } |U| \leq \tfrac{1}{2}|\Delta_i|\,\}.$$

Then, applying Lemma 6.6, we see that there exists s such that $1 \leq r_i \leq s$ for all $i \geq i_0$. Furthermore, choosing i_0 so that $|\Delta_{i_0}| \geq 4s$, it follows that for all $i \geq i_0$, there exists a unique H_i-invariant subset $U_i \subseteq \Delta_i$ of cardinality r_i and that H_i acts transitively on $\Sigma_i = \Delta_i \smallsetminus U_i$. Let $\bar{H}_i$ be the subgroup of $\mathrm{Sym}(\Sigma_i)$ induced by the action of H_i on Σ_i. Then, arguing as above, we first see that $\bar{H}_i$ must act primitively on Σ_i for all but finitely many $i \geq i_0$, and then that $\mathrm{Alt}(\Sigma_i) \leqslant \bar{H}_i$ for all but finitely many $i \geq i_0$. □

In particular, it follows that for every prime p, there exists arbitarily large $i \in \mathbb{N}$ such that there exists an element $g \in H_i$ of order p with $|\,\mathrm{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/(p+1)$.

Lemma 7.12. *If $g \in H$ has prime order $p > s$, then* $\liminf |\,\mathrm{Fix}_{\Delta_i}(g)|/|\Delta_i| \neq 0$.

Proof. Suppose that $\liminf |\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i| = 0$ for some element $g \in H$ of prime order $p > s$. Let θ_i, ψ_i be the normalized permutation characters of the actions $G_i \curvearrowright G_i/H_i$ and $G_i \curvearrowright [\Delta_i]^{r_i}$. Since $p > s \geq r_i$, it follows that

$$\operatorname{Fix}_{[\Delta_i]^{r_i}}(g) = [\operatorname{Fix}_{\Delta_i}(g)]^{r_i}.$$

Hence, combining Lemma 7.11 and Corollary 2.3, we obtain that

$$\theta_i(g) \leq \psi_i(g) = \frac{|\,[\operatorname{Fix}_{\Delta_i}(g)]^{r_i}|}{|\,[\Delta_i]^{r_i}|}$$

and it follows that $\chi(g) = \lim_{i\to\infty} \theta_i(g) = 0$, which is a contradiction. □

Lemma 7.13. $\lim_{i\to\infty} s_{i+1}n_i/n_{i+1} = 1$.

Proof. Let p be a prime with $p > s$, let $a = 1/(p+1)$ and let $n_{p,a}$ be the integer given by Lemma 6.1. Then there exists i_0 such that $|\Delta_{i_0}| \geq \max\{\, n_{p,a}, 5(p+1)\,\}$ and such that H_{i_0} contains an element g of order p such that $|\operatorname{Fix}_{\Delta_{i_0}}(g)| \leq |\Delta_{i_0}|/(p+1)$. Applying Corollary 7.6, it follows that $|\operatorname{Fix}_{\Delta_i}(g)| \leq |\Delta_i|/(p+1)$ for all $i \geq i_0$. Furthermore, by Lemma 7.12, we can assume that $|\operatorname{Fix}_{\Delta_i}(g)| \geq 10$ for all $i \geq i_0$. Suppose that Ω is a non-natural orbit of $G_i = \operatorname{Alt}(\Delta_i)$ on Δ_{i+1}. Then, applying Corollary 2.3 and Lemmas 6.1, 6.2 and 6.3, it follows that

$$\frac{|\operatorname{Fix}_\Omega(g)|}{|\Omega|} < \max\left\{ \frac{5}{|\Delta_i|}, \frac{|\operatorname{Fix}_{\Delta_i}(g)|}{2|\Delta_i|} \right\} = \frac{|\operatorname{Fix}_{\Delta_i}(g)|}{2|\Delta_i|};$$

and it follows that

$$\begin{aligned} |\operatorname{Fix}_{\Delta_{i+1}}(g)| &\leq s_{i+1}n_i \frac{|\operatorname{Fix}_{\Delta_i}(g)|}{|\Delta_i|} + (n_{i+1} - s_{i+1}n_i)\frac{|\operatorname{Fix}_{\Delta_i}(g)|}{2|\Delta_i|} \\ &= (s_{i+1}n_i + n_{i+1})\frac{|\operatorname{Fix}_{\Delta_i}(g)|}{2|\Delta_i|}. \end{aligned}$$

In particular, since $s_{i+1}n_i \leq |\Delta_{i+1}|$, it follows that for all $i \in \mathbb{N}$,

$$|\operatorname{Fix}_{\Delta_{i+1}}(g)|/|\Delta_{i+1}| \leq |\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i|. \tag{7.1}$$

