NEW SYMMETRIES FOR OVERPARTITION RANK AND CRANK FUNCTIONS

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ABSTRACT. Continuing our study of the symmetry phenomenon in rank and crank of partitions, we turn our attention to overpartitions. Building on our previous joint work on the partition rank generating function and that of the second author on the partition crank generating function, we investigate the modularity, transformation and symmetry of the overpartition rank and crank generating functions. Let $\mathcal{O}_{\mathcal{R}}(z,q)$ and $\mathcal{O}_{\mathcal{C}}(z,q)$ be the two-variable generating functions of the overpartition rank and the first residual crank statistics respectively and ζ_p be a primitive p-th root of unity. By considering the action of the group $\Gamma_0(2p)$ on the elements of the p-dissection of $\mathcal{O}_{\mathcal{R}}(\zeta_p,q)$ and $\mathcal{O}_{\mathcal{C}}(\zeta_p,q)$, we discover new symmetries for the rank and first residual crank functions for overpartitions. In the process, we improve upon the results of Chris Jennings Shaffer and that of Bringmann and Lovejoy on the modularity and transformation of the overpartition rank function. We also find lower bounds for the orders of the elements of p-dissection of the said generating functions at the cusps of $\Gamma_1(2p)$. Using these orders, we are working on developing an algorithmic approach comprising of techniques coming from automorphic forms to find new identities for explicit dissections of overpartition rank and the first residual crank generating functions modulo 11 in terms of generalized eta products.

1. INTRODUCTION

The rank statistic for partitions was discovered by Dyson in 1944. The Dyson rank of a partition is the largest part minus the number of parts. This statistic decomposes the partitions of $5n+4$ and $7n+5$ into 5 and 7 equinumerous classes, as conjectured by Dyson and proved by Atkin and Swinnerton-Dyer, thus resolving the mod 5 and mod 7 congruences of Ramanujan. The crank of a partition was introduced by Andrews and Garvan in 1988. It is defined as the largest part if the partition contains no ones, and otherwise as the number of parts larger than the number of ones minus the number of ones. In their paper, Andrews and Garvan show that the crank simultaneously decomposes the partitions of $5n+4$, $7n+5$ and $11n+6$ into 5, 7 and 11 equinumerous classes respectively. The significance of these two statistics lies not only in the aspect that it renders a partial combinatorial interpretation to the famous congruences of Ramanujan, but also in the fact that their generating functions have a rich modular structure. Several mathematicians have worked in this direction to study the modular structure underlying these two statistics following the seminal work done by Zwegers in his thesis, where he shows how Ramanujan's mock theta functions occur as the holomorphic part of certain real analytic modular forms.

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In 2004, Sylvie Corteel and Jeremy Lovejoy [6] introduced the concept of overpartitions. An overpartition is a partition in which the first occurrence of a number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1,$$
$$\overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1.$$

Analogous to the crank of an ordinary partition, Bringmann, Lovejoy and Osburn [4] introduced the first and second residual crank of an overpartition. In this paper, the authors deduce congruence properties for combinatorial functions which can be expressed in terms of the second overpartition rank moment and the corresponding residual crank moment. Several authors have since considered these residual cranks and have worked on their generalizations, finding and proving inequalities between the moments of these functions in conjunction with other overpartition statistics, among other problems. The combinatorial interpretation leading to explicit definitions of these cranks was given by us recently in [12].

Let $M(m,n)$ denote the number of overpartitions of n with first residual crank equal to m. Throughout, we assume $q = e^{2\pi i\tau}$ where $\tau \in \mathcal{H}$, the upper half complex plane. We let $\overline{M}(z,q)$ denote the two-variable generating function for the first residual crank so that

$$\mathcal{O}_{\mathcal{C}}(z,\tau) = \sum_{n=0}^{\infty} \sum_{m} \overline{M}(m,n)\, z^m\, q^n.$$

It is the transformation, modularity and symmetry of this function under special congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, which we hereinafter call the overpartition crank function $\mathcal{O}_{\mathcal{C}}(z,\tau)$, when z is a primitive p-th root of unity, that we study in the first half of this paper. This residual crank can be expressed in terms of a Klein form whose transformation is well known in literature and helps us establish our results. We summarize our two major results for the overpartition crank without delving into further details here.

Theorem 1.1. *Let $p > 3$ be prime. Then the function*

$$\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)} \mathcal{O}_{\mathcal{C}}(\zeta_p, \tau)$$

is a modular form of weight 1 on $\Gamma_0(p^2) \cap \Gamma_1(2p)$.

The modularity and symmetry of the elements of the p-dissection of $\dfrac{\eta(p^2\tau)^2}{\eta(2p^2\tau)} \mathcal{O}_{\mathcal{C}}(\zeta_p, \tau)$ is our other point of interest in the paper. These elements are defined below.

Definition 1.2. Let $p > 3$ be prime, $0 \le m \le p-1$ and $1 \le \ell \le p-1$. Define

$$\mathcal{OK}^{(C)}_{p,m}(\zeta_p^\ell, z) := q^{m/p} \prod_{n=1}^{\infty} \frac{(1-q^{pn})^2}{(1-q^{2pn})} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{p-1} \overline{M}(k,p,pn+m)\, \zeta_p^{k\ell} \right) q^n. \tag{1.1}$$

The result on the modularity of these elements is deferred to Section 4. Below is our other major result of the paper for the crank of overpartitions. The action of the congruence subgroup $\Gamma_0(2p)$ on these elements leads to our symmetry result, analogous to the symmetry result for crank of partitions [18, Theorem 1.6].

Theorem 1.3. *Let $p > 3$ be prime and $0 \leq m \leq p-1$. Then*

$$\mathcal{OK}_{p,m}^{(C)}(\zeta_p^\ell, z) \mid [A]_1 = \frac{\sin(\ell\pi/p)}{\sin(\ell d\pi/p)} \zeta_p^{mak} \mathcal{OK}_{p,ma^2}^{(C)}(\zeta_p^{\ell d}, z) \tag{1.2}$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

In the second half of this paper, we consider the Dyson's rank of an overpartition. It is defined as the largest part minus the number of parts, in particular the rank does not depend on whether or not a part is overlined.

Let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m. Let

$$\mathcal{O}_{\mathcal{R}}(z,\tau) = \sum_{n=0}^{\infty} \sum_{m} \overline{N}(m,n) z^m q^n.$$

In a fashion similar to the work of Bringmann and Ono [5] for the rank of partitions, Bringmann and Lovejoy [3] observed that $\mathcal{O}_{\mathcal{R}}(z,\tau)$ is the holomorphic part of a harmonic weak Maass form of weight $\frac{1}{2}$ on when $z \neq -1$ is a root of unity. The following theorem summarizes their main result in the case where z is a p-th root of unity.

Theorem 1.4. *Let $p > 3$ be prime and $0 < a < p$. Define*

$$\theta(\alpha,\beta;\tau) := \sum_{n \equiv \alpha \pmod{\beta}} n q^{n^2/2\beta},$$

$$\Theta_{a,p}(\tau) := \theta\left(4a+p, 2p; \frac{\tau}{4p}\right), \text{ and}$$

$$J\left(\frac{a}{p}; z\right) := \frac{\pi i \tan\left(\frac{\pi a}{p}\right)}{4p} \int_{-\overline{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}.\Theta_{a,p}\left(-\frac{1}{\tau}\right) d\tau}{\sqrt{-i(\tau+z)}}.$$

Then

$$M\left(\frac{a}{p}; z\right) := \mathcal{O}_{\mathcal{R}}\left(\zeta_p^a; \tau\right) - J\left(\frac{a}{p}; z\right)$$

is a weak Maass form of weight $\frac{1}{2}$ on the congruence subgroup

$$\tilde{\Gamma} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \alpha \equiv \delta \equiv 1 \pmod{4p}, \gamma \equiv 0 \pmod{16p^2} \right\}.$$

In our study of the overpartition rank function $\mathcal{O}_{\mathcal{R}}(\zeta_p,\tau)$, we build around the functions considered by Jennings-Shaffer [13]. In this paper, Jennings-Shaffer reconsiders the modularity of this function and bypassing the approach of Bringmann and Lovejoy, he constructs a completion of $\mathcal{O}_{\mathcal{R}}(\zeta_p,\tau)$ to a harmonic Maass form which is the sum of an easily understood modular form and a harmonic Maass form of Zwegers. We record this work here.

For $u, v, z \in \mathbb{C}, \tau \in \mathcal{H}$, and $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$ we have

$$\vartheta(z;\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \exp\left(\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)\right),$$

$$\mu(u,v;\tau) = \frac{\exp(\pi i u)}{\vartheta(v;\tau)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \exp(\pi i n(n+1)\tau + 2\pi i n v)}{1-\exp(2\pi i n\tau + 2\pi i u)}.$$

Next for $u, z \in \mathbb{C}$, $y = \operatorname{Im}(\tau)$, and $a = \operatorname{Im}(u)/\operatorname{Im}(\tau)$ we define

$$E(z) = 2\int_0^z \exp\left(-\pi w^2\right) dw$$

$$R(u;\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} (\operatorname{sgn}(n) - E((n+a)\sqrt{2y}))(-1)^{n-\frac{1}{2}} \exp\left(-\pi i n^2\tau - 2\pi i n u\right).$$

For $a, b \in \mathbb{R}$ we set

$$g_{a,b}(\tau) = \sum_{n\in\mathbb{Z}+a} n \exp\left(\pi i n^2\tau + 2\pi i n b\right).$$

Finally for $u, v \notin \mathbb{Z} + \tau\mathbb{Z}$ we set

$$\tilde{\mu}(u,v;\tau) = \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau).$$

In his revolutionary PhD thesis [20], Zwegers studied these functions and gave their transformation formulas.

Theorem 1.5. *(Jennings-Shaffer, Corollary 2.2,* [13]*) Suppose a and c are integers, $c > 0$ and $c \nmid 2a$. Then,*

$$\begin{aligned}\mathcal{O}_{\mathcal{R}}\left(\zeta_c^a;\tau\right) &= \frac{2\zeta_c^a\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}P(a,c;\tau) - \frac{i2\zeta_c^a\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}N(a,c;\tau)\\ &\quad + i\sqrt{2}\frac{\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}\int_{-\bar{\tau}}^{i\infty} \frac{g_{0,\frac{1}{2}-\frac{2a}{c}}(2z)}{\sqrt{-i(z+\tau)}}dz \end{aligned} \tag{1.3}$$

where

$$N(a,c;\tau) = q^{-\frac{1}{4}}\tilde{\mu}\left(\frac{2a}{c},\tau;2\tau\right), \quad P(a,c;\tau) = \frac{(-q;q)_\infty \left(q^2;q^2\right)_\infty^2}{(q;q)_\infty \left[\zeta_c^{2a};q^2\right]_\infty}.$$

Jennings-Shaffer gives explicit and compact transformation formulas to determine a larger subgroup of $\mathrm{SL}_2(\mathbb{Z})$ on which the completion of $\mathcal{O}_{\mathcal{R}}(\zeta_p^a,\tau)$ viz.

$$\begin{aligned}M(a,c;\tau) &= \mathcal{O}_{\mathcal{R}}\left(\zeta_c^a;\tau\right) - i\sqrt{2}\frac{\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}\int_{-\bar{\tau}}^{i\infty} \frac{g_{0,\frac{1}{2}-\frac{2a}{c}}(2z)}{\sqrt{-i(z+\tau)}}dz\\ &= \frac{2\zeta_c^a\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}P(a,c;\tau) - \frac{i2\zeta_c^a\left(1-\zeta_c^a\right)}{\left(1+\zeta_c^a\right)}N(a,c;\tau)\end{aligned} \tag{1.4}$$

is a Harmonic Maass form, and reproves the modularity result of Bringmann and Lovejoy.

Theorem 1.6. *Suppose a and c are integers, $c > 0$, and $c \nmid 2a$, then $M(a,c;\tau)$ is a harmonic weak Maass form of weight $\frac{1}{2}$ on $\Gamma_0(16c^2) \cap \Gamma_1(4c)$.*

The case $a = 1$ and $c = p$ for prime $p > 3$ is the result of Bringmann and Lovejoy stated previously in Theorem 1.4.

In our work, with the introduction of a simple eta multiplier to $\mathcal{O}_{\mathcal{R}}(\zeta_p, \tau)$, we narrow down the transformation of all the functions involved in the expression for $\mathcal{O}_{\mathcal{R}}(\zeta_p, \tau)$ (Theorem 1.5) to the simpler congruence subgroup $\Gamma_0(2p^2) \cap \Gamma_1(p)$ sans the multipliers appearing in the results of Jennings-Shaffer (Corollaries 3.3 and 4.1, [13]). This problem also involves finding the overpartition analogue of the completed rank function $\mathcal{R}_p(z)$ given by Garvan [10].

$$\mathcal{R}_p(z) := q^{-\frac{1}{24}} R(\zeta_p, q) - \chi_{12}(p) \sum_{a=1}^{\frac{1}{2}(p-1)} (-1)^a \left(\zeta_p^{3a+\frac{1}{2}(p+1)} + \zeta_p^{-3a-\frac{1}{2}(p+1)} - \zeta_p^{3a+\frac{1}{2}(p-1)} - \zeta_p^{-3a-\frac{1}{2}(p-1)}\right) q^{\frac{a}{2}(p-3a)-\frac{p^2}{24}} \Phi_{p,a}(q^p),$$

where

$$\Phi_{p,a}(q) := \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } 0 < 6a < p, \\ \displaystyle -1 + \sum_{n=0}^{\infty} \frac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & \text{if } p < 6a < 3p, \end{cases}$$

and

$$\chi_{12}(n) := \left(\frac{12}{n}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

This in turn involves finding the overpartition rank analogue of the correction factor $\Phi_{p,a}(q^p)$. We find that this correction factor for the overpartition rank function includes a Lambert series which

i) appears in the dissection elements of the overpartition rank generating function $\mathcal{O}(\zeta_p, \tau)$ of Jennings-Shaffer and Lovejoy and Osburn (Theorem 1.1, [13] & Theorem 1.1, 1.2, [16]), and
ii) has a connection with the cancellation of the Mordell integral of the theta function which is the non-holomorphic part of $\mathcal{O}(\zeta_p, \tau)$ (Corollary 2.2, [13]).

We will eventually see in the course of this paper that this factor allows us to complete the overpartition rank function to a weakly holomorphic modular form on a larger and nicer congruence subgroup improving the results of Bringmann and Lovejoy [3] and Jennings-Shaffer [13]. To that end,

Definition 1.7. Let $p > 3$ be prime and $1 \leq k \leq \frac{1}{2}(p-1)$. Define

$$\Phi_{p,k}(q) := q^{k-\frac{k^2}{p}} \frac{(q^{2p};q^{2p})_\infty}{(q^p;q^p)_\infty^2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{pn(n+1)}}{1-q^{pn+k}} \tag{1.5}$$

and

$$\mathfrak{O}_R(\zeta_p,\tau) := \mathcal{O}_{\mathcal{R}}(\zeta_p;\tau) - \frac{2\,(1-\zeta_p)}{1+\zeta_p}\sum_{k=1}^{\frac{1}{2}(p-1)}(-1)^k\,\left(\zeta_p^{-2k}-\zeta_p^{2k}\right)\,\Phi_{p,k}(q^p). \tag{1.6}$$

Thus one of our main results

Theorem 1.8. *Let $p>3$ be prime. Then the function*

$$\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}\mathfrak{O}_R(\zeta_p,\tau)$$

is a weakly holomorphic modular form of weight 1 on $\Gamma_0(2p^2)\cap\Gamma_1(p)$.

