The Cyclic Behavior of Cosubnormal Operators

Nathan S. Feldman
Washington & Lee University
Lexington, VA

John Conway Day
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University of Florida
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Some History

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- 1955: J. Bram: **Theorem:** $N_\mu = M_z$ on $L^2(\mu)$ is cyclic.

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Nathan S. Feldman
Washington & Lee University
www.wlu.edu/~feldmann
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  - **Corollary:** If \(S = M_z\) on \(\mathcal{H} \subseteq L^2(\mu)\), then \(S^*\) is cyclic.
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- Q: Does every pure subnormal operator have a cyclic adjoint?

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- $\exists$ Common cyclic vectors for $\left\{ M_{\ast}f : f \in H^\infty(\mathcal{D}) \setminus C \right\}$ on $H$, where $H \subseteq \text{Hol}(\mathcal{G})$ & $\mathcal{G} \subseteq \mathbb{C}^n$.

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  **Q:** Does every pure subnormal operator have a cyclic adjoint?
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  $\exists$ Common cyclic vectors for $\{M_f^* : f \in H^\infty(G) \setminus \mathbb{C}\}$ on $\mathcal{H}$
  where $\mathcal{H} \subseteq Hol(G)$ & $G \subseteq \mathbb{C}^n$. 
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  \[ \text{span}\{ \ker(T - \lambda)^* : \lambda \in \sigma(T) \setminus \sigma_{ap}(T) \} = \mathcal{H} \Rightarrow T^* \text{ is cyclic.} \]
  \( \exists \) Common cyclic vectors for \( \{ M_f^* : f \in H^\infty(G) \setminus \mathbb{C} \} \) on \( \mathcal{H} \)  
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Theorem (Feldman ’98) Every Pure SNO has a cyclic adjoint.

Strategy of Proof:
Theorem (Feldman ’98) Every Pure SNO has a cyclic adjoint.

**Strategy of Proof:** Show $\exists$ a one-to-one linear map $A : \mathcal{H} \rightarrow L^2(\mu)$ s.t.

$$AS = N_\mu A$$
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then

$$S^* A^* = A^* N^*_\mu$$
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Strategy of Proof: Show \( \exists \) a one-to-one linear map

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A : \mathcal{H} \rightarrow L^2(\mu) \text{ s.t.} \quad AS = N_\mu A
\]

then

\[
S^* A^* = A^* N^*_\mu
\] & 

\( A^* \) maps cyclic vectors for \( N^*_\mu \) to cyclic vectors for \( S^* \).
The Proof: How do we find $A : \mathcal{H} \rightarrow L^2(\mu)$?

If $S$ is a pure SNO on $\mathcal{H}$, then there exist pure SNOs $S_n = M_z$ on $\mathcal{H}_n \subseteq L^2(\mu_n)$.

One-to-one intertwining maps $A_1$ and $A_2$ such that:

$H \xrightarrow{A_1} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \xrightarrow{A_2} L^2(\mu)$

Is every pure SNO quasi-similar to $(N \bigoplus_{n=1}^{\infty} M_z)$ on $(N \bigoplus_{n=1}^{\infty} \mathcal{H}_n)$?

A Multiplicity Theory?

If $S$ is a pure SNO, does there exist a 1-1 map $A$ such that $(S, \mathcal{H}) \xrightarrow{A} (M_z, \mathcal{H}_1 \subseteq \text{pure } L^2(\mu_1))$?
The Proof: How do we find $A : \mathcal{H} \rightarrow L^2(\mu)$?