Suppose that $\lim_{i\to\infty} s_{i+1}n_i/n_{i+1} \neq 1$. Then there exists a fixed $\varepsilon > 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that for all $i \in I$, we have that $s_{i+1}n_i/n_{i+1} < 1 - \varepsilon$ and hence

$$\frac{|\operatorname{Fix}_{\Delta_{i+1}}(g)|}{|\Delta_{i+1}|} \leq \frac{(2-\varepsilon)}{2}\frac{|\operatorname{Fix}_{\Delta_i}(g)|}{|\Delta_i|}. \tag{7.2}$$

Clearly the inequalities (7.1) and (7.2) imply that $\lim_{i\in\mathbb{N}} |\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i| = 0$, which contradicts Lemma 7.12. □

Lemma 7.14. *G has linear natural orbit growth.*

Proof. Suppose not. Then $a_i = \lim_{j\to\infty} s_{ij}/n_j = 0$ for all $i \in \mathbb{N}$. Hence, for each $\ell \in \mathbb{N}$, we can choose an increasing sequence $(\, k_i \mid i \in \mathbb{N}\,)$ such that:

- $k_0 \geq \ell$ is such that $|\Delta_{k_0}| \geq \max\{\, n_{3,1/4}, 20\,\}$ and G_{k_0} contains an element g of order 3 with $|\operatorname{Fix}_{\Delta_{k_0}}(g)| \leq |\Delta_{k_0}|/4$.
- $k_{i+1} > k_i$ is such that $s_{k_i k_{i+1}} n_{k_i}/n_{k_{i+1}} < 1/2^{i+1}$.

Next define the subsets $\Delta'_{k_i} \subseteq \Delta_{k_i}$ and subgroups $G'_{k_i} = \operatorname{Alt}(\Delta'_{k_i})$ inductively by:

- $\Delta'_{k_0} = \Delta_{k_0}$;
- $\Delta'_{k_{i+1}} = \Delta_{k_{i+1}} \smallsetminus \operatorname{Fix}_{\Delta_{k_{i+1}}}(G'_{k_i})$.

Then clearly $G'(\ell) = \bigcup_{i\in\mathbb{N}} G'_{k_i}$ is a full limit of finite alternating groups with associated parameters $n'_i = |\Delta'_{k_i}|$ and $s'_{i+1} = s_{k_i k_{i+1}}$. Applying Lemma 7.5, it follows that $|\operatorname{Fix}_{\Delta_j}(g)| \leq |\Delta_j|/4$ for all $j \geq k_0$ and hence

$$n'_i = |\Delta'_{k_i}| \geq |\Delta_{k_i} \smallsetminus \operatorname{Fix}_{\Delta_{k_i}}(g)| \geq \frac{3}{4}|\Delta_{k_i}| = \frac{3}{4}n_{k_i}.$$

It follows that

$$s'_{i+1}n'_i/n'_{i+1} \leq \frac{4}{3}s_{k_i k_{i+1}} n_{k_i}/n_{k_{i+1}} \leq \frac{4}{3}\left(\frac{1}{2}\right)^{i+1}$$

and so $\lim_{i\to\infty} s'_{i+1}n'_i/n'_{i+1} = 0$. Thus, applying Lemma 7.13, it follows that the only ergodic IRS of $G'(\ell)$ are δ_1 and $\delta_{G'(\ell)}$. Let $f_\ell : \mathrm{Sub}_G \to \mathrm{Sub}_{G'(\ell)}$ be the Borel map defined by $H \mapsto H \cap G'(\ell)$. Then the map f_ℓ is $G'(\ell)$-equivariant and hence $\nu_{G'(\ell)} = (f_\ell)_*\nu$ is a (not necessarily ergodic) IRS of $G'(\ell)$. It follows that for ν-a.e. $H \in \mathrm{Sub}_G$, for all $\ell \in \mathbb{N}$, either $H \cap G'(\ell) = 1$ or $G'(\ell) \leqslant H$. Clearly we can perform the above construction so that if $\ell < m$, then $G'(\ell) \leqslant G'(m)$. Furthermore, since $G_\ell \leqslant G'(\ell)$, it follows that $G = \bigcup_{\ell\in\mathbb{N}} G'(\ell)$. But this implies that for ν-a.e. $H \in \mathrm{Sub}_G$, either $H = 1$ or $H = G$, which is a contradiction. □