Furthermore, we also define the elements of the p-dissection of $\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}\mathfrak{O}_R(\zeta_p,\tau)$. These elements feature the functions $\mathcal{OK}_{p,m}^{(R)}(\zeta_p^d;z)$ defined below into two cases for when m is a quadratic residue or non-residue modulo p, wherein fragments of the correction factor above (Equation (1.5)) appears in the quadratic residue case. We study their modularity and symmetry, akin to the case of rank of partitions (Definition 1.4, [11]).

Definition 1.9. Let $p>3$ be prime and $0\le m\le p-1$. Define $\mathcal{OK}_{p,m}^{(R)}(\zeta_p^d;z)$ as follows :

(i) For $m=0$ or $\left(\frac{-m}{p}\right)=-1$ define

$$\mathcal{OK}_{p,m}^{(R)}(\zeta_p^d;z) := q^{m/p}\,\frac{\prod_{n=1}^{\infty}(1-q^{pn})^2}{\prod_{n=1}^{\infty}(1-q^{2pn})}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{p-1}\overline{N}(k,p,pn)\,\zeta_p^{kd}\right)q^n, \tag{1.7}$$

where $q=\exp(2\pi iz)$.

(ii) For $\left(\frac{-m}{p}\right)=1$ define

$$\begin{aligned}\mathcal{OK}_{p,m}^{(R)}(\zeta_p^d;z) := q^{m/p}\,\frac{\prod_{n=1}^{\infty}(1-q^{pn})^2}{\prod_{n=1}^{\infty}(1-q^{2pn})}\Bigg(&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{p-1}\overline{N}(k,p,pn+m)\,\zeta_p^{kd}\right)q^n\\ &- q^{\frac{1}{p}(a(p-a)-m)}\frac{2\,(1-\zeta_p)}{1+\zeta_p}(-1)^a(\zeta_p^{-2a}-\zeta_p^{2a})\Phi_{p,a}(q)\Bigg),\end{aligned} \tag{1.8}$$

where $1\le a\le\frac{1}{2}(p-1)$ has been chosen so that

$$-m\equiv a^2 \pmod{p}.$$

The result on the modularity of these elements is deferred to Section 5. Below is our other major result of the paper for the rank of overpartitions. The action of the congruence subgroup $\Gamma_0(2p)$ on these elements leads to our symmetry result, similar to the symmetry result for crank of overpartitions Theorem 1.3 and analogous to the symmetry result for rank of partitions [11, Theorem 1.6].

Theorem 1.10. *Let $p>3$ be prime and $0\le m\le p-1$. Then*

$$\mathcal{OK}_{p,m}^{(R)}(\zeta_p,\tau)\mid[A]_1 = \frac{1-\zeta_p}{1-\zeta_p^d}\,\frac{1+\zeta_p^d}{1+\zeta_p}\,\zeta_p^{mak}\,\mathcal{OK}_{p,ma^2}^{(R)}(\zeta_p^d,\tau) \tag{1.9}$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Remarks :

I. In this paper, we strengthen the results of Bringmann and Lovejoy [3] and Jennings-Shaffer [13] for the rank of overpartitions. In particular

   i. For the case $p > 3$ prime, we strengthen Theorems 1.4 and 1.6. We show that the group in these results can be enlarged to $\Gamma_0(2p^2) \cap \Gamma_1(p)$ but with a simple eta multiplier $\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}$, on which our completed rank function is a weakly holomorphic modular form of order 1. The idea for choosing this special congruence subgroup comes from the work of Jennings-Shaffer which we explain below.

   ii. The introduction of the eta multiplier $\frac{\eta(\tau)^2}{\eta(2\tau)}$ to the functions $P(a, c, \tau)$ and $N(a, c, \tau)$ involved in the expression for $\mathcal{O}_{\mathcal{R}}(\zeta_p, \tau)$ gets rid of the multipliers appearing in the results of Jennings-Shaffer (Corollaries 3.3 and 4.1, [13]), where he is able to significantly simplify the transformation of these two functions to the much simpler congruence subgroup $\Gamma_0(2p^2) \cap \Gamma_1(p)$. However, what we realize is that considering a generalization of Jennings-Shaffer's function $N_7(k; \tau)$ [13, Page 6] extended to any prime p leads us to our improved modularity result described in the point above. The resulting function $N_p(k; \tau)$ appears in our modified expression (with the correction factor and eta multipliers) for $\mathcal{O}_{\mathcal{R}}(\zeta_p, \tau)$ in Equation (1.3). See Equation (5.3). The non-holomorphic part of this function helps annihilate the non-holomorphic part arising from the Mordell integral of the theta function in Equation (1.3) and leads us to our completion of the overpartition rank function in Equation (1.6) and eventually Theorem 1.8. Our investigation of the transformation and modularity of $N_p^*(k; \tau) = \frac{\eta(\tau)^2}{\eta(2\tau)} N_p(k; \tau)$ also generalizes and improves the transformation result of Jennings- Shaffer for $N_7(k; \tau)$ [13, Proposition 5.2] to any prime $p > 3$. This is documented in Corollary 5.4.

II. A majority of our modularity and transformation results in this paper are direct analogies of those in [11] and [18].

III. The reason behind considering the study of the first residual crank of overpartitions in this paper is because we observe that the respective modular and transformation properties in the case of Dyson's rank of overpartitions hold for the exact same congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. To be precise, the modularity result for the generating function of the modified overpartition statistics holds under $\Gamma_0(2p^2) \cap \Gamma_1(p)$, the modularity and transformation results for the dissection elements $\mathcal{OK}_{p,m}(\zeta_p, \tau)$ under $\Gamma(2p)$ and $\Gamma_1(2p)$ and finally the symmetry result for the dissection elements $\mathcal{OK}_{p,m}(\zeta_p, \tau)$ under $\Gamma_0(2p)$, for both the first residual crank and the Dyson's rank of overpartitions. However, the second residual crank of overpartitions also appears in the hindsight in our study of the rank. It arises in the formula for the two variable overpatition rank generating function $\mathcal{O}_{\mathcal{R}}\left(\zeta_c^a;\tau\right)$ due to Jennings-Shaffer in Equation (1.3). This is the function $P(a,c,\tau)$ whose modularity and transformation alongwith a simple eta eta multiplier, when ζ_c^a is a p-th root of unity $\zeta_p, p>3$ is investigated in Propositions 5.1 and 5.10 and Corollary 5.2.

In the final part of this paper, we find explicit p-dissections of the overpartition rank and crank functions in terms of generalized eta products. Identities for the dissection elements of the overpartition rank function for the case $p=7$ were found and studied by Jennings-Shaffer [13]. However we find that for higher primes $p>7$, the computational techniques in Maple that we employed to find dissections for the partition counterparts in [11] and [18] fall short. Consequently, the idea here is to consider the ring of modular/cusp forms for $\Gamma_1(2p)$, and eventually find a simpler basis consisting of Jacobi products which we accomplish using a combination of Maple and Sage. We present and prove such identities for the cases $p=7,11$ in the last section.

The paper is organized as follows. In Section 2, we review and build a database of the necessary notations, definitions and fundamental transformation results which we will use in the course of establishing our main results in the subsequent sections. In Section 3, we give conditions for the modularity of the generalized eta-quotients analogous to the conditions developed for the rank of partitions [11, Section 3]. In Sections 4 and 5, we consider the first residual crank of overpartitions and Dyson's rank of overpartitions respectively and deduce our transformation, modularity and symmetry results. Section 6 is devoted to calculating lower bounds for the orders of $\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z)$ and $\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z)$ at the cusps of $\Gamma_1(2p)$ which we utilize to prove identities for the same in the subsequent section. Section 7, currently under development, is devoted to developing an algorithm that uses the Valence formula for proving generalized eta-quotient identities for $\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z)$ and $\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z)$ for $p=7,11$.

2. PRELIMINARIES

2.1. **Theta functions and transformations.** We define a class of generalized eta products and functions that we will use frequently in expressions for our overpartition rank and crank identities and to study their transformations in the subsequent sections.

Definition 2.1. Following Biagioli (see [2]), define

$$f_{N,\rho}(z):=(-1)^{\lfloor\rho/N\rfloor}q^{(N-2\rho)^2/(8N)}\left(q^\rho,q^{N-\rho},q^N;q^N\right)_\infty. \tag{2.1}$$

Then, for a vector $\overrightarrow{n}=(n_0,n_1,n_2,\cdots,n_p)\in\mathbb{Z}^{p+1}$, define

$$j(z)=j(p,\overrightarrow{n},z)=\eta(2pz)^{n_0}\prod_{k=1}^{p}f_{2p,k}(z)^{n_k}. \tag{2.2}$$

We note that

$$f_{N,\rho}(z)=f_{N,N+\rho}(z)=f_{N,-\rho}(z), \tag{2.3}$$

and

$$f_{N,\rho}(z) = \eta(Nz)\, \eta_{N,\rho}(z) \tag{2.4}$$

where

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

and

$$\eta_{N,k}(z) = q^{\frac{N}{2}P_2(k/N)} \prod_{\substack{m>0 \\ m\equiv \pm k \pmod N}} (1-q^m)$$

where $z \in \mathfrak{h}$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, and $\{t\} = t - [t]$ is the fractional part of t.

The transformations for $\eta(z)$ due to Knopp [14, Theorem 2, p.51] and the theta function $f_{N,\rho}(z)$ due to Biagioli [2, Lemma 2.1, p.278] are well known and have been used previously by us to study transformations in [10] and [11].

2.2. **Klein forms.** Our transformation results for the first residual crank of overpartitions are realized by writing them in terms of a Klein form which we define here. These functions are also considered by Jennings-Shaffer in his study of the transformation of the function $P(a, c; \tau)$ described in the introduction section. Here we correct a minor mistake in the definition adopted by him [13, Section 4] and instead state and use the definition adopted by Eum, Koo and Shin [9] as follows.

Definition 2.2. [9, Equation 1.4] For $(a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and $\zeta = \exp(2\pi i(a_1\tau + a_2))$, the Klein form $t_{(a_1,a_2)}(\tau)$ is defined by the following infinite product expansion

$$t_{(a_1,a_2)}(\tau) = \exp\left(\pi i a_2(a_1-1)\right) q^{\frac{1}{2}(a_1(a_1-1))} (1-\zeta) \frac{(\zeta q;q)_\infty (\zeta^{-1}q;q)_\infty}{(q;q)_\infty^2}. \tag{2.5}$$

The transformation and order of this Klein form necessary for our study is recorded in the proposition below.

Proposition 2.3. [9, Proposition 2.1]

(i) For $(a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and $(b_1, b_2) \in \mathbb{Z}^2$, we have

$$t_{(a_1+b_1,a_2+b_2)}(\tau) = (-1)^{b_1b_2+b_1+b_2} \exp(-\pi i(b_1a_2 - b_2a_1)) t_{(a_1,a_2)}(\tau).$$

(ii) For $(a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$t_{(a_1,a_2)}(\tau)|[A]_1 = t_{(a_1,a_2).A}(\tau) = t_{(a_1a+a_2c,a_1b+a_2d)}(\tau).$$

(iii) For $(a_1, a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, we have

$$ord_q t_{(a_1,a_2)}(\tau) = \frac{1}{2}\{a_1\}(\{a_1\}-1),$$

where $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

2.3. **A useful permutation.**

Definition 2.4. Let $p > 3$ be prime. For $1 \leq r \leq p$, we define a permutation $\pi_r : [p] \to [p]$, where $[p] = \{1, 2, \cdots, p\}$ by $\pi_r(i) = i'$ where $ri' \equiv \pm i \pmod{2p}$.

π_r induces a permutation on $\mathbb{Z}^p$. For $\overrightarrow{n} = (n_0, n_1, n_2, \cdots, n_p), \pi_r(\overrightarrow{n})$ permutes the components to $\pi_r(\overrightarrow{n}) = (n_0, n_{\pi_r(1)}, n_{\pi_r(2)}, \cdots, n_{\pi_r(p)})$.

Lemma 2.5. *Let $p > 3$ be prime and $\overrightarrow{n}$ and $j(z)$ be defined as in Definition 2.1. Then,*

$$j(p, \pi_r(\overrightarrow{n}), z) = \eta(pz)^{n_0} \prod_{k=1}^{p} f_{2p,rk}(z)^{n_k}.$$

The proof of the lemma follows as in [11, Lemma 3.5].

2.4. **The Atkin operator.** We define the (weight k) Atkin U_p operator by

$$F \mid [U_p]_k := \frac{1}{p} \sum_{r=0}^{p-1} F\left(\frac{z+r}{p}\right) = p^{\frac{k}{2}-1} \sum_{n=0}^{p-1} F \mid [T_r]_k \,, \tag{2.6}$$

where

$$T_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix},$$

and the more general $U_{p,m}$ is defined as

$$F \mid [U_{p,m}]_k := \frac{1}{p} \sum_{r=0}^{p-1} \exp\left(-\frac{2\pi irm}{p}\right) F\left(\frac{z+r}{p}\right) = p^{\frac{k}{2}-1} \sum_{r=0}^{p-1} \exp\left(-\frac{2\pi irm}{p}\right) F \mid [T_r]_k \,. \tag{2.7}$$

We note that $U_p = U_{p,0}$. In addition, if

$$F(z) = \sum_n a(n) q^n = \sum_n a(n) \exp(2\pi i z n),$$

then

$$F \mid [U_{p,m}]_k = q^{m/p} \sum_n a(pn+m)\, q^n = \exp(2\pi i m z/p) \sum_n a(pn+m) \exp(2\pi i n z).$$

3. Modularity conditions for generalized eta-quotients

We present a general criteria for an eta-quotient $j(p, \overrightarrow{n}, z)$ in our Definition 2.1 to be a weakly holomorphic modular form of weight 1 on $\Gamma(2p)$ in the form of a theorem. This result is analogous to the criteria for the eta quotients that we had defined to study the partition rank [11, Theorem 3.1].