If $S$ is a pure SNO on $\mathcal{H} \Rightarrow \exists$ pure SNOs $S_n = M_z$ on $\mathcal{H}_n \subseteq L^2(\mu_n)$ and one-to-one intertwining maps $A_1$ and $A_2$ s.t.
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$$
\begin{align*}
\mathcal{H} &\overset{A_1}{\underset{1-1}{\longrightarrow}} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \\
S &\overset{A_1}{\underset{1-1}{\longrightarrow}} \bigoplus_{n=1}^{\infty} M_z
\end{align*}

\bigoplus_{n=1}^{\infty} M_z \overset{A_2}{\underset{1-1}{\longrightarrow}} N_\mu \overset{A_2}{\underset{1-1}{\longrightarrow}} L^2(\mu)
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Is every pure SNO quasi-similar to $\left( \bigoplus_{n=1}^{N} M_z \right)$ on $\left( \bigoplus_{n=1}^{N} \mathcal{H}_n \right)$?
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$$\mathcal{H} \xrightarrow{A_1 \ 1-1} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \xrightarrow{A_2 \ 1-1} L^2(\mu)$$

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A Multiplicity Theory? If $S$ is a pure SNO, does $\exists$ a 1-1 map $A$ s.t.

$$(S, \mathcal{H}) \xrightarrow{A \ 1-1} (M_z, \mathcal{H}_1 \subseteq_{\text{pure}} L^2(\mu_1))?$$
Non Pure SNOs

- What if $S$ is not pure?

Theorem: If $S = S^p \oplus N$, then $S^*$ is cyclic if and only if $N$ is cyclic.

Corollary: If $S = \text{SNO}$, then $S^*$ is cyclic $\iff S^*$ is $\ast$-cyclic.

Some Open Questions

1. If $S$ is a pure SNO, is there a common cyclic vector for the pure operators in $P_\infty(S^*)$?

2. If $S = (S_1, S_2, \ldots, S_n)$ is a pure subnormal tuple, then is $S^* = (S_1^*, S_2^*, \ldots, S_n^*)$ cyclic?

3. If $T$ is a pure hyponormal operator, then is $T^*$ cyclic?

Nathan S. Feldman
Washington & Lee University
www.wlu.edu/~feldmann
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Stronger Forms of Cyclicity

**Definition**

If \( x \in \mathcal{H} \) and \( T \in \mathcal{B}(\mathcal{H}) \), then the orbit of \( x \) under \( T \) is

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\text{Orb}(x, T) = \{ T^n x : n \geq 0 \} = \{ x, Tx, T^2 x, \ldots \}.
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Definition
1. Let \( T \in \mathcal{B}(\mathcal{H}) \), then \( T \) is (weakly) hypercyclic if there is an \( x \in \mathcal{H} \) such that \( Orb(x, T) \) is (weakly) dense in \( \mathcal{H} \).
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2. \( T \) is (weakly) supercyclic if there is an \( x \in \mathcal{H} \) such that \( \mathbb{C} \cdot \text{Orb}(x, T) \) is (weakly) dense in \( \mathcal{H} \).
Theorem (G. Godefrey & J. Shapiro (1991))

If \( G \) is a bounded region in \( \mathbb{C} \), then \( M^*_z \) is hypercyclic on \( H^2(G) \) or \( L^2_a(G) \) if and only if \( G \cap \partial \mathbb{D} \neq \emptyset \).
Some Hypercyclic Operators

**Theorem (G. Godefrey & J. Shapiro (1991))**

If $G$ is a bounded region in $\mathbb{C}$, then $M_z^*$ is hypercyclic on $H^2(G)$ or $L^2_a(G)$ if and only if $G \cap \partial \mathbb{D} \neq \emptyset$.

**Corollary**

If $G$ is any bounded region in $\mathbb{C}$, then $M_z^*$ is supercyclic on $H^2(G)$ or $L^2_a(G)$.
Theorem (K. Chan & R. Sanders (2002))

If $G = \{z \in \mathbb{C} : 1 < |z| < r\}$, then $M_z^*$ on $H^2(G)$ is weakly hypercyclic, but not norm hypercyclic.
Corollary

If \( \{ z \in \mathbb{C} : 1 < |z| < r \} \subseteq G \) and \( G \cap \mathbb{D} = \emptyset \), then \( M_z^* \) on \( H^2(G) \) is weakly hypercyclic, but not norm hypercyclic.

Open Question

For which open sets \( G \) is \( M_z^* \) weakly hypercyclic on \( H^2(G) \)?
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What if...