Note that Proposition 7.3 is an immediate consequence of Proposition 3.17 and Lemma 7.14. Continuing our analysis, suppose that $H \in \mathrm{Sub}_G$ be a ν-generic subgroup. Let i_0 be an integer such that for all $i \geq i_0$, there exists a unique H_i-invariant subset $U_i \subseteq \Delta_i$ of cardinality $1 \leq r_i \leq s$ such that H_i induces at least $\mathrm{Alt}(\Sigma_i)$ on $\Sigma_i = \Delta_i \smallsetminus U_i$ and such that $|\mathrm{Alt}(\Sigma_{i_0})| \gg s!$. For each $i \geq i_0$, let $\pi_i : H_i \to \mathrm{Sym}(U_i)$ be the homomorphism defined by $g \mapsto g \restriction U_i$ and let $K_i = \ker \pi_i$. Since $[\,H_i : K_i\,] \leq s!$, it follows that $K_i = \mathrm{Alt}(\Sigma_i)$. Also note that since $[\,H_{i+1} : K_{i+1}\,] \leq s!$, it follows that $[\,K_i : K_i \cap K_{i+1}\,] \leq s!$ and hence $K_i \leqslant K_{i+1}$. Let $K = \bigcup_{i\geq i_0} K_i$. Since K_i is the unique largest factor of the socle of H_i, it follows that the map $H \mapsto K$ is G-equivariant and hence there is a corresponding ergodic IRS $\tilde{\nu}$ which concentrates on the corresponding subgroups $K \leqslant H$. Applying Theorem 3.21, it follows that there exists $1 \leq r \leq s$ such that $\tilde{\nu}$ is the stabilizer distribution ν_r of $G \curvearrowright (\,\Delta^r, m^{\otimes r}\,)$, where $G \curvearrowright (\,\Delta, m\,)$ is the canonical ergodic action. Hence, in order to complete the proof of Proposition 7.4, it is enough to show that $H = K$ for ν-a.e. $H \in \mathrm{Sub}_G$. To see this, let $H \in \mathrm{Sub}_G$ be such that the corresponding subgroup K is the stabilizer of the sequence $(x_1, \cdots x_r) \in \Delta^r$. For each $j \in \mathbb{N}$, let $U_j = \{\, x_\ell \restriction \Delta_j \mid 1 \leq \ell \leq r\,\}$. Then

$$K_j = \mathrm{Alt}(\Delta_j \smallsetminus U_j) \trianglelefteq H_j \leqslant \mathrm{Sym}(\Delta_j \smallsetminus U_j) \times \mathrm{Sym}(U_j),$$

and hence $K \trianglelefteq H$. In the proof of Corollary 3.19, we showed that $G_{\bar{x}}$ is self-normalizing for $m^{\otimes r}$-a.e. $\bar{x} \in \Delta^r$ and this means that $H = K$ for ν-a.e. $H \in \mathrm{Sub}_G$.

8. Arbitrary limits of finite alternating groups

In this section, we will complete the complete the classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups G such that $G \not\cong \mathrm{Alt}(\mathbb{N})$. The ergodic IRSs of $\mathrm{Alt}(\mathbb{N})$ will be described in Section 9. Throughout this section, let $G = \bigcup_{i\in\mathbb{N}} G_i$ be the (not necessarily full) union of the increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and suppose that $G \not\cong \mathrm{Alt}(\mathbb{N})$.

Lemma 8.1. *For each $i \in \mathbb{N}$, the number c_{ij} of nontrivial G_i-orbits on Δ_j is unbounded as $j \to \infty$.*

Proof. By Hall [5, Theorem 5.1], if there exist i, $c \in \mathbb{N}$ such that G_i has at most c nontrivial orbits on Δ_j for all $j > i$, then $G \cong \mathrm{Alt}(\mathbb{N})$, which is a contradiction. □

Hence, after passing to a suitable subsequence, we can suppose that each G_i has at least 2 nontrivial orbits on Δ_{i+1}. Of course, since G_i is simple, this implies that if $1 \neq G'_i \leqslant G_i$, then G'_i also has at least 2 nontrivial orbits on Δ_{i+1}. For each $\ell \in \mathbb{N}$, we can define sequences of subsets $\Sigma^\ell_j \subseteq \Delta_j$ and subgroups $G(\ell)_j = \mathrm{Alt}(\Sigma^\ell_j)$ for $j \geq \ell$ inductively as follows:

- $\Sigma^\ell_\ell = \Delta_\ell$;
- $\Sigma^\ell_{j+1} = \Delta_{j+1} \smallsetminus \mathrm{Fix}_{\Delta_{j+1}}(G(\ell)_j)$.

Clearly each $G(\ell)_j$ is strictly contained in $G(\ell)_{j+1}$ and $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ is the full limit of the $G(\ell)_j = \mathrm{Alt}(\Sigma^\ell_j)$. It is also easily checked that if $\ell < m$, then $G(\ell) \leqslant G(m)$ and that $G = \bigcup_{\ell \in \mathbb{N}} G(\ell)$. For the rest of this section, suppose that $\nu \neq \delta_1$, δ_G is an ergodic IRS of G.