Theorem 3.1. *Let $p > 3$ be prime and suppose $\overrightarrow{n} = (n_0, n_1, n_2, \cdots, n_p) \in \mathbb{Z}^{p+1}$. Then $j(p, \overrightarrow{n}, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(2p)$ satisfying the modularity condition*

$$j(p, \overrightarrow{n}, z) \mid [A]_1 = \exp(\tfrac{2\pi ibm}{p}) j(p, \overrightarrow{n}, z)$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2p)$ *provided the following conditions are met :*

$$(1)\ n_0 + \sum_{k=1}^{p} n_k = 2,$$

$$(2)\ n_0 + 3\sum_{k=1}^{p} n_k \equiv 0 \pmod{24},$$

$$(3)\ \sum_{k=1}^{p} k^2 n_k \equiv 4m \pmod{4p}.$$

Proof. The Dedekind eta function is a modular form of weight $\frac{1}{2}$. Thus, $\eta(2pz)^{n_0}$ contributes $\frac{n_0}{2}$ and each of the $f_{2p,k}(z)^{n_k}$ contributes $\frac{n_k}{2}$ to the weight and the weight of $j(p, \overrightarrow{n}, z)$ is $\frac{n_0}{2} + \sum_{k=1}^{p} \frac{n_k}{2}$. Condition (1) implies that this weight is 1.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2p)$. Then, by [10, Theorem 6.14, p.243], we have

$$\eta(2pz) \ \Big|\ [A]_{1/2} = \nu_\eta({}^{2p}A)\, \eta(2pz),$$

where $\nu_\eta({}^{2p}A)$ is the eta-multiplier,

$${}^{2p}A = \begin{pmatrix} a & 2bp \\ c/2p & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then

$$\eta(2pAz) = \nu_\eta({}^{2p}A)\sqrt{cz+d}\, \eta(2pz)$$

and using the Biagioli transformation [10, Theorem 6.12, p.243] for $f_{p,k}(z)$, we have

$$\begin{aligned} f_{2p,k}(z) \ \Big|\ [A]_{1/2} &= (-1)^{kb+\lfloor ka/2p \rfloor + \lfloor k/2p \rfloor} \exp\left(\frac{\pi i a b}{2p} k^2\right) \nu_\eta^3\left({}^{2p}A\right) f_{2p,ka}(z) \\ &= (-1)^{kb+\lfloor ka/2p \rfloor} \exp\left(\frac{\pi i a b}{2p} k^2\right) \nu_\eta^3\left({}^{2p}A\right) f_{2p,k}(z), \end{aligned}$$

assuming $1 \le k \le p$. Therefore

$$\begin{aligned} j(p, \overrightarrow{n}, z) \mid [A]_1 &= (-1)^{L_1(A)} \exp\left(\frac{\pi i a b}{2p} L_2(A)\right) \nu_\eta^{L_3(A)}\left({}^{2p}A\right) j(z) \\ &= \exp\left(\pi i L_1(A) + \frac{\pi i a b}{2p} L_2(A)\right) \nu_\eta^{L_3(A)}\left({}^{2p}A\right) j(z), \end{aligned}$$

where

$$L_1(A) = b\sum_{k=1}^{p} k n_k + \sum_{k=1}^{p} \left\lfloor \frac{ka}{2p} \right\rfloor n_k,$$

$$L_2(A) = \sum_{k=1}^{p} k^2 n_k,$$
$$L_3(A) = n_0 + 3\sum_{k=1}^{p} n_k.$$

Now assume conditions $(1) - (3)$ hold, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2p)$. Since $L_3(A) \equiv 0 \pmod{24}$, the modularity condition holds if we can show that $L_1(A) + \frac{ab}{2p}L_2(A) - \frac{2bm}{p}$ is an even integer.

Since $a \equiv 1 \pmod{2p}$, we have $abL_2(A) \equiv bL_2(A) \pmod{2p}$.

Also, using (3), we have $bL_2(A) \equiv 4bm \pmod{4p}$.

Combining the two congruences, we can conclude that $L_1(A)+\frac{ab}{2p}L_2(A)-\frac{2bm}{p}$ is an integer.

Now
$$L_1(A) + \frac{ab}{2p}L_2(A) - \frac{2bm}{p} = \frac{1}{p}(pL_1(A) + \frac{ab}{2}L_2(A) - 2bm).$$
Also, condition (3) also implies that
$$L_2(A) \equiv 0 \pmod{4}$$
$$\frac{ab}{2}L_2(A) \equiv 0 \pmod{2}.$$
Thus, to show that $L_1(A) + \frac{ab}{2p}L_2(A) - \frac{2bm}{p}$ is an even integer, it is sufficient to show that $L_1(A) \equiv 0 \pmod{2}$.

We have $a \equiv 1 \pmod{2p}$. Let $a = 2pr + 1$.

Then
$$\frac{ka}{2p} = \frac{2kpr}{2p} + \frac{k}{2p} = kr + \frac{k}{2p}.$$
So
$$\left\lfloor \frac{ka}{2p} \right\rfloor = kr.$$
Thus
$$L_1(A) = \sum_{k=1}^{p}(bk + \left\lfloor \frac{ka}{2p} \right\rfloor)n_k$$
$$\equiv (b+r)\sum_{k=1}^{p} kn_k \pmod{2}$$

$$\equiv (b+r)\sum_{k=1}^{p} k^2 n_k \pmod 2$$
$$\equiv 0 \pmod 2,$$

where the last congruence follows because $\sum_{k=1}^{p} k^2 n_k = L_2(A) \equiv 0 \pmod 4$. □

Definition 3.2. Let $\mathfrak{F}(m,p)$ be the set of functions $j(p,\overrightarrow{n},z)$ that satisfy the conditions of Theorem 3.1.

Theorem 3.3. *Let $p>3$ be prime and $0 \le m \le p-1$. Suppose $j(p,\overrightarrow{n},z) \in \mathfrak{F}(m,p)$. Then,*

$$j(p,\overrightarrow{n},z) \mid [A]_1 = (-1)^{L(\overrightarrow{n},a,p)} \exp\left(\frac{2\pi i abm}{p}\right) \eta(2pz)^{n_0} \prod_{k=1}^{p} f_{2p,ka}(z)^{n_k}, \tag{3.1}$$

where

$$L(\overrightarrow{n},a,p) = \sum_{k=1}^{p} \left\lfloor \frac{ka}{2p} \right\rfloor n_k.$$

for

$$A = \begin{pmatrix} a & b \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Also

$$j(p,\pi_r(\overrightarrow{n}),z) \in \mathfrak{F}(m',p), \tag{3.2}$$

where $1 \le r \le p$ and $m' \equiv r^2 m \pmod{2p}$.

Proof. Following the proof of Theorem 3.1, we use the transformation for $\eta(z)$ and $f_{p,k}(z)$ to get

$$j(p,\overrightarrow{n},z) \mid [A]_1 = (-1)^{L_1(A)} \exp\left(\frac{\pi i ab}{2p} L_2(A)\right) \nu_\eta^{L_3(A)}\left({}^{2p}A\right) \eta(2pz)^{n_0} \prod_{k=1}^{p} f_{2p,ka}(z)^{n_k}$$

where

$$L_1(A) = b\sum_{k=1}^{p} k n_k + \sum_{k=1}^{p} \left\lfloor \frac{ka}{2p} \right\rfloor n_k,$$
$$L_2(A) = \sum_{k=1}^{p} k^2 n_k,$$
$$L_3(A) = n_0 + 3\sum_{k=1}^{p} n_k.$$

Since $j(p,\overrightarrow{n},z) \in \mathfrak{F}(m,p)$, we have that $L_3(A) \equiv 0 \pmod{24}$ and we can also deduce from the third modularity condition that

$$\sum_{k=1}^{p} k n_k \pmod 2 \equiv \sum_{k=1}^{p} k^2 n_k \pmod 2 \equiv 0 \pmod 2.$$

It suffices to prove that

$$\frac{ab}{2}L_2(A) - 2abm$$

is an even multiple of p, which is easy to see since $L_2(A) \equiv 4\,m \pmod{4p}$. Equation (3.1) follows.

Now suppose $1 \le r \le p$ so that for $1 \le i \le p$ we have $\pi_r(i) = i'$ where $1 \le i' \le p$ and $ri' \equiv \pm i \pmod{2p}$. We note that

$$\sum_{k=1}^{p} n_k = \sum_{k=1}^{p} n_{\pi_r(k)}, \text{ and}$$
$$\sum_{k=1}^{p} k^2 n_{\pi_r(k)} = \sum_{k=1}^{p} k^2 n_{k'} \equiv r^2 \sum_{k'=1}^{p} (k')^2 n_{k'} \pmod{2p}.$$

Equation (3.2) follows easily. □

4. Transformation, modularity and symmetry of the first residual crank of overpartitions

The two variable generating function for the first residual crank of an overpartition given by Bringmann, Lovejoy and Osburn [4] is

$$\mathcal{O}_{\mathcal{C}}(z,\tau) = \sum_{n=0}^{\infty}\sum_{m} \overline{M}(m,n)\, z^m\, q^n = (-q;q)_\infty \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}.$$

We write this function in terms of the Klein form in Definition 2.2.

$$\begin{aligned}
\mathcal{O}^*_{\mathcal{C}}(\zeta_p^a,\tau) &= \frac{\eta(\tau)^2}{\eta(2\tau)}\,\mathcal{O}_{\mathcal{C}}(\zeta_p^a,\tau)\\
&= \frac{\eta(\tau)^2}{\eta(2\tau)}\,\frac{(-q;q)_\infty}{(q;q)_\infty}\,\frac{(q;q)_\infty^2}{(\zeta_p^a q;q)_\infty(\zeta_p^{-a}q;q)_\infty}\\
&= \frac{(q;q)_\infty^2}{(\zeta_p^a q;q)_\infty(\zeta_p^{-a}q;q)_\infty}\\
&= \left(\exp\left(\frac{\pi i a}{p}\right) - \exp\left(\frac{-\pi i a}{p}\right)\right)\frac{1}{t_{(0,\frac{a}{p})}(\tau)},
\end{aligned}$$

where

$$t_{(0,\frac{a}{p})}(\tau) = -\exp\left(-\frac{\pi i a}{p}\right)(1-\zeta_p^a)\frac{(\zeta_p^a q;q)_\infty(\zeta_p^{-a}q;q)_\infty}{(q;q)_\infty^2}. \tag{4.1}$$

4.1. **Transformation and modularity of the crank and associated functions.**

Theorem 4.1. *Let $p > 3$ be prime. Then $\mathcal{O}^*_{\mathcal{C}}(\zeta_p^a,\tau)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_0(p^2)\cap\Gamma_1(2p)$.*

Proof. Let
$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(2p).$$
Using the transformations in Proposition 2.3, we have
$$\begin{aligned} t_{(0,\frac{a}{p})}(\tau)|[A]_1 &= t_{(\frac{a\gamma}{p},\frac{a\delta}{p})}(\tau) \\ &= t_{(0+\frac{a\gamma}{p},\frac{a\delta}{p}+0)}(\tau) \\ &= (-1)^{\frac{a\gamma}{p}} \exp\left(-\pi i\left(\frac{a^2\gamma\delta}{p^2} - 0\right)\right) t_{(0,\frac{a\delta}{p})}(\tau). \end{aligned}$$
Then
$$\begin{aligned} \mathcal{O}^*_{\mathcal{C}}(\zeta_p^a,\tau)|[A]_1 &= \frac{\exp\left(\frac{\pi i a}{p}\right) - \exp\left(\frac{-\pi i a}{p}\right)}{\exp\left(\frac{\pi i a\delta}{p}\right) - \exp\left(\frac{-\pi i a\delta}{p}\right)}(-1)^{a\gamma}\exp\left(\frac{\pi i a^2\gamma\delta}{p^2}\right)\mathcal{O}^*_{\mathcal{C}}(\zeta_p^{a\delta},\tau) \\ &= \frac{\sin\left(\frac{a\pi}{p}\right)}{\sin\left(\frac{a\delta\pi}{p}\right)}(-1)^{a\gamma}\exp\left(\frac{\pi i a^2\gamma\delta}{p^2}\right)\mathcal{O}^*_{\mathcal{C}}(\zeta_p^{a\delta},\tau) \\ &= (-1)^{a\gamma}\exp\left(\frac{\pi i a^2\gamma\delta}{p^2}\right)\mathcal{O}^*_{\mathcal{C}}(\zeta_p^{a\delta},\tau), \end{aligned}$$
since
$$\sin\left(\frac{a\pi}{p}\right) = \sin\left(\frac{a\delta\pi}{p}\right) \text{ when } \delta \equiv 1 \pmod{2p}.$$
□

Theorem 4.2. *Let*
$$X_p(\tau) = \frac{\eta(p^2\tau)^2\eta(2\tau)}{\eta(2p^2\tau)\eta(\tau)^2}.$$
Then $X_p(\tau)$ is a modular function on $\Gamma_0(2p^2)$.

Proof. $f(z) = \prod_{0<\delta|N} \eta(q^\delta)^{r_\delta}$ is a weak modular form of weight $k = \frac{1}{2}\sum_{0<\delta|N} r_\delta$ on $\Gamma_0(N)$ if it satisfies the following :
$$\sum_{0<\delta|N} \delta r_\delta \equiv 0 \pmod{24},$$
$$\sum_{0<\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$
Let $\delta|2p^2$. Then for $f(z) = X_p(\tau)$, we have
$$\sum_{0<\delta|2p^2} \delta r_\delta = -2 + 2 + 2p^2 - 2p^2 = 0,$$

$$\sum_{0<\delta|N} \frac{2p^2}{\delta} r_\delta = -4p^2 + p^2 + 4 - 1 = 3\,(1-p^2) \equiv 0 \pmod{24}.$$

Thus, $X_p(\tau)$ is a modular function on $\Gamma_0(2p^2)$. □

The modularity of the overpartition crank function stated in the introduction section as Theorem 1.1 now deduces easily as a corollary. We restate it here.

Corollary 4.3. *Let $p > 3$ be prime. Then $X_p(\tau)\,\mathcal{O}^*_{\mathcal{C}}(\zeta_p^a, \tau)$ is a modular form of weight 1 on $\Gamma_0(p^2) \cap \Gamma_1(2p)$.*

Next, we consider the elements of the p-dissection of $X_p(\tau)\,\mathcal{O}^*_{\mathcal{C}}(\zeta_p^a, \tau)$ defined in Definition 1.2. Using the definition of the Atkin operator Equation (2.7), we can equivalently write them as

$$\mathcal{OK}^{(C)}_{p,m}(\zeta_p^\ell, z) = X_p(\tau)\,\mathcal{O}^*_{\mathcal{C}}(\zeta_p^\ell, \tau) \;\Big|\; [U_{p,m}]_1\,. \tag{4.2}$$

The following theorem accounts for the modularity of $\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z)$ under special congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 4.4. *Let $p > 3$ be prime and $0 \le m \le p-1$. Then*

(i) $\mathcal{OK}^{(C)}_{p,0}(\zeta_p, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_1(2p)$.

(ii) If $1 \le m \le (p-1)$ then $\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(2p)$. In particular,

$$\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z) \mid [A]_1 = \exp\left(\frac{2\pi i b m}{p}\right) \mathcal{OK}^{(C)}_{p,m}(\zeta_p, z),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2p)$.

Proof. We let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2p)$$

so that $a \equiv d \equiv 1 \pmod{2p}$ and $c \equiv 0 \pmod{2p}$. Let $0 \le k \le p-1$. We take $k' \equiv b + k \pmod{p}$ so that

$$T_k\,A = B_k\,T_{k'},$$

and

$$B_k = T_k\,A\,T_{k'}^{-1} = \begin{pmatrix} a + ck & \frac{1}{p}(-k'(a+kc) + b + kd) \\ pc & d - k'c \end{pmatrix} \in \Gamma_0(p^2) \cap \Gamma_1(2p).$$

Thus,

$$\begin{aligned}
&\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z) \mid [A]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \exp\left(-\frac{2\pi i k m}{p}\right) X_p(\tau)\,\mathcal{O}^*_{\mathcal{C}}(\zeta_p, \tau) \mid [(T_k\,A)]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \exp\left(-\frac{2\pi i (k'-b) m}{p}\right) X_p(\tau)\,\mathcal{O}^*_{\mathcal{C}}(\zeta_p, \tau) \mid [(B_k\,T_{k'})]_1
\end{aligned}$$

$$= \frac{1}{\sqrt{p}} \exp\left(\frac{2\pi i b m}{p}\right) \sum_{k'=0}^{p-1} \exp\left(-\frac{2\pi i k' m}{p}\right) X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p, \tau) \mid [T_{k'}]_1$$

(by Corollary 4.3 since $B_k \in \Gamma_0(p^2) \cap \Gamma_1(2p)$)

$$= \exp\left(\frac{2\pi i b m}{p}\right) \mathcal{OK}^{(C)}_{p,m}(\zeta_p, z),$$

as required. Thus each function $\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z)$ has the desired transformation property. It is clear that each $\mathcal{OK}^{(C)}_{p,m}(\zeta_p, z)$ is holomorphic on $\mathcal{H}$. The cusp conditions follow by a standard argument. We examine orders at each cusp in more detail in a later section. □

4.2. **Overpartition crank symmetry.** We finally present a proof of our result on the observation of symmetry among the elements of the p-dissection of the overpartition crank function, which was stated in the introduction section in Theorem 1.3. We restate it here.