Is $M^*_z$ weakly hypercyclic on $H^2(G)$?
$X = \text{Banach Space}$

A basis for the weak topology on $X$

$$N(x_0, \mathcal{F}, \epsilon) = \{x \in X : |f(x - x_0)| < \epsilon \text{ for all } f \in \mathcal{F}\}$$

where $\mathcal{F} \subseteq X^*$ is a finite set
$X = $ Banach Space

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where $\mathcal{F} \subseteq X^*$ is a finite set

A set $E \subseteq X$ is \textit{n-weakly dense} in $X$ if $E \cap N(x_0, \mathcal{F}, \epsilon) \neq \emptyset$

$\forall x_0 \in X, \epsilon > 0, \text{ and all finite sets } \mathcal{F} \subseteq X^* \text{ with } |\mathcal{F}| \leq n$
Theorem

If $\mathcal{H}$ is a Hilbert space and $E \subseteq \mathcal{H}$, then the following are equivalent:

1. $E$ is $n$-weakly dense in $\mathcal{H}$.

Definition

1. An operator $T$ is $n$-weakly hypercyclic if $\exists x \in \mathcal{H}$ such that $\text{Orb}(x, T)$ is $n$-weakly dense in $\mathcal{H}$.

2. $T$ is $n$-weakly supercyclic if $\exists x \in \mathcal{H}$ such that $C \cdot \text{Orb}(x, T)$ is $n$-weakly dense in $\mathcal{H}$. 
Theorem

If $\mathcal{H}$ is a Hilbert space and $E \subseteq \mathcal{H}$, then the following are equivalent:

1. $E$ is $n$-weakly dense in $\mathcal{H}$.
2. $F(E)$ is dense in $\mathbb{C}^n$ for every onto continuous linear map $F : \mathcal{H} \rightarrow \mathbb{C}^n$. 

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**Theorem**

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2. $F(E)$ is dense in $\mathbb{C}^n$ for every onto continuous linear map $F: \mathcal{H} \to \mathbb{C}^n$.
3. $E$ has a dense orthogonal projection onto every subspace with dimension at most $n$. 
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Theorem (Feldman 2010)

Suppose that $B_1, B_2, \ldots, B_m$ are each hypercyclic backward weighted shifts and $1 \leq n \leq m$. Then $B_1 \oplus \cdots \oplus B_n$ is $n$-weakly hypercyclic if and only if the direct sum of any $n$ of the operators $\{B_1, \ldots, B_m\}$ is hypercyclic.

Corollary (Feldman 2010)

There exist operators that are $n$-weakly hypercyclic, but not $(n+1)$-weakly hypercyclic, for any $n \geq 1$. 
Theorem (Feldman 2010)

Suppose that \( B_1, B_2, \ldots, B_m \) are each hypercyclic backward weighted shifts and \( 1 \leq n \leq m \).
Then \( B = \bigoplus_{k=1}^{m} B_k \) is \( n \)-weakly hypercyclic if and only if the direct sum of any \( n \) of the operators \( \{ B_1, \ldots, B_m \} \) is hypercyclic.

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There exist operators that are $n$-weakly hypercyclic, but not $(n + 1)$-weakly hypercyclic, for any $n \geq 1$. 
The Matrix Case

Theorem (Feldman 2010)

There are matrices that are 2-weakly supercyclic on $\mathbb{R}^n$ if and only if $n$ is even.
The Matrix Case

Theorem (Feldman 2010)

There are matrices that are 2-weakly supercyclic on $\mathbb{R}^n$ if and only if $n$ is even.

Theorem (Feldman 2010)

If $\{\pi, \theta_1, \theta_2, \ldots, \theta_n\}$ are linearly independent over $\mathbb{Q}$, then $T = R(\theta_1) \oplus R(\theta_2) \oplus \cdots \oplus R(\theta_n)$ is 2-weakly supercyclic on $\mathbb{R}^{2n}$, where $R(\theta)$ is the $2 \times 2$ matrix that rotates by $\theta$. 
1-Weakly Hypercyclic?

Open Question

Is $M_z^*$ 1-weakly hypercyclic on $H^2(G)$?
Thanks for your Time!
Nathan Feldman

Best wishes to John!