Lemma 8.2. *$G(\ell)$ has linear natural orbit growth for all but finitely many $\ell \in \mathbb{N}$.*

Proof. Otherwise, by Proposition 7.3, there exist infinitely many $\ell \in \mathbb{N}$ such that the only ergodic IRS of $G(\ell)$ are δ_1 and $\delta_{G(\ell)}$. Arguing as in the proof of Lemma 7.14, we easily reach a contradiction. □

Hence we can suppose that $G(\ell)$ has linear natural orbit growth for all $\ell \in \mathbb{N}$. Let $G(\ell) \curvearrowright (\Delta_\ell, m_\ell)$ be the canonical ergodic action and for each $r \in \mathbb{N}^+$, let $\nu(\ell)_r$ be the stabilizer distribution of $G(\ell) \curvearrowright (\Delta^r_\ell, m^{\otimes r}_\ell)$. Let $\nu_{G(\ell)}$ be the (not necessarily ergodic) IRS of $G(\ell)$ arising from the $G(\ell)$-equivariant map $\mathrm{Sub}_G \to \mathrm{Sub}_{G(\ell)}$ defined by $H \mapsto H \cap G(\ell)$. Then Proposition 7.4 implies that there exist $\alpha(\ell)$, $\beta(\ell)$, $\gamma(\ell)_r \in [0,1]$ with $\alpha(\ell) + \beta(\ell) + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r = 1$ such that

$$\nu_{G(\ell)} = \alpha(\ell)\delta_1 + \beta(\ell)\delta_{G(\ell)} + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r \nu(\ell)_r. \tag{8.1}$$

Let $H \in \mathrm{Sub}_G$ be a ν-generic subgroup and let $\ell_0 \in \mathbb{N}$ be the least integer such that $1 < H \cap G(\ell_0) < G(\ell_0)$. Then equation (8.1) implies that for each $\ell \geq \ell_0$, there exist $i_\ell \geq \ell$ and $r_\ell \geq 1$ such that for all $j \geq i_\ell$, there exists $U^\ell_j \in [\Sigma^\ell_j]^{r_\ell}$ such that

$$H \cap G(\ell)_j = H \cap \mathrm{Alt}(\Sigma^\ell_j) = \mathrm{Alt}(\Sigma^\ell_j \smallsetminus U^\ell_j)$$

and such that U^ℓ_k is contained in the union of the natural $G(\ell)_j$-orbits on Σ^ℓ_k for all $k > j$. Define $i_\ell = \ell$ for $0 \leq \ell < \ell_0$ and let $f_H \in \mathbb{N}^\mathbb{N}$ be the function defined by $f_H(\ell) = i_\ell$. Applying the Borel-Cantelli Lemma, it follows easily that there exists a *fixed* function $f \in \mathbb{N}^\mathbb{N}$ such that for ν-a.e. $H \in \mathrm{Sub}_G$, for all but finitely many $\ell \in \mathbb{N}$, we have that $f_H(\ell) \leq f(\ell)$. Let $(j_\ell \mid \ell \in \mathbb{N})$ be a strictly increasing sequence of integers such that $j_\ell \geq \max\{ f(k) \mid k \leq \ell \}$. For each $\ell \in \mathbb{N}$, let $\Delta'_\ell = \Sigma^\ell_{j_\ell}$ and let $G'_\ell = \mathrm{Alt}(\Delta'_\ell)$. Then it is easily checked that if $k < \ell$, then

$$G'_k = G(k)_{j_k} \leqslant G(\ell)_{j_\ell} = G'_\ell.$$

Also, since $G_\ell \leqslant G(\ell)_{j_\ell} = G'_\ell$, it follows that $G = \bigcup_{\ell \in \mathbb{N}} G'_\ell$.

Suppose that $H \in \mathrm{Sub}_G$ is a ν-generic subgroup. Then there exists an integer $\ell_H \in \mathbb{N}$ such that $i_\ell \leq f(\ell)$ and for all $\ell \geq \ell_H$. Suppose that $\ell \geq \ell_H$. Then, since

$$j_{\ell+1} \geq \max\{ f(\ell), f(\ell+1) \} \geq \max\{ i_\ell, i_{\ell+1} \}$$

and $\Sigma^{\ell}_{j_{\ell+1}} \subseteq \Sigma^{\ell+1}_{j_{\ell+1}} \subseteq \Delta_{j_{\ell+1}}$, it follows that there exist subsets $U^{\ell}_{j_{\ell+1}} \in [\Sigma^{\ell}_{j_{\ell+1}}]^{r_\ell}$ and $U^{\ell+1}_{j_{\ell+1}} \in [\Sigma^{\ell+1}_{j_{\ell+1}}]^{r_{\ell+1}}$ such that