Theorem 4.5. *Let $p > 3$ be prime and $0 \leq m \leq p-1$. Then*

$$\mathcal{OK}^{(C)}_{p,m}(\zeta_p^\ell, z) \mid [A]_1 = \frac{\sin(\ell\pi/p)}{\sin(\ell d\pi/p)}\, \zeta_p^{mak}\, \mathcal{OK}^{(C)}_{p,ma^2}(\zeta_p^{\ell d}, z) \tag{4.3}$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. We undergo the same matrix transformations as we did in the case of $\mathcal{K}_{p,m}(\zeta_p, z)$ ([11], Proposition 4.7), and for $0 \leq r \leq p-1$ let $T_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}$, and

$$B_r = \begin{pmatrix} a + 2pr & \frac{1}{p}(k + rd - r'(a+2pr)) \\ 2p^2 & d - 2r'p \end{pmatrix},$$

where $0 \leq r' \leq p-1$ is chosen so that $r' \equiv rd^2 + dk \pmod{p}$. Then

$$T_r\, A = B_r\, T_{r'}, \quad r \equiv r'a^2 - ak \pmod{p}, \quad \text{and} \quad B_r \in \Gamma_0(2p^2).$$

$$\begin{aligned}
\mathcal{OK}^{(C)}_{p,m}(\zeta_p^\ell, z) \mid [A]_1 &= X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^\ell, \tau) \;\Big|\; [U_{p,m}]_1 \mid [A]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \zeta_p^{-rm}\, X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^\ell, \tau) \mid [T_r]_1 \mid [A]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \zeta_p^{-rm}\, X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^\ell, \tau) \mid [B_r]_1 \mid [T_{r'}]_1 \\
&= \frac{1}{\sqrt{p}} \frac{\sin\left(\frac{\ell\pi}{p}\right)}{\sin\left(\frac{\ell\,(d-2r'p)\,\pi}{p}\right)} (-1)^{\ell\,(2p^2)} \exp\left(\frac{\pi i \ell^2\, 2p^2\,(d-2r'p)}{p^2}\right)
\end{aligned}$$

$$\sum_{r=0}^{p-1} \zeta_p^{-rm}\, X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^{\ell(d-2r'p)},\tau) \mid [T_{r'}]_1$$

(using the second step of the transformation of $\mathcal{O}^*_{\mathcal{C}}(\zeta_p^a,\tau)|[A]_1$ in the proof of Theorem 4.1)

$$= \frac{1}{\sqrt{p}} \frac{\sin\left(\frac{\ell\pi}{p}\right)}{\sin\left(\frac{\ell\, d\,\pi}{p}\right)} \sum_{r=0}^{p-1} \zeta_p^{-rm}\, X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^{\ell d},\tau) \mid [T_{r'}]_1$$

$$= \frac{\sin\left(\frac{\ell\pi}{p}\right)}{\sin\left(\frac{\ell\, d\,\pi}{p}\right)} \zeta_p^{mak} \frac{1}{\sqrt{p}} \sum_{r'=0}^{p-1} \zeta_p^{-r'ma^2} X_p(\tau)\, \mathcal{O}^*_{\mathcal{C}}(\zeta_p^{\ell d},\tau) \mid [T_{r'}]_1\ ,$$

since

$$\zeta_p^{-rm} = \zeta_p^{m(-r'a^2+ak)} = \zeta_p^{mak}\zeta_p^{-mr'a^2},$$

and as r runs through a complete residue system mod p so does r'. The result follows. □

5. Transformation, modularity and symmetry of Dyson's rank of overpartitions

Similar to the transformation, modularity and symmetry of the rank generating function for partitions, in this section, we establish analogous results for the overpartition rank generating function.

5.1. **Transformation and modularity of the rank and associated functions.**

Proposition 5.1. *Let $p > 3$ be prime and $1 \le \ell \le (p-1)$. Define*

$$N^*(\ell,p;\tau) := \frac{\eta(\tau)^2}{\eta(2\tau)} N(\ell,p;\tau),$$

$$P^*(\ell,p;\tau) := \frac{\eta(\tau)^2}{\eta(2\tau)} P(\ell,p;\tau).$$

Then

$$N^*(\ell,p;\tau) \mid [A]_1 = \mu(A,\ell)\, N^*(d\ell,p;\tau),$$

$$P^*(\ell,p;\tau) \mid [A]_1 = \mu(A,\ell)\, P^*(d\ell,p;\tau),$$

where

$$\mu(A,\ell) = \exp\left(-\pi i \left(\frac{2\ell(1-d)}{p} - \frac{2cd\ell^2}{p^2}\right)\right)$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. Knopp's formula for the eta multiplier gives

$$\frac{\eta(\tau)^2}{\eta(2\tau)}|[A]_1 = \frac{\nu_\eta(A)^2}{\nu_\eta(^2A)}\frac{\eta(\tau)^2}{\eta(2\tau)} = \nu_\eta(^2A)^3(-1)^{-(b+\frac{a-1}{2})}i^{ab}.$$

Using [13, Corollary 3.2] we get

$$N(\ell,p;\tau)|[A]_{\frac{1}{2}} = \nu_\eta(^2A)^{-3}(-1)^{b+\frac{a-1}{2}+\frac{ac}{p}}\exp\left(-\pi i\left(\frac{-2\ell^2cd}{p^2}+\frac{2\ell(1-d)}{p}+\frac{ab}{2}\right)\right)N(d\ell,p;\tau).$$

Therefore,

$$N^*(\ell,p;\tau)|[A]_{\frac{1}{2}} = \exp\left(-\pi i\left(\frac{2\ell(1-d)}{p}-\frac{2cd\ell^2}{p^2}\right)\right)N^*(d\ell,p;\tau).$$

And, writing $P(\ell,p,\tau)$ in terms of the Klein form (Equation (2.5)), we get

$$P(\ell,p,\tau) = -\zeta_p^{-\ell}\frac{\eta(2\tau)}{\eta(\tau)^2}\frac{1}{t_{(0,\frac{2\ell}{p})}(2\tau)}.$$

Therefore, using the transformations in Proposition 2.3, we get

$$\begin{aligned}
P^*(\ell,p;\tau)|[A]_1 &= -\zeta_p^{\ell}\frac{1}{t_{(0,\frac{2\ell}{p})}(2\tau)}|[A]_1\\
&= -\zeta_p^{\ell}\frac{1}{(-1)^{\frac{\ell c}{p}}\exp\left(-\pi i\left(\frac{2\ell^2c(d-1)}{p^2}\right)\right)t_{(\frac{\ell c}{p},\frac{2\ell d}{p})}(2\tau)}\\
&= -\zeta_p^{\ell}(-1)^{\frac{\ell c}{p}}\exp\left(\pi i\left(\frac{2\ell^2cd}{p^2}\right)\right)\frac{1}{t_{(0,\frac{2\ell d}{p})}(2\tau)}\\
&= \exp\left(-\pi i\left(\frac{2\ell(1-d)}{p}-\frac{2cd\ell^2}{p^2}\right)\right)P^*(d\ell,p;\tau).
\end{aligned}$$

□

The following corollary now follows easily.

Corollary 5.2. *Let $p>3$ be prime and $1\le\ell\le(p-1)$. Then*

$$N^*(\ell,p;\tau)\mid[A]_1 = N^*(\ell,p;\tau),$$
$$P^*(\ell,p;\tau)\mid[A]_1 = P^*(\ell,p;\tau),$$

for

$$A\in\Gamma_0(2p^2)\cap\Gamma_1(p).$$

Now, using Theorem 1.5 and Proposition 3.5 [13], we have

$$\begin{aligned}
N(\ell,p;\tau) &= \frac{-\left(1+\zeta_p^\ell\right)}{2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}\mathcal{O}_{\mathcal{R}}\left(\zeta_p^\ell;\tau\right)+\frac{1}{i}P(\ell,p;\tau)+\frac{\zeta_p^{-\ell}}{\sqrt{2}}\int_{-\bar\tau}^{i\infty}\frac{g_{0,\frac{1}{2}-\frac{2\ell}{p}}(2z)}{\sqrt{-i(z+\tau)}}dz\\
&= \frac{-\left(1+\zeta_p^\ell\right)}{2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}\mathcal{O}_{\mathcal{R}}\left(\zeta_p^\ell;\tau\right)+\frac{1}{i}P(\ell,p;\tau)+\frac{i\zeta_p^{-\ell}}{2\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n\left(\zeta_p^{-2\ell n}-\zeta_p^{2\ell n}\right)\Gamma\left(\frac{1}{2};4\pi yn^2\right)q^{-n^2}
\end{aligned}$$

$$= \frac{-\left(1+\zeta_p^\ell\right)}{2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}\,\mathcal{O}_{\mathcal{R}}\left(\zeta_p^\ell;\tau\right) + \frac{1}{i}P(\ell,p;\tau) + \frac{i\zeta_p^{-\ell}}{2\sqrt{\pi}}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)\cdot$$
$$\left(\sum_{n=0}^{\infty}(-1)^n q^{-(pn+k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn+k)^2\right) - \sum_{n=1}^{\infty}(-1)^n q^{-(pn-k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn-k)^2\right)\right).$$

In Section 2 [13], the author defines

$$N_7(k;\tau) = q^{-k^2+7k-\frac{49}{4}}\tilde{\mu}(14k\tau,49\tau;98\tau).$$

For an arbitrary prime $p>3$, we define the generalized function

$$N_p(k;\tau) = q^{-(k-p/2)^2}\tilde{\mu}(2pk\tau,p^2\tau;2p^2\tau).$$

Then

$$\begin{aligned} N_p(k;\tau) &= q^{-2p^2\frac{1}{2}(\frac{k}{p}-\frac{1}{2})^2}\tilde{\mu}(2pk\tau,p^2\tau;2p^2\tau) \\ &= M(k,p;2p^2\tau), \end{aligned} \tag{5.1}$$

where $M(a,c;\tau) = q^{-\frac{1}{2}(\frac{a}{c}-\frac{1}{2})^2}\tilde{\mu}(\frac{a\tau}{c},\frac{\tau}{2};\tau)$ is defined by Jennings-Shaffer in Section 5 [13]. This helps in determining the transformation of $N_p(k;\tau)$ under matrices in $SL_2(\mathbb{Z})$ using Proposition 5.1 [13].

Also, using Proposition 1.3 and 1.4 [20], we have

$$\mu(2pk\tau,p^2\tau;2p^2\tau) = iq^{pk+\frac{p^2}{4}}\frac{(q^{2p^2};q^{2p^2})_\infty}{(q^{p^2};q^{p^2})_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{p^2n(n+2)}}{1-q^{2p(pn+k)}}.$$

Further, we define

$$N_p^*(k;\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}N_p(k;\tau).$$

We deduce the transformation of this function under $\Gamma_0(2p^2)\cap\Gamma_1(p)$ like the previous two functions in Corollary 5.2.

Proposition 5.3. *Let $p>3$ be prime. Then*

$$N_p^*(\ell;\tau)\mid[A]_1 = N_p^*(a\ell;\tau)$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2p^2).$$

Proof. We have

$$N_p^*(k;\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}N_p(k;\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}M(k,p;2p^2\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)}q^{-(k-\frac{p}{2})^2}\tilde{\mu}(2pk\tau,p^2\tau;2p^2\tau).$$

Knopp's formula for the eta multiplier gives

$$\frac{\eta(\tau)^2}{\eta(2\tau)}|[A]_{\frac{1}{2}} = \frac{\nu_\eta(A)^2}{\nu_\eta(^2A)}\frac{\eta(\tau)^2}{\eta(2\tau)} = \nu_\eta(^2A)^3(-1)^{-(b+\frac{a-1}{2})}i^{ab}.$$

Using the transformation for $\mu(u,v;\tau)$ [20, Theorem 1.1 (1)] due to Zwegers, we get

$$\begin{aligned}
N_p(k;\tau)|[A]_{\frac{1}{2}} &= M(k,p;2p^2\tau)|[A]_{\frac{1}{2}} \\
&= M\left(k,p;2p^2\frac{a\tau+b}{c\tau+d}\right) \\
&= M\left(k,p;\widetilde{A}(2p^2\tau)\right) \text{ where } \widetilde{A}=\begin{pmatrix} a & 2p^2b \\ c/2p^2 & d\end{pmatrix} \\
&= \nu_\eta(\widetilde{A})^{-3}\exp\left(-\pi i a.2p^2 b\left(\frac{k}{p}-\frac{1}{2}\right)^2 - \pi i a^2.2p^2\tau\left(\frac{k}{p}-\frac{1}{2}\right)^2\right) \\
&\quad \tilde{\mu}\left(\frac{ka.2p^2\tau+k.2p^2b}{p},\frac{a.2p^2\tau+2p^2b}{2};2p^2\tau\right) \\
&= \nu_\eta(\widetilde{A})^{-3}\exp\left(-2\pi iabk^2-\pi i\frac{p^2ab}{2}+2\pi ipabk-\pi ia^2.2p^2\tau\left(\frac{k}{p}-\frac{1}{2}\right)^2\right) \\
&\quad \tilde{\mu}\left(2kap\tau+2apb,ap^2\tau+p^2b;2p^2\tau\right) \\
&= \nu_\eta(\widetilde{A})^{-3}(-i)^{p^2ab}\exp\left(\pi ia^2.2p^2\tau\left(\frac{k}{p}-\frac{1}{2}\right)^2\right)(-1)^{p^2b}\,\tilde{\mu}\left(2kap\tau,ap^2\tau;2p^2\tau\right) \\
&= \nu_\eta(\widetilde{A})^{-3}(-i)^{p^2ab}\exp\left(-\pi ia^2.2p^2\tau\left(\frac{k}{p}-\frac{1}{2}\right)^2\right)(-1)^{p^2b} \\
&\quad \tilde{\mu}\left(2kap\tau,p^2\tau+\frac{a-1}{2}.2p^2\tau;2p^2\tau\right) \\
&= \nu_\eta(\widetilde{A})^{-3}(-i)^{p^2ab}\exp\left(-\pi ia^2.2p^2\tau\left(\frac{k}{p}-\frac{1}{2}\right)^2\right)(-1)^{p^2b} \\
&\quad (-1)^{\frac{a-1}{2}}\exp\left(\pi i\left(\frac{a-1}{2}\right)^2.2p^2\tau-2\pi i\left(\frac{a-1}{2}\right)(2kap\tau-p^2\tau)\right) \\
&\quad \tilde{\mu}\left(2kap\tau,p^2\tau;2p^2\tau\right) \\
&= \nu_\eta(\widetilde{A})^{-3}(-1)^{b+\frac{a-1}{2}+ab}\,i^{p^2ab}\,q^{-p^2\left(\frac{ak}{p}-\frac{1}{2}\right)^2}\,\tilde{\mu}\left(2pka\tau,p^2\tau;2p^2\tau\right).
\end{aligned}$$

Combining the two transformation results, we have the statement of the proposition. □

Corollary 5.4. *Let $p>3$ be prime. Then*

$$N_p^*(\ell;\tau)\mid[A]_1 = N_p^*(\ell;\tau).$$

for $A\in\Gamma_0(2p^2)\cap\Gamma_1(p)$.

Next, following the proof of Proposition 5.4 [13], we can deduce that

$$N_p(k;\tau)=q^{-(k-p/2)^2}\mu(2pk\tau,p^2\tau;2p^2\tau)+\frac{i}{2\sqrt{\pi}}.$$

$$\left(\sum_{n=0}^{\infty}(-1)^n q^{-(pn+k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn+k)^2\right)-\sum_{n=1}^{\infty}(-1)^n q^{-(pn-k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn-k)^2\right)\right)$$

$$=iq^{2pk-k^2}\frac{(q^{2p^2};q^{2p^2})_\infty}{(q^{p^2};q^{p^2})_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{p^2n(n+2)}}{1-q^{2p(pn+k)}}+$$

$$\frac{i}{2\sqrt{\pi}}\left(\sum_{n=0}^{\infty}(-1)^n q^{-(pn+k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn+k)^2\right)-\sum_{n=1}^{\infty}(-1)^n q^{-(pn-k)^2}\Gamma\left(\frac{1}{2};4\pi y(pn-k)^2\right)\right).$$

Subtracting the expressions for $N(\ell,p;\tau)$ and $N_p(k;\tau)$ with k summed over 1 to $\frac{p-1}{2}$, the non-holomorphic parts cancel out and we get

$$N(\ell,p;\tau)-\zeta_p^{-\ell}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)N_p(k;\tau)=\frac{-\left(1+\zeta_p^\ell\right)}{2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}\mathcal{O}_{\mathcal{R}}\left(\zeta_p^\ell;\tau\right)+\frac{1}{i}P(\ell,p;\tau)+$$

$$i\,\zeta_p^{-\ell}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)q^{pk-k^2}\frac{(q^{2p^2};q^{2p^2})_\infty}{(q^{p^2};q^{p^2})_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{p^2n(n+1)}}{1-q^{p^2n+k}},$$

which is weakly holomorphic. Rearranging the terms we get

$$\mathcal{O}_{\mathcal{R}}\left(\zeta_p^\ell;\tau\right)-\frac{2\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)q^{pk-k^2}\frac{(q^{2p^2};q^{2p^2})_\infty}{(q^{p^2};q^{p^2})_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{p^2n(n+1)}}{1-q^{p^2n+k}}$$

$$=\frac{-2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}N(\ell,p;\tau)+\frac{2\,i\,\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)N_p(k;\tau)+\frac{2\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}P(\ell,p;\tau).$$

Let

$$\mathcal{J}\left(\frac{\ell}{p};\tau\right):=\frac{-2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}N(\ell,p;\tau)+\frac{2\,i\,\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)N_p(k;\tau)$$

$$+\frac{2\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}P(\ell,p;\tau). \tag{5.2}$$

Definition 5.5. For $p>3$ prime and $1\le\ell\le p-1$ define

$$\mathcal{J}^*\left(\frac{\ell}{p};\tau\right):=X_p(\tau)\mathfrak{O}_R^*(\zeta_p^\ell,\tau) \tag{5.3}$$

where recall $X_p(\tau)=\dfrac{\eta(p^2\tau)^2\eta(2\tau)}{\eta(2p^2\tau)\eta(\tau)^2}$, and

$$\mathfrak{O}_R^*(\zeta_p^\ell,\tau)=\frac{-2\,i\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}N^*(\ell,p;\tau)+\frac{2\,i\,\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\left(\zeta_p^{-2\ell k}-\zeta_p^{2\ell k}\right)N_p^*(k;\tau)$$

(5.4)
$$+\frac{2\,\zeta_p^\ell\left(1-\zeta_p^\ell\right)}{\left(1+\zeta_p^\ell\right)}P^*(\ell,p;\tau).$$

Now, with the correction factor $\Phi_{p,k}(q)$ and the corrected overpartition rank function $\mathfrak{O}_R(\zeta_p,\tau)$ defined as in Definitions (1.5) and (1.6) respectively, we can easily see that

$$\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}\,\mathfrak{O}_R(\zeta_p^\ell,\tau)=\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}\,\mathcal{J}\left(\frac{\ell}{p};\tau\right)=X_p(\tau)\mathfrak{O}^*_R(\zeta_p^\ell,\tau)=\mathcal{J}^*\left(\frac{\ell}{p};\tau\right). \tag{5.5}$$

Thus we rewrite one of our main results, Theorem 1.8, in the equivalent form :

Theorem 5.6. *Let $p>3$ be prime. Then the function $\mathcal{J}^*\left(\frac{1}{p};\tau\right)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_0(2p^2)\cap\Gamma_1(p)$.*

Proof. This follows easily from combining Theorem 4.2, Corollary 5.2 and Corollary 5.4. □

Again, the functions $\mathcal{OK}^{(R)}_{p,m}(\zeta_p^\ell;\tau)$ as in Definition 1.9 can be equivalently written as

Proposition 5.7. *Let $p>3$ be prime and $0\le m\le p-1$. Then*

$$\mathcal{OK}^{(R)}_{p,m}(\zeta_p^\ell;\tau)=\mathcal{J}^*\left(\frac{\ell}{p};\tau\right)\ \mid[U_{p,m}]_1\,. \tag{5.6}$$

Next, the following theorem accounts for the modularity of $\mathcal{OK}^{(R)}_{p,m}(\zeta_p^\ell,z)$ under special congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 5.8. *Let $p>3$ be prime and $0\le m\le p-1$. Then*

(i) $\mathcal{OK}^{(R)}_{p,0}(z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_1(2p)$.
(ii) If $1\le m\le(p-1)$ then $\mathcal{OK}^{(R)}_{p,m}(z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma(2p)$. In particular,

$$\mathcal{OK}^{(R)}_{p,m}(z)\mid[A]_1=\exp\left(\frac{2\pi ibm}{p}\right)\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z)$$

for $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1(2p)$.

Proof. We consider the matrix in $\Gamma_1(2p)$ and go through the same matrix transformations as in the proof of Theorem 4.4 in the previous section to arrive at our result using Theorem 5.6 and Proposition 5.7. □

5.2. **Overpartition rank symmetry result.** We now investigate how the Atkin operator and a matrix in $\Gamma_0(2p)$ acts on each function of (5.3) to arrive at our symmetry result for the overpartition rank.

Proposition 5.9. *Let $p>3$ be prime. Then*

$$X_p(\tau)\,N^*(1,p;\tau)\ \mid[U_{p,m}]_1\ \mid[A]_1=\zeta_p^{mak+d-1}X_p(\tau)\,N^*(d,p;\tau)\ \Big|\ \Big[U_{p,\overline{ma^2}}\Big]_1$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. For $0 \le r \le p-1$ let $T_r = \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix}$, and $B_r = \begin{pmatrix} a+2pr & \frac{1}{p}(k+rd-r'(a+2pr)) \\ 2p^2 & d-2r'p \end{pmatrix}$, where $0 \le r' \le p-1$ is chosen so that $r' \equiv rd^2 + dk \pmod p$. Then

$$T_r\, A = B_r\, T_{r'}, \quad r \equiv r'a^2 - ak \pmod p, \quad \text{and} \quad B_r \in \Gamma_0(2p^2).$$

We apply Proposition 5.1 and Theorem 4.2. We have

$$\begin{aligned}
X_p(\tau)\, N^*(1,p;\tau) \ \big|\ [U_{p,m}]_1 \ |\, [A]_1 &= \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \zeta_p^{-rm} X_p(\tau)\, N^*(1,p;\tau) \mid [T_r]_1 \ |\, [A]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \zeta_p^{-rm} X_p(\tau)\, N^*(1,p;\tau) \mid [B_r]_1 \ |\, [T_{r'}]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \zeta_p^{-rm} X_p(\tau)\, \mu(B_r,1)\, N^*(d,p;\tau) \mid [T_{r'}]_1 \\
&= \frac{1}{\sqrt{p}} \zeta_p^{mak+d-1} \sum_{r'=0}^{p-1} \zeta_p^{-r'ma^2} X_p(\tau)\, N^*(d,p;\tau) \mid [T_{r'}]_1\,,
\end{aligned}$$

$$\begin{aligned}
\text{since} \quad &\mu(B_r,1) = \exp\left(-\pi i \left(\frac{2\,(1-d+2r'p)}{p} - \frac{4p^2(d-2r'p)}{p^2}\right)\right), \\
&\zeta_p^{-rm} = \zeta_p^{m(-r'a^2+ak)} = \zeta_p^{mak}\zeta_p^{-mr'a^2},
\end{aligned}$$

and as r runs through a complete residue system mod p so does r'. The result follows. □

Proposition 5.10. *Let $p > 3$ be prime. Then*

$$X_p(\tau)\, P^*(1,p;\tau) \ \big|\ [U_{p,m}]_1 \ |\, [A]_1 = \zeta_p^{mak+d-1} X_p(\tau)\, P^*(d,p;\tau) \ \Big|\ \left[U_{p,\overline{ma^2}}\right]_1$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. The proof follows similarly using Proposition 5.1. □

Proposition 5.11. *Let $p > 3$ be prime. Then*

$$X_p(\tau)\, N_p^*(\ell;\tau) \ \big|\ [U_{p,m}]_1 \ |\, [A]_1 = \zeta_p^{mak} X_p(\tau)\, N_p^*(a\ell;\tau) \ \Big|\ \left[U_{p,\overline{ma^2}}\right]_1$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. The proof follows using Proposition 5.3 and by going through the same series of matrix transformations and steps as it did for the function $N^*(1,p;\tau)$ in the proof of Proposition 5.9, except for the appearance of the root of unity multiplier $\mu(B_r,1)$ in that proof which does not appear here due to the exact transformation of $N_p^*(\ell;\tau)$ as seen in Proposition 5.3. $\square$

Proposition 5.12. *Let $p>3$ be prime, $1\le \ell,\ell'\le p-1$, $\ell^2\equiv -m \pmod p$, $\ell'^2\equiv -ma^2 \pmod p$. Then*

$$\begin{aligned}&(-1)^{\ell}\left(\zeta_p^{-2\ell}-\zeta_p^{2\ell}\right)X_p(\tau)\,N_p^*(\ell;\tau)\ \Big|\ [U_{p,m}]_1\ \big|\,[A]_1\\ &=(-1)^{\ell'}\left(\zeta_p^{-2\ell'd}-\zeta_p^{2\ell'd}\right)\zeta_p^{mak}X_p(\tau)\,N_p^*(\ell';\tau)\ \Big|\ \left[U_{p,\overline{ma^2}}\right]_1\end{aligned}$$

for

$$A=\begin{pmatrix}a & k\\ 2p & d\end{pmatrix}\in\Gamma_0(2p).$$

Proof. The proof follows from the previous proposition and the fact that $\ell'\equiv\pm a\ell \pmod p$, a is odd and $ad-2pk=1$. $\square$

Proposition 5.13. *Let $1\le k\le \frac{1}{2}(p-1)$. Then*

$$X_p(\tau)\,N_p^*(k,\tau)\ \Big|\ [U_{p,m}]_1=\begin{cases}\dfrac{\eta(p\tau)^2}{\eta(2p\tau)}\,N_p(k,\frac{\tau}{p}) & \textit{if } k^2\equiv -m \pmod p\\ 0 & \textit{otherwise}.\end{cases}$$

Proof.

$$\begin{aligned}X_p(\tau)\,N_p^*(k,\tau)\ \Big|\ [U_{p,m}]_1&=\frac{1}{p}\sum_{r=0}^{p-1}\zeta_p^{-rm}X_p\left(\frac{\tau+r}{p}\right)N_p^*\left(k,\frac{\tau+r}{p}\right)\\ &=\frac{1}{p}\sum_{r=0}^{p-1}\zeta_p^{-rm}\frac{\eta(p\tau+pr)^2}{\eta(2p\tau+2pr)}\exp\left(-2\pi i\,\frac{\tau+r}{p}\,\left(k-\frac{p}{2}\right)^2\right)\\ &\quad\tilde{\mu}\left(2pk\,\frac{\tau+r}{p},p^2\,\frac{\tau+r}{p};2p^2\,\frac{\tau+r}{p}\right)\\ &=\frac{1}{p}\sum_{r=0}^{p-1}\zeta_p^{-rm}\frac{\eta(p\tau)^2}{\eta(2p\tau)}\exp\left(-2\pi i\,\frac{\tau+r}{p}\,\left(k-\frac{p}{2}\right)^2\right)\\ &\quad\exp\left(\frac{-\pi i}{4}.2pr+\pi ipr+\pi i\right)\tilde{\mu}(2k\tau,p\tau;2p\tau)\\ &=\frac{1}{p}\sum_{r=0}^{p-1}\zeta_p^{-rm-rk^2+2rkp}q^{-\frac{(k-\frac{p}{2})^2}{p}}\frac{\eta(p\tau)^2}{\eta(2p\tau)}\tilde{\mu}(2k\tau,p\tau;2p\tau)\\ &=\frac{1}{p}\sum_{r=0}^{p-1}\zeta_p^{-r(m+k^2)}\frac{\eta(p\tau)^2}{\eta(2p\tau)}N_p\left(k,\frac{\tau}{p}\right).\end{aligned}$$

The result follows since $-r(m+k^2)\equiv 0 \pmod p$ if and only if $k^2\equiv -m \pmod p$. $\square$

We finally present a proof of our result on the observation of symmetry among the elements of the p-dissection of the overpartition rank function, which was stated in the introduction section Theorem 1.10. We restate it here.

Theorem 5.14. *Let* $p > 3$ *be prime and* $0 \le m \le p-1$. *Then*

$$\mathcal{OK}^{(R)}_{p,m}(\zeta_p,\tau) \mid [A]_1 = \frac{1-\zeta_p}{1-\zeta_p^d}\,\frac{1+\zeta_p^d}{1+\zeta_p}\,\zeta_p^{mak}\,\mathcal{OK}^{(R)}_{p,ma^2}(\zeta_p^d,\tau) \tag{5.7}$$

for

$$A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p).$$

Proof. Proposition 5.7 gives

$$\mathcal{OK}^{(R)}_{p,m}(\zeta_p;\tau) = \mathcal{J}\left(\frac{1}{p};\tau\right) \ \mid [U_{p,m}]_1 \,.$$

We consider the following two cases.

CASE 1. $m = 0$ or $\left(\frac{-m}{p}\right) = -1$. In this case

$$\ell^2 \not\equiv -m \pmod p, \quad \text{and} \quad \ell'^2 \not\equiv -ma^2 \pmod p,$$

for $1 \le \ell, \ell' \le \frac{1}{2}(p-1)$. The result then follows from Proposition 5.9, 5.10 and 5.13.