$$\mathrm{Alt}(\Sigma^{\ell}_{j_{\ell+1}} \smallsetminus U^{\ell}_{j_{\ell+1}}) = H \cap \mathrm{Alt}(\Sigma^{\ell}_{j_{\ell+1}}) \leqslant H \cap \mathrm{Alt}(\Sigma^{\ell+1}_{j_{\ell+1}}) = \mathrm{Alt}(\Sigma^{\ell+1}_{j_{\ell+1}} \smallsetminus U^{\ell+1}_{j_{\ell+1}}).$$

This implies that $U^{\ell}_{j_{\ell+1}} = U^{\ell+1}_{j_{\ell+1}} \cap \Sigma^{\ell}_{j_{\ell+1}}$. Since $j_\ell \geq f(\ell) \geq i_\ell$, it follows that $U^{\ell}_{j_{\ell+1}}$ is contained in the union of the natural $G(\ell)_{j_\ell}$-orbits on $\Sigma^{\ell}_{j_\ell+1}$; and since $\Delta_{\ell+1} \smallsetminus \Sigma^{\ell}_{j_\ell+1} \subseteq \mathrm{Fix}_{\Delta_{\ell+1}}(G(\ell)_{j_\ell})$, it follows that $U^{\ell+1}_{j_{\ell+1}}$ is contained in the union of the natural and trivial $G(\ell)_{j_\ell}$-orbits on $\Sigma^{\ell+1}_{j_{\ell+1}}$. In other words, writing $U'_\ell = U^{\ell}_{j_\ell}$, we have shown that for all $\ell \geq \ell_H$,

(i) $H \cap G'_\ell = \mathrm{Alt}(\Delta'_\ell \smallsetminus U'_\ell)$; and
(ii) $U'_{\ell+1}$ is contained in the union of the natural G'_ℓ-orbits on $\Delta'_{\ell+1}$.

Applying Theorem 5.1, we obtain that G has almost diagonal type. At this point in our analysis, we have completed the proof of Theorem 3.4. The next result completes the proof of Theorem 3.18.

Lemma 8.3. *If G has linear natural orbit growth, then there exists $r \in \mathbb{N}^+$ such that $\nu = \nu_r$.*

Proof. Suppose that $G = \bigcup_{\ell \in \mathbb{N}} G'_\ell$ has linear natural orbit growth with parameters n'_ℓ, $s'_{\ell k}$, etc. Then we can suppose that $a'_\ell = \lim_{k\to\infty} s'_{\ell k}/n'_k > 0$ for all $\ell \in \mathbb{N}$. If $1 \neq g \in G'_\ell$ and $k > \ell$, then

$$|\operatorname{supp}_{\Delta'_k}(g)| \geq \frac{s'_{\ell k}}{n'_k} |\operatorname{supp}_{\Delta'_\ell}(g)|\, n'_k \geq a'_\ell |\operatorname{supp}_{\Delta'_\ell}(g)|\, n'_k.$$

Since $1 \neq g \in G$ was arbitrary, Lemma 6.6 implies that there exists a constant s such that $1 \leq r_\ell \leq s$ for all $\ell \in \mathbb{N}$. Applying Theorem 3.21, it follows that $\nu = \nu_r$ for some $1 \leq r \leq s$. □

The next result is an immediate consequence of Theorem 5.5.

Lemma 8.4. *If G has sublinear natural orbit growth, then there exists $\beta_0 \in (0, \infty)$ such that $\nu = \nu_{\beta_0}$.*

9. The ergodic IRS of Alt(ℕ)

In this section, adapting and slightly correcting Vershik's analysis of the ergodic IRSs of the group Fin(ℕ) of finitary permutations of the natural numbers, we will state the classification of the ergodic IRSs of the infinite alternating group Alt(ℕ) and we will characterize the ergodic actions $\mathrm{Alt}(\mathbb{N}) \curvearrowright (Z, \mu)$ such that the associated character $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is indecomposable.

Recall that $\mathrm{Fin}(\mathbb{N}) = \{ g \in \mathrm{Sym}(\mathbb{N}) \mid |\operatorname{supp}(g)| < \infty \}$. Throughout this section, if $g \in \mathrm{Fin}(\mathbb{N})$, then $c_n(g)$ denotes the number of cycles of length $n > 1$ in the cyclic decomposition of the permutation g and $\mathrm{sgn} : \mathrm{Fin}(\mathbb{N}) \to C = \{\pm 1\}$ is the homomorphism defined by

$$\mathrm{sgn}(g) = \begin{cases} 1, & \text{if } g \in \mathrm{Alt}(\mathbb{N}); \\ -1, & \text{otherwise.} \end{cases}$$

Vershik's analysis of the ergodic IRSs of Fin(ℕ) is based upon the following two insights.