CASE 2. $\left(\frac{-m}{p}\right) = 1$. In this case choose $1 \le \ell, \ell' \le \frac{1}{2}(p-1)$ such that

$$\ell^2 \equiv -m \pmod p, \quad \text{and} \quad \ell'^2 \equiv -ma^2 \pmod p.$$

We have

$$\begin{aligned}
&\mathcal{J}^*\left(\frac{1}{p};\tau\right) \ \mid [U_{p,m}]_1 \ \mid [A]_1 \\
&= X_p(\tau)\Bigg(\frac{-2\,i\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} N^*(1,p;\tau) + \frac{2\,i\,(1-\zeta_p)}{(1+\zeta_p)} \sum_{\ell=1}^{\frac{p-1}{2}} (-1)^\ell \left(\zeta_p^{-2\ell} - \zeta_p^{2\ell}\right) N_p^*(\ell;\tau) \\
&\qquad + \frac{2\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} P^*(1,p;\tau)\Bigg) \ \mid [U_{p,m}]_1 \ \mid [A]_1 \\
&= \left(\frac{-2\,i\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, N^*(d,p;\tau) + \frac{2\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, P^*(d,p;\tau)\right) \ \Big| \ \left[U_{p,\overline{ma^2}}\right]_1 \\
&\qquad + \frac{2\,i\,(1-\zeta_p)}{(1+\zeta_p)} (-1)^\ell \left(\zeta_p^{-2\ell} - \zeta_p^{2\ell}\right) N_p^*(\ell;\tau) \ \mid [U_{p,m}]_1 \ \mid [A]_1 \text{ (by Propositions 5.9, 5.10 and 5.13)} \\
&= \left(\frac{-2\,i\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, N^*(d,p;\tau) + \frac{2\,\zeta_p\,(1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, P^*(d,p;\tau)\right) \ \Big| \ \left[U_{p,\overline{ma^2}}\right]_1 \\
&\qquad + \frac{2\,i\,(1-\zeta_p)}{(1+\zeta_p)} (-1)^{\ell'} \left(\zeta_p^{-2\ell' d} - \zeta_p^{2\ell' d}\right) \zeta_p^{mak} X_p(\tau) N_p^*(\ell';\tau) \ \Big| \ \left[U_{p,\overline{ma^2}}\right]_1 \text{ (by Proposition 5.12)}
\end{aligned}$$

$$= \left(\frac{-2\, i\, \zeta_p\, (1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, N^*(d,p;\tau) + \frac{2\, \zeta_p\, (1-\zeta_p)}{(1+\zeta_p)} \zeta_p^{mak+d-1} X_p(\tau)\, P^*(d,p;\tau) \right) \; \Big| \; \left[U_{p,\overline{ma^2}} \right]_1$$

$$+ \frac{2\, i\, (1-\zeta_p)}{(1+\zeta_p)} \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \left(\zeta_p^{-2jd} - \zeta_p^{2jd} \right) \zeta_p^{mak} X_p(\tau) N_p^*(j;\tau) \; \Big| \; \left[U_{p,\overline{ma^2}} \right]_1$$

$$= \frac{(1-\zeta_p)}{(1+\zeta_p)} \frac{\left(1+\zeta_p^d\right)}{\left(1-\zeta_p^d\right)} \zeta_p^{mak} \mathcal{J}^* \left(\frac{d}{p}; \tau \right) \; \big| \, [U_{p,ma^2}]_1 \, .$$

This completes the proof. □

6. Lower bounds for order of at cusps

In this section, we calculate lower bounds for the orders of $\mathcal{OK}_{p,m}^{(C)}(\zeta_p, z)$ and $\mathcal{OK}_{p,m}^{(R)}(\zeta_p, z)$ at the cusps of $\Gamma_1(2p)$, which we use in proving the identities for the same in the subsequent section.

For any cusp $\frac{a}{c}$ with $(a,c) = 1$ we define

$$\text{ord}\left(\mathcal{F}; \frac{a}{c} \right) := \text{ord}(\mathcal{F} \; | \, [A]_1 \, ; \infty),$$

where $A \in \text{SL}_2(\mathbb{Z})$ and $A\infty = \frac{a}{c}$.

The following result is necessary to calculate $\text{ord}\left(X_p(z); \frac{a}{c}\right)$, $\text{ord}\left(\frac{\eta(p^2 z)^2}{\eta(2p^2 z)}; \frac{a}{c} \right)$, $\text{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}; \frac{a}{c} \right)$ appearing in our transformations.

Proposition 6.1. [15, Corollary 2.2]. *Let $N \geq 1$ and*

$$F(z) = \prod_{m|N} \eta(mz)^{r_m},$$

where each $r_m \in \mathbb{Z}$. Then for $(a,c) = 1$,

$$\text{ord}\left(F(z); \frac{a}{c} \right) = \sum_{m|N} \frac{(m,c)^2 r_m}{24m}.$$

We also need the following result in the context of calculating the orders for $\mathcal{OK}_{p,m}^{(R)}(\zeta_p, z)$ at the cusps of $\Gamma_1(2p)$. Using Equation (5.1), we can write $N_p(j; \frac{z}{p}) = M(j,p;2pz)$. Jennings-Shaffer gives lower bounds for orders of $N(a,c;z)$, $M(a,c;mz)$ and $P(a,c;z)$ at cusps $\frac{\alpha}{\gamma}$ (see [13, Propositions 6.3, 6.4, 6.5]). We state these orders below for our functions $N(1,p;z)$, $N_p(j;z)$ and $P(1,p;z)$.

Proposition 6.2. *For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\{x\}$ the fractional part of x. For non-negative integers α and γ with $(\alpha,\gamma) = 1$, we have*

i)

$$\text{ord}_{holo}\left(N(1,p;z); \frac{\alpha}{\gamma} \right) \geq \begin{cases} -\frac{\gamma^2}{p^2} + \frac{\gamma}{p} - \frac{1}{4} + 2\tilde{\nu}\left(\frac{\gamma}{p}, \frac{1}{2}\right) & \text{if } \gamma \text{ is even}, \\ -\frac{\gamma^2}{p^2} + \frac{\alpha\gamma}{p} - \frac{\alpha^2}{4} + \frac{1}{2}\tilde{\nu}\left(\frac{2\gamma}{p}, \alpha\right) & \text{if } \gamma \text{ is odd}, \end{cases} \tag{6.1}$$

$$\textit{where } \tilde{\nu}(u,w) = \frac{1}{2}(\lfloor u\rfloor - \lfloor w\rfloor)^2 + (\lfloor u\rfloor - \lfloor w\rfloor)(\{u\}-\{w\}) + k(u,w),$$

$$k(u,w) = \begin{cases} \nu(\{u\},\{w\}) & \textit{if } \{u\}-\{w\} \neq \pm\frac{1}{2}, \\ \min(\frac{1}{8}, \nu(\{u\},\{w\})) & \textit{if } \{u\}-\{w\} = \pm\frac{1}{2}, \end{cases}$$

$$\nu(u,w) = \begin{cases} \frac{u+w}{2} - \frac{1}{8} & \textit{if } u+w \leq 1, \\ \frac{7}{8} - \frac{u+w}{2} & \textit{if } u+w > 1. \end{cases}$$

ii)

(6.2)

$$\textit{ord}_{\textit{holo}}\left(N_p(j;\frac{z}{p});\frac{\alpha}{\gamma}\right) \geq -\frac{g^2x^2}{4p}\left(\frac{j}{p}-\frac{1}{2}\right)^2 + \frac{g^2}{2p}\tilde{\nu}\left(\frac{jx}{p},\frac{x}{2}\right) \quad \textit{where } g = (2p,\gamma) \textit{ and } x = \frac{2p\alpha}{g}.$$

iii)

$$\textit{ord}_{\textit{holo}}\left(P(1,p;z);\frac{\alpha}{\gamma}\right) \geq \begin{cases} \left\{\frac{\gamma}{p}\right\} - \left\{\frac{\gamma}{p}\right\}^2 & \textit{if } \gamma \textit{ is even}, \\ \frac{1}{4}\left\{\frac{\gamma}{p}\right\} - \frac{1}{4}\left\{\frac{\gamma}{p}\right\}^2 - \frac{1}{16} & \textit{if } \gamma \textit{ is odd}. \end{cases} \tag{6.3}$$

Proposition 6.3. [7, Corollary 4, p.930] *Let $p > 3$ be prime. Then a set of inequivalent cusps $\mathcal{S}_p$ for $\Gamma_1(2p)$ is given by*

$$i\infty,\ 0,\ \frac{1}{2},\ \frac{1}{3},\ \ldots,\ \frac{1}{p-1},\ \frac{2}{p},\ \frac{3}{p},\ \ldots,\ \frac{\frac{1}{2}(p-1)}{p},\ \frac{3}{2p},\ \frac{5}{2p},\ \ldots,\ \frac{p-2}{2p}.$$

We now calculate lower bounds of the invariant order of $\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z)$ and $\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z)$ at each cusp of $\Gamma_1(2p)$.

Theorem 6.4. *Let $p > 3$ be prime and $0 \leq m \leq p-1$. Then*

(i)

$$\textit{ord}\left(\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z);0\right) \quad \begin{cases} \geq 0 & \textit{if } p = 3,5,7, \\ = -\frac{1}{16p}(p-1)(p-7) & \textit{otherwise}; \end{cases}$$

(ii)

$$\textit{ord}\left(\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z);\frac{1}{n}\right) \quad \begin{cases} \geq 0 \textit{ if } n \textit{ is even}, \\ \geq 0 \textit{ if } n \textit{ is odd}, p \leq 7, \\ \geq -\frac{1}{16p}(p-1)(p-7) \textit{ if } n \textit{ is odd}, p > 7; \end{cases}$$

(iii)

$$\textit{ord}\left(\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z);\frac{n}{p}\right) \geq \left(\frac{p^2-1}{16p}\right),\ 1 \leq n \leq \frac{p-1}{2};$$

(iv)

$$\textit{ord}\left(\mathcal{OK}^{(C)}_{p,m}(\zeta_p,z);\frac{n}{p}\right) \geq 0 \textit{ if } n \textit{ is odd, } 3 \leq n \leq p-2.$$

Proof. We derive lower bounds for ord $\left(\mathcal{OK}_{p,m}^{(C)}(\zeta_p, z); \zeta\right)$ for each cusp ζ of $\Gamma_1(2p)$ not equivalent to $i\infty$. From Equation (4.2), we have

$$\mathcal{OK}_{p,m}^{(C)}(\zeta_p, z) = X_p(z)\,\mathcal{O}_{\mathcal{C}}^{*}(\zeta_p, z) \;\big|\; [U_{p,m}]_1 = \frac{1}{\sqrt{p}}(\zeta_p^{\frac{1}{2}} - \zeta_p^{\frac{-1}{2}}) \sum_{k=0}^{p-1} \zeta_p^{-rm}\, X_p(z)\, \frac{1}{t_{(0,\frac{1}{p})}(z)} \mid [T_k]_1 \, .$$

We calculate

$$X_p(z)\, \frac{1}{t_{(0,\frac{1}{p})}(z)} \mid [T_k\, A]_1$$

for each $0 \le k \le p-1$ and each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and then pick A suitably to cover all the cusps in Proposition 6.3.

Then

$$\mathrm{ord}\left(X_p(z)\, \frac{1}{t_{(0,\frac{1}{p})}(z)}; \frac{a}{c} \right) = \mathrm{ord}\left(X_p(z); \frac{a}{c} \right) + \mathrm{ord}\left(\frac{1}{t_{(0,\frac{1}{p})}(z)}; \frac{a}{c} \right).$$

Following the proof of Theorem 4.1, we have

$$t_{(0,\frac{1}{p})}(z) | [A]_1 = t_{(\frac{c}{p},\frac{d}{p})}(z)$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Case 1. $a + kc \not\equiv 0 \pmod{p}$. Choose $0 \le k' \le p-1$ such that

$$(a + kc)\, k' \equiv (b + kd) \pmod{p}.$$

Then

$$T_k\, A = C_k\, T_{k'},$$

where

$$C_k = T_k\, A\, T_{k'}^{-1} = \begin{pmatrix} a + ck & \frac{1}{p}(-k'(a + kc) + b + kd) \\ pc & d - k'c \end{pmatrix} \in \Gamma_0(p).$$

Then

$$\begin{aligned} (6.4) \qquad X_p(z)\, \frac{1}{t_{(0,\frac{1}{p})}(z)} \mid [T_k\, A]_1 &= X_p(z)\, \frac{1}{t_{(0,\frac{1}{p})}(z)} \mid [C_k\, T_{k'}]_1 \\ &= (X_p(z) \mid [C_k\, T_{k'}]_0) \left(\frac{1}{t_{(c,\frac{d-k'c}{p})}(z)} \mid [T_{k'}]_1 \right). \end{aligned}$$

Case 2. $a+kc\equiv 0 \pmod p$. In this case we consider

$$T_k\,A=\begin{pmatrix}a+kc & b+kd\\ pc & pd\end{pmatrix}.$$

In this case we find that

$$T_k\,A=D_k\,P,$$

where

$$P=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix},$$

$$D_k=\begin{pmatrix}\frac{1}{p}(a+kc) & b+kd\\ c & pd\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}).$$

Then

$$X_p(z)\,\frac{1}{t_{(0,\frac{1}{p})}(z)}\mid[D_k\,P]_1=(X_p(z)\mid[D_k\,P]_0)\left(\frac{1}{t_{(0,\frac{1}{p})}(z)}\mid[D_k\,P]_1\right) \tag{6.5}$$
$$=(X_p(z)\mid[D_k\,P]_0)\left(\frac{1}{t_{(\frac{c}{p},d)}(z)}\right)|[P]_1.$$

Now we are ready to examine each cusp ζ of $\Gamma_1(2p)$. We choose

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}),\quad \text{so that } A(\infty)=\tfrac{a}{c}=\zeta.$$

(i) $\zeta=0$. Let $A=\begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix}$ so that $A(\infty)=0$. We assume $0\le k\le p-1$.

If $k\neq 0$ then applying (6.4) with $C_k=\begin{pmatrix}k & *\\ p & d-k'\end{pmatrix}$ we have

$$\begin{aligned}\mathrm{ord}\left(X_p\left(\frac{z+k}{p}\right)\,\frac{1}{t_{(0,\frac{1}{p})}\left(\frac{z+k}{p}\right)};0\right)&=\frac{1}{p}\,\mathrm{ord}\left(X_p(z);\frac{k}{p}\right)+\frac{1}{p}\,\mathrm{ord}\left(\frac{1}{t_{(1,\frac{d-k'c}{p})}(z)};i\infty\right)\\&=0+0\\&=0,\end{aligned}$$

where the two orders are calculated using Propositions 6.1 and 2.3 (iii) respectively.

Next applying (6.5) with $k=0$ we have

$$\begin{aligned}\mathrm{ord}\left(X_p\left(\frac{z}{p}\right)\,\frac{1}{t_{(0,\frac{1}{p})}\left(\frac{z}{p}\right)};0\right)&=p\,\mathrm{ord}\,(X_p(z);0)+p\,\mathrm{ord}\left(\frac{1}{t_{(\frac{1}{p},d)}(z)};i\infty\right)\\&=\frac{p}{24}\left(-2+\frac{1}{2}+\frac{2}{p^2}+\frac{-1}{2p^2}\right)+\frac{p}{2}.\frac{1}{p}\left(1-\frac{1}{p}\right)\\&=-\frac{1}{16p}(p-1)(p-7)\end{aligned}$$

again by using Propositions 6.1 and 2.3 (iii). The result (i) follows.

(ii) $\zeta = \frac{1}{n}$, *where* $1 \leq n \leq p-1$. Let $A = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ so that $A(\infty) = \frac{1}{n}$.