(i) If $H \leqslant \mathrm{Fin}(\mathbb{N})$ is a random subgroup, then the corresponding H-orbit decomposition $\mathbb{N} = \bigsqcup_{i \in I} B_i$ is a random partition of $\mathbb{N}$, and these have been classified by Kingman [7].
(ii) The induced action of H on an infinite orbit B_i can be determined via an application of Wielandt's theorem [21, Satz 9.4], which states that $\mathrm{Alt}(\mathbb{N})$ and $\mathrm{Fin}(\mathbb{N})$ are the only primitive subgroups of $\mathrm{Fin}(\mathbb{N})$.

With minor modifications, the same ideas apply to the ergodic IRSs of $\mathrm{Alt}(\mathbb{N})$, which can be classified as follows. Suppose that $\alpha = (\alpha_i)_{i \in \mathbb{N}} \in [0,1]^{\mathbb{N}}$ is a sequence such that:

- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0$; and
- $\sum_{i=0}^{\infty} \alpha_i = 1$.

Then we can define a probability measure p_α on $\mathbb{N}$ by $p_\alpha(\{i\}) = \alpha_i$. Let μ_α be the corresponding product probability measure on $\mathbb{N}^{\mathbb{N}}$. Then $\mathrm{Alt}(\mathbb{N})$ acts ergodically on $(\mathbb{N}^{\mathbb{N}}, \mu_\alpha)$ via the shift action $(\gamma \cdot \xi)(n) = \xi(\gamma^{-1}(n))$. For each $\xi \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$, let $B_i^\xi = \{ n \in \mathbb{N} \mid \xi(n) = i \}$. Then for μ_α-a.e. $\xi \in \mathbb{N}^{\mathbb{N}}$, the following statements are equivalent for all $i \in \mathbb{N}$.

(a) $\alpha_i > 0$.
(b) $B_i^\xi \neq \emptyset$.
(c) B_i^ξ is infinite.
(d) $\lim_{n \to \infty} |B_i^\xi \cap \{0, 1, \cdots, n-1\}|/n = \alpha_i$.

In this case, we say that ξ is *μ_α-generic*.

First suppose that $\alpha_0 \neq 1$, so that $I = \{ i \in \mathbb{N}^+ \mid \alpha_i > 0 \} \neq \emptyset$. Let $S_\alpha = \bigoplus_{i \in I} C_i$, where each $C_i = \{\pm 1\}$ is cyclic of order 2, and let $E_\alpha \leqslant S_\alpha$ be the subgroup consisting of the elements $(\varepsilon_i)_{i \in I}$ such that $|\{ i \in I \mid \varepsilon_i = -1 \}|$ is even. Then for each subgroup $A \leqslant E_\alpha$, we can define a corresponding $\mathrm{Alt}(\mathbb{N})$-equivariant Borel map

$$\begin{aligned} f_\alpha^A : \mathbb{N}^{\mathbb{N}} &\to \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \\ \xi &\mapsto H_\xi \end{aligned}$$

as follows. If ξ is μ_α-generic, then $H_\xi = s_\xi^{-1}(A)$, where s_ξ is the homomorphism

$$\begin{aligned} s_\xi : \bigoplus_{i \in I} \mathrm{Fin}(B_i^\xi) &\to \bigoplus_{i \in I} C_i \\ (\pi_i) &\mapsto (\mathrm{sgn}(\pi_i)). \end{aligned}$$

Otherwise, if ξ is not μ_α-generic, then we let $H_\xi = 1$. Let $\nu_\alpha^A = (f_\alpha^A)_* \mu_\alpha$ be the corresponding ergodic IRS of $\mathrm{Alt}(\mathbb{N})$. Finally, if $\alpha_0 = 1$, then we define $\nu_\alpha^\emptyset = \delta_1$.

Theorem 9.1. *If ν is an ergodic IRS of* $\mathrm{Alt}(\mathbb{N})$*, then there exists α, A as above such that $\nu = \nu_\alpha^A$.*

There exist examples of sequences α and distinct subgroups A, $A' \leqslant E_\alpha$ such that $\nu_\alpha^A = \nu_\alpha^{A'}$. However, since $\lim_{n \to \infty} |B_i^\xi \cap \{0, 1, \cdots, n-1\}|/n = \alpha_i$ for μ_α-a.e. $\xi \in \mathbb{N}^{\mathbb{N}}$, it follows that if $\alpha \neq \alpha'$ and A, A' are subgroups of E_α, $E_{\alpha'}$, then $\nu_\alpha^A \neq \nu_{\alpha'}^{A'}$. In particular, $\mathrm{Alt}(\mathbb{N})$ has uncountably many ergodic IRSs. The remainder of this section is devoted to the proof of the following result.

Theorem 9.2. *If* $\mathrm{Alt}(\mathbb{N}) \curvearrowright (Z, \mu)$ *is an nontrivial ergodic action and $\nu \neq \delta_1$ is the corresponding stabilizer distribution, then the following are equivalent.*

(i) *The associated character* $\chi(g) = \mu(\operatorname{Fix}_G(g))$ *is indecomposable.*
(ii) *There exists* α *such that* $\nu = \nu_\alpha^{E_\alpha}$.

The proof of Theorem 9.2 makes use of the following results of Thoma [16].

Theorem 9.3. (Thoma [16, Satz 6]) *The indecomposable characters of* $\operatorname{Alt}(\mathbb{N})$ *are precisely the restrictions* $\chi \restriction \operatorname{Alt}(\mathbb{N})$ *of the indecomposable characters* χ *of* $\operatorname{Fin}(\mathbb{N})$.

Theorem 9.4. (Thoma [16, Satz 1]) *If* χ *is a character of* $\operatorname{Fin}(\mathbb{N})$*, then* χ *is indecomposable if and only if there exists a sequence* $(s_n \mid n \geq 2)$ *of real numbers with each* $|s_n| \leq 1$ *such that* $\chi(g) = \prod_{n \geq 2} s_n^{c_n(g)}$.

Lemma 9.5. *If* $\operatorname{Alt}(\mathbb{N}) \curvearrowright (Z, \mu)$ *is an ergodic action and there exists* α *such that the corresponding stabilizer distribution is* $\nu_\alpha^{E_\alpha}$*, then the associated character* $\chi(g) = \mu(\operatorname{Fix}_G(g))$ *is indecomposable.*

Proof. With the above notation, $\operatorname{Fin}(\mathbb{N})$ acts ergodically on $(\mathbb{N}^\mathbb{N}, \mu_\alpha)$ and we can define a $\operatorname{Fin}(\mathbb{N})$-equivariant Borel map

$$\begin{aligned} \varphi_\alpha : \mathbb{N}^\mathbb{N} &\to \operatorname{Sub}_{\operatorname{Fin}(\mathbb{N})} \\ \xi &\mapsto \bigoplus_{i \in I} \operatorname{Fin}(B_i^\xi). \end{aligned}$$

Let $\nu_\alpha^+ = (\varphi_\alpha)_* \mu_\alpha$ be the corresponding ergodic IRS of $\operatorname{Fin}(\mathbb{N})$ and let χ_α^+ be the character of of $\operatorname{Fin}(\mathbb{N})$ defined by

$$\chi_\alpha^+(g) = \mu_\alpha(\{\xi \in \mathbb{N}^\mathbb{N} \mid g \in \bigoplus_{i \in I} \operatorname{Fin}(B_i^\xi)\}).$$

Then it is easily checked that

$$\chi_\alpha^+(g) = \prod_{n > 1} (\sum_{i \in I} \alpha_i^n)^{c_n(g)}.$$

Hence, by Theorem 9.4, it follows that χ_α^+ is an indecomposable character of of $\operatorname{Fin}(\mathbb{N})$. Notice that if $g \in \operatorname{Alt}(\mathbb{N})$, then

$$\begin{aligned} \chi(g) &= \mu(\operatorname{Fix}_Z(g)) \\ &= \nu_\alpha^{E_\alpha}(\{H \in \operatorname{Sub}_{\operatorname{Alt}(\mathbb{N})} \mid g \in H\}) \\ &= \mu_\alpha(\{\xi \in 2^\mathbb{N} \mid g \in \operatorname{Alt}(\mathbb{N}) \cap \bigoplus_{i \in I} \operatorname{Fin}(B_i^\xi)\}) = \chi_\alpha^+(g). \end{aligned}$$

Applying Theorem 9.3, it follows that χ is an indecomposable character of $\operatorname{Alt}(\mathbb{N})$. □