If $kn \not\equiv -1 \pmod{p}$ then applying (6.4) with $C_k = \begin{pmatrix} 1+kn & * \\ pn & 1-k'n \end{pmatrix}$ we have

$$\operatorname{ord}\left(X_p\left(\frac{z+k}{p}\right)\frac{1}{t_{(0,\frac{1}{p})}\left(\frac{z+k}{p}\right)}; \frac{1}{n}\right) = \frac{1}{p}\operatorname{ord}\left(X_p(z); \frac{1+kn}{pn}\right) + \frac{1}{p}\operatorname{ord}\left(\frac{1}{t_{(n,\frac{1-k'n}{p})}(z)}; i\infty\right)$$

From Proposition 2.3, we have

$$\operatorname{ord}\left(\frac{1}{t_{(n,\frac{1-k'n}{p})}(z)}; i\infty\right) = 0.$$

And to calculate $\operatorname{ord}\left(X_p(z); \frac{1+kn}{pn}\right)$ using Proposition 6.1, we split the cusps into two cases :

Case I : When n is odd, we have

$$\operatorname{ord}\left(X_p(z); \frac{1+kn}{pn}\right) = \frac{1}{24}\left(-2 + \frac{1}{2} + 2 - \frac{1}{2}\right) = 0.$$

Case II : When n is even, we have

$$\operatorname{ord}\left(X_p(z); \frac{1+kn}{pn}\right) = \frac{1}{24}(-2+2+2-2) = 0.$$

Next assuming $kn \equiv -1 \pmod{p}$ and applying (6.5), we have

$$\operatorname{ord}\left(X_p\left(\frac{z+k}{p}\right)\frac{1}{t_{(0,\frac{1}{p})}\left(\frac{z+k}{p}\right)}; \frac{1}{n}\right) = p\operatorname{ord}\left(X_p(z); \frac{(1+kn)/p}{n}\right) + p\operatorname{ord}\left(\frac{1}{t_{(\frac{n}{p},1)}(z)}; i\infty\right)$$

From Proposition 2.3, we have

$$\operatorname{ord}\left(\frac{1}{t_{(\frac{n}{p},1)}(z)}; i\infty\right) = \frac{1}{2}.\frac{n}{p}\left(1 - \frac{n}{p}\right).$$

And to calculate $\operatorname{ord}\left(X_p(z); \frac{(1+kn)/p}{n}\right)$ using Proposition 6.1, we split the cusps into two cases :

Case I : When n is odd, we have

$$\operatorname{ord}\left(X_p(z); \frac{(1+kn)/p}{n}\right) = \frac{1}{24}\left(-2 + \frac{1}{2} + \frac{2}{p^2} + \frac{-1}{2p^2}\right) = \frac{-1}{16}\left(1 - \frac{1}{p^2}\right).$$

Case II : When n is even, we have

$$\operatorname{ord}\left(X_p(z); \frac{(1+kn)/p}{n}\right) = \frac{1}{24}\left(-2 + 2 + \frac{2}{p^2} + \frac{-2}{p^2}\right) = 0.$$

Combining these results, we have

$$\operatorname{ord}\left(X_p\left(\frac{z+k}{p}\right)\ \frac{1}{t_{(0,\frac{1}{p})}\left(\frac{z+k}{p}\right)};\frac{1}{n}\right)=\begin{cases}-\frac{(p-1)(p-7)}{16p} & \text{if } n \text{ is odd,}\\ \frac{n}{2}\left(1-\frac{n}{2}\right) & \text{if } n \text{ is even.}\end{cases}$$

The result (ii) follows.

(iii) $\zeta=\frac{n}{p}$*, where* $1\le n\le\frac{1}{2}(p-1)$. Choose b,d so that $A=\begin{pmatrix} n & b\\ p & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ and $A(\infty)=n/p$.

Since $n\not\equiv 0 \pmod p$ we apply (6.4) for each k with $C_k=\begin{pmatrix} n+kp & *\\ p^2 & d-k'p\end{pmatrix}$. We have

$$\begin{aligned}\operatorname{ord}\left(X_p\left(\frac{z+k}{p}\right)\ \frac{1}{t_{(0,\frac{1}{p})(\frac{z+k}{p})}};\frac{n}{p}\right)&=\frac{1}{p}\operatorname{ord}\left(X_p(z);\frac{n+kp}{p^2}\right)+\frac{1}{p}\operatorname{ord}\left(\frac{1}{t_{(p,\frac{d-k'p}{p})}(z)};i\infty\right)\\ &=\frac{1}{24p}\left(-2+\frac{1}{2}+2p^2+\frac{-1}{2}p^2\right)+0\\ &=\frac{p^2-1}{16p}.\end{aligned}$$

The result (iii) follows.

(iv) $\zeta=\frac{n}{2p}$*, where* n *is odd and* $3\le n\le p-2$. Let $A=\begin{pmatrix} n & b\\ 2p & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$ so that $A(\infty)=n/2p$.

Since $n\not\equiv 0 \pmod p$ we apply (6.4) for each k with $C_k=\begin{pmatrix} n+2kp & *\\ 2p^2 & d-2k'p\end{pmatrix}$. We have

$$\begin{aligned}\operatorname{ord}\left(X_p\left(\frac{z+k}{p}\right)\ \frac{1}{t_{(0,\frac{1}{p})(\frac{z+k}{p})}};\frac{n}{2p}\right)&=\frac{1}{p}\operatorname{ord}\left(X_p(z);\frac{n+2kp}{2p^2}\right)+\frac{1}{p}\operatorname{ord}\left(\frac{1}{t_{(2p,\frac{d-2k'p}{p})}(z)};i\infty\right)\\ &=\frac{1}{24}\left(-2+2+2p^2-2p^2\right)+0\\ &=0.\end{aligned}$$

The result (iv) follows. ☐

Theorem 6.5. *Let* $p>3$ *be prime and* $0\le m\le p-1$. *Then*

(i)

$$\operatorname{ord}\left(\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z);0\right)\ge\begin{cases}\frac{1}{8}\left(\frac{1}{2p}-1\right) \text{ if } \left(\frac{-m}{p}\right)=1 \text{ and } p=5,\\ -\frac{(p-1)(p-3)}{16p} \text{ otherwise;}\end{cases}$$

(ii)

$$\operatorname{ord}\left(\mathcal{OK}^{(R)}_{p,m}(\zeta_p,z);\frac{1}{n}\right)\ge\begin{cases}0 \text{ if } n \text{ is even,}\\ \frac{-1}{16p} \text{ if } n \text{ is odd;}\end{cases}$$

(iii)

$$ord\left(\mathcal{OK}^{(R)}_{p,m}(\zeta_p, z); \frac{n}{p}\right) \geq \begin{cases} \frac{p}{2} - \frac{(p-1)^4}{4p} - \frac{(p-1)^2}{4} \text{ if } \left(\frac{-m}{p}\right) = 1, \\ \frac{p}{16} - \frac{1}{16p} \text{ otherwise}; \end{cases}$$

(iv)

$$ord\left(\mathcal{OK}^{(R)}_{p,m}(\zeta_p, z); \frac{n}{2p}\right) \geq \begin{cases} p - \frac{(p-1)^4}{4p} \text{ if } \left(\frac{-m}{p}\right) = 1, \\ 0 \text{ otherwise}; \end{cases}$$

Proof. We derive lower bounds for ord $\left(\mathcal{OK}^{(R)}_{p,m}(\zeta_p, z); \zeta\right)$ for each cusp ζ of $\Gamma_1(2p)$ not equivalent to $i\infty$. From Equation (5.6), we have

$$\begin{aligned}
\mathcal{OK}^{(R)}_{p,m}(\zeta_p; z) &= \mathcal{J}^*\left(\frac{1}{p}; z\right) \ \mid [U_{p,m}]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km}\, \mathcal{J}^*\left(\frac{1}{p}; z\right) \ \mid [T_k]_1 \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km}\, X_p(z)\mathfrak{O}^*_R(\zeta_p, z) \mid [T_k]_1 \text{ (using Equation (5.3))} \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km}\, \frac{\eta(p^2z)^2}{\eta(2p^2z)} \mathcal{J}\left(\frac{1}{p}; z\right) \ \mid [T_k]_1 \text{ (using Equation (5.5))} \\
&= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km}\, \frac{\eta(p^2z)^2}{\eta(2p^2z)} \Bigg[\frac{-2\,i\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} N(1,p;z) \\
&\qquad + \frac{2\,i\,\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \left(\zeta_p^{-2j} - \zeta_p^{2j}\right) N_p(j;z) \\
&\qquad + \frac{2\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} P(1,p;z)\Bigg] \mid [T_k]_1 \text{ (using Equation (5.2))} \\
&= \begin{cases} \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km} \frac{\eta(p^2z)^2}{\eta(2p^2z)} \left[\frac{-2\,i\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} N(1,p;z) + \frac{2\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} P(1,p;z)\right] \mid [T_k]_1 \\ \quad \text{if } m = 0 \text{ or } \left(\frac{-m}{p}\right) = -1, \\ \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \zeta_p^{-km} \frac{\eta(p^2z)^2}{\eta(2p^2z)} \left[\frac{-2\,i\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} N(1,p;z) + \frac{2\,\zeta_p\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} P(1,p;z)\right] \mid [T_k]_1 \\ \quad + \frac{2\,i\left(1-\zeta_p\right)}{\left(1+\zeta_p\right)} (-1)^\ell \left(\zeta_p^{-2\ell} - \zeta_p^{2\ell}\right) \frac{\eta(pz)^2}{\eta(2pz)} N_p(\ell; \frac{z}{p}), \\ \quad \text{where } 1 \leq \ell \leq \frac{1}{2}(p-1), \ell^2 \equiv -m \pmod{p}, \text{and } \left(\frac{-m}{p}\right) = 1. \end{cases}
\end{aligned}$$

We calculate

$$\frac{\eta(p^2z)^2}{\eta(2p^2z)} N(1,p;z) \mid [T_k\, A]_1\ ,$$

$$\frac{\eta(pz)^2}{\eta(2pz)} N_p(\ell; \frac{z}{p}) \mid [A]_1 \text{ and}$$
$$\frac{\eta(p^2z)^2}{\eta(2p^2z)} P(1,p;z) \mid [T_k\, A]_1$$

for each $0 \le k \le p-1$ and each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and then pick A suitably to cover all the cusps in Proposition 6.3.

Case 1. $a + kc \not\equiv 0 \pmod{p}$. Choose $0 \le k' \le p-1$ such that

$$(a+kc)\, k' \equiv (b+kd) \pmod{p}.$$

Then

$$T_k\, A = C_k\, T_{k'},$$

where

$$C_k = T_k\, A\, T_{k'}^{-1} = \begin{pmatrix} a+ck & \frac{1}{p}(-k'(a+kc)+b+kd) \\ pc & d-k'c \end{pmatrix} \in \Gamma_0(p).$$

Then

$$\begin{aligned}
\text{(6.6)} \qquad \frac{\eta(p^2z)^2}{\eta(2p^2z)} Y(1,p;z) \mid [T_k\, A]_1 &= \frac{\eta(p^2z)^2}{\eta(2p^2z)} Y(1,p;z) \mid [C_k\, T_{k'}]_1 \\
&= \left(\frac{\eta(p^2z)^2}{\eta(2p^2z)} \mid [C_k\, T_{k'}]_0 \right) \left(Y(1,p;z) \mid [C_k\, T_{k'}]_1 \right).
\end{aligned}$$

where $Y = N$ or P.

Case 2. $a + kc \equiv 0 \pmod{p}$. In this case we consider

$$T_k\, A = \begin{pmatrix} a+rc & b+rd \\ pc & pd \end{pmatrix}.$$

In this case we find that

$$T_k\, A = D_k\, P,$$

where

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

$$D_k = \begin{pmatrix} \frac{1}{p}(a+kc) & b+kd \\ c & pd \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then

$$\text{(6.7)} \qquad \frac{\eta(p^2z)^2}{\eta(2p^2z)} Y(1,p;z) \mid [D_k\, P]_1 = \left(\frac{\eta(p^2z)^2}{\eta(2p^2z)} \mid [D_k\, P]_0 \right) \left(Y(1,p;z) \mid [D_k\, P]_1 \right)$$

where $Y = N$ or P.

Now we are ready to examine each cusp ζ of $\Gamma_1(2p)$. We choose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad \text{so that } A(\infty) = \tfrac{a}{c} = \zeta.$$

(i) $\zeta = 0$. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that $A(\infty) = 0$. We assume $0 \leq k \leq p-1$.

If $k \neq 0$ then applying (6.6) with $C_k = \begin{pmatrix} k & * \\ p & d-k' \end{pmatrix}$ we have

$$\begin{aligned}
\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[T_k]_1;0\right) &= \frac{1}{p}\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{k}{p}\right) + \frac{1}{p}\operatorname{ord}\left(N(1,p,z);\frac{k}{p}\right) \\
&\geq \frac{1}{24p}(2-\frac{1}{2}) + \frac{1}{p}(-1+k-\frac{k^2}{4}+\frac{1}{2}\tilde{\nu}(2,k)) \\
&= \frac{1}{16p} + \frac{1}{p}(-1+k-\frac{k^2}{4}+\frac{1}{2}\tilde{\nu}(2,k)) \\
&= 0.
\end{aligned}$$

And,

$$\begin{aligned}
\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[T_k]_1;0\right) &= \frac{1}{p}\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{k}{p}\right) + \frac{1}{p}\operatorname{ord}\left(P(1,p,z);\frac{k}{p}\right) \\
&\geq \frac{1}{16p} + \frac{1}{p}(0-0-\frac{1}{16}) \\
&= 0,
\end{aligned}$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

If $k = 0$ then applying (6.7) with $D_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have

$$\begin{aligned}
\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[D_p]_1;0\right) &= p\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};0\right) + p\operatorname{ord}\left(N(1,p,z);0\right) \\
&\geq \frac{p}{24}\left(\frac{2}{p^2}-\frac{1}{2p^2}\right) + p\left(\frac{-1}{p^2}+\frac{1}{2}\tilde{\nu}\left(\frac{2}{p},0\right)\right) \\
&= \frac{1}{16p} + p\left(\frac{-1}{p^2}+\frac{1}{2}\nu\left(\frac{2}{p},0\right)\right) \\
&= \frac{1}{16p} + p\left(\frac{-1}{p^2}+\frac{1}{2}\left(\frac{1}{p}-\frac{1}{8}\right)\right) \\
&= \frac{1-2p}{2p^2}.
\end{aligned}$$

And,

$$\begin{aligned}
\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[D_p]_1;0\right) &= p\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};0\right) + p\operatorname{ord}\left(P(1,p,z);0\right) \\
&\geq \frac{1}{16p} + p\left(\frac{1}{4p}-\frac{1}{4p^2}-\frac{1}{16}\right) \\
&= -\frac{(p-1)(p-3)}{16p},
\end{aligned}$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

Finally

$$\begin{aligned}\operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}N_p\left(\ell,\frac{z}{p}\right);0\right) &= \operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)};0\right)+\operatorname{ord}\left(N_p\left(\ell,\frac{z}{p}\right);0\right)\\ &\geq \frac{1}{24}\left(\frac{2}{p}-\frac{1}{2p}\right)+(0+\frac{1}{2p}\tilde{\nu}(0,0))\\ &= \frac{1}{16p}-\frac{1}{8},\end{aligned}$$

where the orders are calculated using Proposition 6.1 and Equation (6.2) respectively.

(ii) $\zeta=\frac{1}{n}$*, where* $1\leq n\leq p-1$. Let $A=\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$ so that $A(\infty)=\frac{1}{n}$.