Proof of Theorem 9.2. Suppose that $\operatorname{Alt}(\mathbb{N}) \curvearrowright (Z, \mu)$ is an nontrivial ergodic action such that associated character $\chi(g) = \mu(\operatorname{Fix}_G(g))$ is indecomposable. Let ν be the corresponding stabilizer distribution and suppose that $\nu \neq \delta_1$. Then there exist α, A as above such that $\nu = \nu_\alpha^A$ and hence

$$\chi(g) = \mu_\alpha(\{\xi \in \mathbb{N}^\mathbb{N} \mid g \in H_\xi\}).$$

Clearly we can suppose that $|I| \geq 2$. For each element $a = (\varepsilon_i)_{i \in I} \in A$, let $\sigma(a) = \{i \in I \mid \varepsilon_i = -1\}$. If $A \neq 0$, let m_A be the least integer m such that there exists an element $0 \neq a \in I$ such that $|\sigma(a)| = m$. If $A = 0$, let $m_A = 0$. Let $g = (1\,2)(3\,4))$ and $h = (1\,2)(3\,4)(5\,6)(7\,8)$. Then Theorem 9.4 implies that $\chi(h) = \chi(g)^2$.

Case 1: Suppose that $m_A > 2$. Then it is easily seen that $\chi(g) = \sum_{i\in I} \alpha_i^4$ and that $\chi(h) \geq \sum_{i\in I} \alpha_i^8 + \binom{4}{2} \sum_{\{i,j\}\in[I]^2} \alpha_i^4\alpha_j^4$. On the other hand, we have that $\chi(g)^2 = \sum_{i\in I} \alpha_i^8 + 2\sum_{\{i,j\}\in[I]^2} \alpha_i^4\alpha_j^4$ and so $\chi(h) > \chi(g)^2$, which is a contradiction.

Case 2: Suppose that $m_A \in \{0,2\}$. Let $\Gamma = (I,E)$ be the graph with vertex set I and edge set E such that $\{j,k\} \in E$ if and only if there exists $a \in A$ with $\sigma(a) = \{j,k\}$. Then it is enough to show that $E = [I]^2$.

In this case, it is clear that $\chi(g) = \sum_{i\in I} \alpha_i^4 + 2\sum_{\{i,j\}\in E} \alpha_i^2\alpha_j^2$ and so

$$\chi(g)^2 = \sum_{i\in I} \alpha_i^8 + 2\sum_{\{i,j\}\in E} \alpha_i^4\alpha_j^4 + 4\sum_{i\in I} \alpha_i^4 \sum_{\{j,k\}\in E} \alpha_j^2\alpha_k^2 + 4\sum_{\substack{\{i,j\}\in E\\ \{k,\ell\}\in E}} \alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2.$$

After rearranging the terms, we obtain that

$$\begin{aligned}\chi(g)^2 = \sum_{i\in I} \alpha_i^8 + 6\sum_{\{i,j\}\in E} \alpha_i^4\alpha_j^4 + 4\sum_{\{i,j\}\in E} \alpha_i^6\alpha_j^2 + 4\sum_{i\notin\{j,k\}\in E} \alpha_i^4\alpha_j^2\alpha_k^2 \\ + 8\sum_{\substack{\{i,j\}\in E\\ \{i,k\}\in E}} \alpha_i^4\alpha_j^2\alpha_k^2 + 8\sum_{\substack{\{i,j\}\in E\\ \{k,\ell\}\in E\\ i,j,k,\ell \text{ distinct}}} \alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\chi(h) = \sum_{i\in I} \alpha_i^8 + 6\sum_{\{i,j\}\in[I]^2} \alpha_i^4\alpha_j^4 + 4\sum_{\{i,j\}\in E} \alpha_i^6\alpha_j^2 \\ + 12\sum_{i\notin\{j,k\}\in E} \alpha_i^4\alpha_j^2\alpha_k^2 + 24\sum_{\{i,j,k,\ell\}\in T} \alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2,\end{aligned}$$

where T is the set of $\{i,j,k,\ell\} \in [I]^4$ such that there exists $a \in A$ with $\sigma(a) = \{i,j,k,\ell\}$. Note that if $\{i,j\}$, $\{k,\ell\} \in E$ are disjoint edges, then $\{i,j,k,\ell\} \in T$. Also, each $\{i,j,k,\ell\} \in T$ can be partitioned into two disjoint edges in at most 3 ways. It follows that

$$8\sum_{\substack{\{i,j\}\in E\\ \{k,\ell\}\in E\\ i,j,k,\ell \text{ distinct}}} \alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2 \leq 24\sum_{\{i,j,k,\ell\}\in T} \alpha_i^2\alpha_j^2\alpha_k^2\alpha_\ell^2. \tag{9.1}$$

Clearly we also have that

$$6\sum_{\{i,j\}\in E} \alpha_i^4\alpha_j^4 \leq 6\sum_{\{i,j\}\in[I]^2} \alpha_i^4\alpha_j^4. \tag{9.2}$$

Since $\chi(h) = \chi(g)^2$, inequalities (9.1) and (9.2) must both be equalities and it follows that $E = [I]^2$, as desired. □

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