If $kn\not\equiv -1 \pmod p$ then applying (6.6) with $C_k=\begin{pmatrix}1+kn & *\\ pn & 1-k'n\end{pmatrix}$ we have

$$\begin{aligned}&\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[T_k]_1;\frac{1}{n}\right)\\ &=\frac{1}{p}\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{1+kn}{pn}\right)+\frac{1}{p}\operatorname{ord}\left(N(1,p,z);\frac{1+kn}{pn}\right)\\ &\geq\begin{cases}\frac{1}{24p}(-2+\frac{1}{2}+2-\frac{1}{2})+\frac{1}{p}\left(-n^2+n-\frac{1}{4}+2\tilde{\nu}\left(n,\frac{1}{2}\right)\right) & \text{if } n \text{ is even,}\\ \frac{1}{24p}(-2+2+2-2)+\frac{1}{p}\left(-n^2+(1+kn)n-\frac{(1+kn)^2}{4}+\frac{1}{2}\tilde{\nu}\left(2n,1+kn\right)\right) & \text{if } n \text{ is odd,}\end{cases}\\ &=\begin{cases}\frac{1}{p}\left(-n^2+n-\frac{1}{4}+2\tilde{\nu}\left(n,\frac{1}{2}\right)\right) & \text{if } n \text{ is even,}\\ \frac{1}{p}\left(-n^2+(1+kn)n-\frac{(1+kn)^2}{4}+\frac{1}{2}\tilde{\nu}\left(2n,1+kn\right)\right) & \text{if } n \text{ is odd,}\end{cases}\\ &=\begin{cases}0 & \text{if } n \text{ is even,}\\ \frac{-1}{16p} & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

And,

$$\begin{aligned}&\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[T_k]_1;\frac{1}{n}\right)\\ &=\frac{1}{p}\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{1+kn}{p}\right)+\frac{1}{p}\operatorname{ord}\left(P(1,p,z);\frac{1+kn}{pn}\right)\\ &\geq\begin{cases}0+\frac{1}{p}.0 & \text{if } n \text{ is even,}\\ 0+\frac{1}{p}.\frac{-1}{16} & \text{if } n \text{ is odd,}\end{cases}\\ &=\begin{cases}0 & \text{if } n \text{ is even,}\\ \frac{-1}{16p} & \text{if } n \text{ is odd,}\end{cases}\end{aligned}$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

If $kn \equiv -1 \pmod p$ then applying (6.7) with $D_k = \begin{pmatrix} \frac{1+kn}{p} & k \\ n & p \end{pmatrix}$ we have

$$\begin{aligned}
&\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[D_p]_1;\frac{1}{n}\right)\\
&= p\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{\frac{1+kn}{p}}{n}\right) + p\operatorname{ord}\left(N(1,p,z);\frac{\frac{1+kn}{p}}{n}\right)\\
&\geq \begin{cases} \frac{p}{24}\left(\frac{2}{p^2}-\frac{2}{p^2}\right) + p\left(\frac{-n^2}{p^2}+\frac{n}{p}-\frac{1}{4}+2\tilde{\nu}\left(\frac{n}{p},\frac{1}{2}\right)\right) & \text{if } n \text{ is even,}\\ \frac{p}{24}\left(\frac{2}{p^2}-\frac{1}{2p^2}\right) + p\left(\frac{-n^2}{p^2}+\frac{\frac{1+kn}{p}.n}{p}-\frac{(\frac{1+kn}{p})^2}{4}+\frac{1}{2}\tilde{\nu}\left(\frac{2n}{p},\frac{1+kn}{p}\right)\right) & \text{if } n \text{ is odd,}\end{cases}\\
&= \begin{cases} p\left(\frac{-n^2}{p^2}+\frac{n}{p}-\frac{1}{4}+2\tilde{\nu}\left(\frac{n}{p},\frac{1}{2}\right)\right) & \text{if } n \text{ is even,}\\ \frac{p}{16} + p\left(\frac{-n^2}{p^2}+\frac{\frac{1+kn}{p}.n}{p}-\frac{(\frac{1+kn}{p})^2}{4}+\frac{1}{2}\tilde{\nu}\left(\frac{2n}{p},\frac{1+kn}{p}\right)\right) & \text{if } n \text{ is odd,}\end{cases}\\
&= \begin{cases} \frac{-n^2}{p}+2n & \text{if } n \text{ is even and } n \leq \frac{p}{2},\\ \frac{-n^2}{p}+p & \text{if } n \text{ is even and } n > \frac{p}{2},\\ \frac{-n^2}{p}+\frac{n}{2} & \text{if } n \text{ is odd and } 1 \leq n \leq \frac{p-1}{2},\\ \frac{-n^2}{p}+\frac{3n}{2}-\frac{1}{2p} & \text{if } n \text{ is odd and } \frac{p-1}{2} < n \leq p-1.\end{cases}
\end{aligned}$$

And,

$$\begin{aligned}
&\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[D_p]_1;\frac{1}{n}\right)\\
&= p\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)};\frac{\frac{1+kn}{p}}{n}\right) + p\operatorname{ord}\left(P(1,p,z);\frac{\frac{1+kn}{p}}{n}\right)\\
&\geq \begin{cases} \frac{p}{24}\left(\frac{2}{p^2}-\frac{2}{p^2}\right) + p\left(\frac{n}{p}-\frac{n^2}{p^2}\right) & \text{if } n \text{ is even,}\\ \frac{p}{24}\left(\frac{2}{p^2}-\frac{1}{2p^2}\right) + p\left(\frac{1}{4}.\frac{n}{p}-\frac{1}{4}.\frac{n^2}{p^2}-\frac{1}{16}\right) & \text{if } n \text{ is odd,}\end{cases}\\
&= \begin{cases} p\left(\frac{n}{p}-\frac{n^2}{p^2}\right) & \text{if } n \text{ is even,}\\ \frac{p}{16} + p\left(\frac{1}{4}.\frac{n}{p}-\frac{1}{4}.\frac{n^2}{p^2}-\frac{1}{16}\right) & \text{if } n \text{ is odd,}\end{cases}\\
&= \begin{cases} n-\frac{n^2}{p} & \text{if } n \text{ is even,}\\ \frac{n}{4}-\frac{n^2}{4p} & \text{if } n \text{ is odd,}\end{cases}
\end{aligned}$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

Finally

$$\operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}N_p\left(\ell,\frac{z}{p}\right);\frac{1}{n}\right) = \operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)};\frac{1}{n}\right) + \operatorname{ord}\left(N_p\left(\ell,\frac{z}{p}\right);\frac{1}{n}\right)$$

$$\geq \begin{cases} \frac{2.1.2-2}{48p} - p\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{2}{p}\tilde{\nu}(\ell,\frac{p}{2}) & \text{if } n \text{ is even,} \\ \frac{2.1.2-1}{48p} - p\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{1}{2p}\tilde{\nu}(2\ell,p) & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{1}{24p} - p\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{2}{p}\tilde{\nu}(\ell,\frac{p}{2}) & \text{if } n \text{ is even,} \\ \frac{1}{16p} - p\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{1}{2p}\tilde{\nu}(2\ell,p) & \text{if } n \text{ is odd,} \end{cases}$$

$$= \begin{cases} \frac{1}{24p} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where the orders are calculated using Proposition 6.1 and Equation (6.2) respectively.

(iii) $\zeta = \frac{n}{p}$, *where* $1 \leq n \leq \frac{1}{2}(p-1)$. Choose b,d so that $A = \begin{pmatrix} n & b \\ p & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $A(\infty) = \frac{n}{p}$.

Since $n \not\equiv 0 \pmod{p}$ we apply (6.6) for each k with $C_k = \begin{pmatrix} n+kp & * \\ p^2 & d-k'p \end{pmatrix}$. We have

$$\begin{aligned}
&\mathrm{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[T_k]_1; \frac{n}{p}\right) \\
&= \frac{1}{p}\,\mathrm{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}; \frac{n+kp}{p^2}\right) + \frac{1}{p}\,\mathrm{ord}\left(N(1,p,z); \frac{n+kp}{p^2}\right) \\
&\geq \frac{1}{24p}(2p^2 - \frac{p^2}{2}) + \frac{1}{p}\left(-p^2 + (n+kp)p - \frac{(n+kp)^2}{4} + \frac{1}{2}\tilde{\nu}(2p, n+kp)\right) \\
&= \frac{p}{16} + \frac{1}{p}\left(-p^2 + (n+kp)p - \frac{(n+kp)^2}{4} + \frac{1}{2}\tilde{\nu}(2p, n+kp)\right), \\
&= \frac{p}{16} - \frac{1}{16p}.
\end{aligned}$$

And,

$$\mathrm{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[T_k]_1; \frac{n}{p}\right) \geq \frac{1}{24p}(2p^2 - \frac{p^2}{2}) + \frac{1}{p}.0 = \frac{p}{16},$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

Finally

$$\begin{aligned}
&\mathrm{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}N_p\left(\ell, \frac{z}{p}\right); \frac{n}{p}\right) \\
&= \mathrm{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}; \frac{n}{p}\right) + \mathrm{ord}\left(N_p\left(\ell, \frac{z}{p}\right); \frac{n}{p}\right) \\
&\geq \frac{(p,p)^2.2}{24p} + \frac{(2p,p)^2.(-1)}{48p} + \left(-pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{p}{2}\tilde{\nu}\left(\frac{2n\ell}{p}, n\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p}{16} - pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{p}{2}\tilde{\nu}\left(\frac{2n\ell}{p}, n\right)\\
&= \begin{cases} \frac{p}{16} - pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{p}{2}\left(\frac{n^2}{2} - \frac{2n^2\ell}{p} + \frac{n\ell}{p} - \frac{1}{8}\right) & \text{if } \frac{2n\ell}{p} \le 1,\\ \frac{p}{16} - pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{p}{2}\left(\frac{1}{2}\left(\lfloor\frac{2n\ell}{p}\rfloor - n\right)^2 + \left(\lfloor\frac{2n\ell}{p}\rfloor - n\right)\{\frac{2n\ell}{p}\} + \frac{7}{8} - \frac{\{\frac{2n\ell}{p}\}}{2}\right) & \text{if } \frac{2n\ell}{p} > 1,\end{cases}\\
&\begin{cases} = \frac{n\ell}{2} - \frac{n^2\ell^2}{p} & \text{if } \frac{2n\ell}{p} \le 1,\\ \ge \frac{p}{16} - pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + \frac{p}{2}\left(\frac{n^2}{2} - \frac{2n^2\ell}{p} + \frac{7}{8} - \frac{n\ell}{p}\right) & \text{if } \frac{2n\ell}{p} > 1,\end{cases}\\
&\begin{cases} \ge \frac{1}{2} - \frac{1}{p} & \text{if } \frac{2n\ell}{p} \le 1,\\ = \frac{p}{2} - \frac{n^2\ell^2}{p} - \frac{n\ell}{2} & \text{if } \frac{2n\ell}{p} > 1,\end{cases}\\
&\ge \frac{p}{2} - \frac{(p-1)^4}{4p} - \frac{(p-1)^2}{4}
\end{aligned}
$$

where the orders are calculated using Proposition 6.1 and Equation (6.2) respectively.

(iv) $\zeta = \frac{n}{2p}$*, where* n *is odd and* $3 \le n \le p-2$*.* Let $A = \begin{pmatrix} n & b\\ 2p & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ so that $A(\infty) = \frac{n}{2p}$.

Since $n \not\equiv 0 \pmod p$ we apply (6.6) for each k with $C_k = \begin{pmatrix} n+2kp & *\\ 2p^2 & d-2k'p\end{pmatrix}$. We have

$$
\begin{aligned}
&\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}N(1,p,z)|[T_k]_1; \frac{n}{2p}\right)\\
&= \frac{1}{p}\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}; \frac{n+2kp}{2p^2}\right) + \frac{1}{p}\operatorname{ord}\left(N(1,p,z); \frac{n+2kp}{2p^2}\right)\\
&\ge \frac{1}{24p}(2p^2-2p^2) + \frac{1}{p}\left(-p^2 + (n+kp)p - \frac{(n+kp)^2}{4} + \frac{1}{2}\tilde{\nu}(2p, n+kp)\right)\\
&= \frac{p}{16} + \frac{1}{p}.\frac{-1}{16}\\
&= 0.
\end{aligned}
$$

And,

$$
\operatorname{ord}\left(\frac{\eta(p^2z)^2}{\eta(2p^2z)}P(1,p,z)|[T_k]_1; \frac{n}{2p}\right) = \frac{1}{24p}(2p^2-2p^2) + \frac{1}{p}.0 = 0,
$$

where the orders are calculated using Proposition 6.1 and Equations (6.1) and (6.3).

Finally

$$
\begin{aligned}
&\operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}N_p\left(\ell, \frac{z}{p}\right); \frac{n}{2p}\right)\\
&= \operatorname{ord}\left(\frac{\eta(pz)^2}{\eta(2pz)}; \frac{n}{2p}\right) + \operatorname{ord}\left(N_p\left(\ell, \frac{z}{p}\right); \frac{n}{2p}\right)
\end{aligned}
$$

$$\geq \frac{(p,2p)^2.2}{24p} + \frac{(2p,2p)^2.(-1)}{48p} + \left(-pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\tilde{\nu}\left(\frac{n\ell}{p},\frac{n}{2}\right)\right)$$

$$= -pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\tilde{\nu}\left(\frac{n\ell}{p},\frac{n}{2}\right)$$

$$= \begin{cases} -pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\left(\frac{1}{2}(\frac{n-1}{2})^2 - \frac{n-1}{2}(\frac{n\ell}{p}-\frac{1}{2}) + \frac{n\ell}{2p} + \frac{1}{8}\right) & \text{if } \frac{n\ell}{p} \leq \frac{1}{2}, \\ -pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\left(\frac{1}{2}(\frac{n-1}{2})^2 - \frac{n-1}{2}(\frac{n\ell}{p}-\frac{1}{2}) + \frac{5}{8} - \frac{n\ell}{2p}\right) & \text{if } \frac{1}{2} < \frac{n\ell}{p} \leq 1, \\ -pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\left(\frac{1}{2}\left(\lfloor\frac{n\ell}{p}\rfloor - \frac{n-1}{2}\right)^2 + \left(\lfloor\frac{n\ell}{p}\rfloor - \frac{n-1}{2}\right)(\{\frac{n\ell}{p}\}-\frac{1}{2}) + \frac{7}{8} - \frac{\{\frac{n\ell}{p}\}+\frac{1}{2}}{2}\right) & \text{if } \frac{n\ell}{p} > 1, \end{cases}$$

$$\begin{cases} = 2n\ell - \frac{n^2\ell^2}{p} & \text{if } \frac{n\ell}{p} \leq \frac{1}{2}, \\ = p - \frac{n^2\ell^2}{p} & \text{if } \frac{1}{2} < \frac{n\ell}{p} \leq 1, \\ \geq -pn^2\left(\frac{\ell}{p}-\frac{1}{2}\right)^2 + 2p\left(\frac{1}{2}(\frac{n-1}{2})^2 - \frac{n-1}{2}(\frac{n\ell}{p}-\frac{1}{2}) + \frac{7}{8} - \frac{n\ell}{2p} - \frac{1}{4}\right) & \text{if } \frac{n\ell}{p} > 1, \end{cases}$$

$$\begin{cases} = 2n\ell - \frac{n^2\ell^2}{p} & \text{if } \frac{n\ell}{p} \leq \frac{1}{2}, \\ = p - \frac{n^2\ell^2}{p} & \text{if } \frac{1}{2} < \frac{n\ell}{p} \leq 1, \\ = p - \frac{n^2\ell^2}{p} & \text{if } \frac{n\ell}{p} > 1, \end{cases}$$

$$\geq p - \frac{(p-1)^4}{4p}$$

where the orders are calculated using Proposition 6.1 and Equation (6.2) respectively. □

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