Combinatorics PhD Exam

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Contents

1 Important Sequences

We list the first several entries in some important sequences:

- *Factorial* - 1, 2, 6, 24, 120 ...
- *Catalan Numbers* - 1, 2, 5, 14, 42 ...
- *Fibonacci Numbers* - 1, 1, 2, 3, 5, 8, 13 ...
- *Bell Numbers* - 1, 2, 5, 15, 52 ...
- *Euler Numbers* - 1, 1, 2, 5, 16, 61 ...
- *Schröder Numbers* - 2, 6, 22, 90 ...
- *Motzkin Numbers* - 1, 2, 4, 9, 21, 51 ...

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2 Some Elementary Principles

Reference: *A Walk Through Combinatorics* - Bóna, Chapters 1-4.

2.1 The Pigeon-Hole Principle

Theorem 2.1 (Pigeon-Hole Principle). *Let $n > k \in \mathbb{Z}^+$. Suppose we place n identical balls into k identical boxes. Then there will be at least one box with at least two balls.*

Theorem 2.2 (Generalized Pigeon-Hole Principle). *Let n, m and r be positive integers so that $n > rm$, and let us distribute n identical balls into m identical boxes. Then there will be at least one box into which we place $r + 1$ balls.*

Problems

Problem 2.3. We have distributed 200 balls into 100 boxes, each box got at least 1 ball and at most 100 balls. Prove that one can find a set of boxes containing together 100 balls.

Proof. If all boxes have the same number of balls, then they each have 2 and the result clearly holds. Suppose not, so then we can find 2 boxes a_1 and a_2 such that $a_1 \neq a_2$. Order the rest of boxes $a_3, \ldots, a_{100}$. For each $i \in [100]$, denote $s_i = a_1 + \cdots + a_i$. If there exists $i > j \in [100]$ such that $s_i \equiv s_j \mod 100$, then we have $100 = s_i - s_j = a_{j+1} + \cdots + a_i$. Otherwise, by the pigeonhole principle there exists some $1 < k \in [100]$ such that $s_k \equiv a_2 \mod 100$ and we have that $100 = s_k - a_2 = a_1 + a_3 + \cdots + a_k$. □

Problem 2.4. The sum of 100 given real numbers is 0. Prove that at least 99 of the pairwise sums of these 100 numbers are non-negative. Is this result the best possible?

Proof. We can partition [100] into blocks of size 2 in 99 ways so that each block happens exaclty once, (i.e. pairings). In each partition, at least one sum is non-negative. This result is the best possible: consider $\{99, -1, -1, \ldots, -1\}$. □

Problem 2.5. Let $a_1, \ldots, a_{10}$ be positive integers not exceeding 100. Prove that there are disjoint nonempty subsets S and T of [10] such that

$$\sum_{i \in S} a_i = \sum_{j \in T} a_j.$$

Proof. It is clear that the sum of any such nonempty set of the a_i's is at most 1000 and at least 1. There are then certainly $2^{10} - 1 = 1023$ nonempty subsets of [10] and therefore by the pigeon-hole principle there are two distinct subsets G and H of [10], neither of which is contained in the other, satisfying $\sum_{i \in G} a_i = \sum_{j \in H} a_j$. Therefore it is clear that $S = G - H$ and $T = H - G$ satisfy our hypothesis. □

Problem 2.6. One afternoon, a mathematics library had several vistors. A librarian noticed that it was impossible to find three visitors so that no two of them met in the library that afternoon. Prove that then it was possible to find two moments of time that afternoon so that each visitor was in the library at one of those two moments.

Proof. For each visitor, either he met all the people that left the library before him or he met all the people that got to the library after him, or both. That is, assume someone did not meet someone who left the library before him and someone who arrived after him, then those are three people none of which met each other that day. Consider the set of people who met everyone who left before him. Then at the moment that the first person from this set was leaving, all the people in this set were present.

Now consider the rest of the people, all of whom met everyone who arrived after them. Then at the moment the last person in the set was arriving, then everyone in the set was present. Thus everyone was present at one of these two times. □

Problem 2.7. Determine the number of binary strings of length n beginning with 0, ending with 1, and such that the number of copies of 00 equals the number of copies of 11.

Proof. The number is then the number of all such words with an equal number of 0's and 1's. This takes a bit of thought. So we just have to pick where the 1's (or 0's) go. That is, there are no strings if n is odd and

$$\binom{n-2}{(\frac{n-2}{2})}$$

strings if n is even. □

2.2 Binomial Theorem and Related Identities

Theorem 2.8 (Binomial Theorem)**.** *For all nonnegative integers n,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proposition 2.9. *For all nonnegative integers n and k, we have the identities*

- ***a.*** $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$*, and*
- ***b.*** $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$.

For **a**, $\binom{n}{k}$ counts the subsets with $n+1$, and $\binom{n}{k+1}$ counts the subsets without $n+1$. For **b**, each summand $\binom{k+i}{k}$ counts the subsets whose largest element is $k+i+1$.

Theorem 2.10 (Vandermonde's Identity)**.** *For all positive integers n, m and k,*

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$

Theorem 2.11 (Multinomial Theorem)**.** *For all nonnegative integers n and k, the equality*

$$(x_1 + \cdots + x_k)^n = \sum_{a_1,\ldots,a_k} \binom{n}{a_1,\ldots,a_k} x_1^{a_1} \ldots x_k^{a_k}.$$

holds. Here the sum is taken over all k-tuples of nonnegative integers $a_1, \ldots, a_k$ satisfying $n = \sum_{i=1}^{k} a_i$.

Definition. For any $x \in \mathbb{R}$, we can define the *generalized binomial coefficient* by $\binom{x}{0} = 1$, and

$$\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$$

for $k \in \mathbb{Z}^+$. This extends to a generalized binomial theorem.

Problems

Problem 2.12. Let $p \geq 3$ be prime, and let m and $k < p^m$ be positive integers. Show that p divides $\binom{p^m}{k}$.

Proof. We have that

$$\binom{p^m}{k} = \frac{p^m}{k} \cdot \prod_{i=1}^{k-1} \frac{p^m - i}{i} = \frac{p^m}{k} \cdot \prod_{i=1}^{k-1} \frac{p^m - p^{j_i} q_i}{p^{j_i} q_i} = \frac{p^m}{k} \cdot \prod_{i=1}^{k-1} \frac{p^{m-j_i} - q_i}{q_i},$$

where $i = p^{j_i} q_i$ and $p \nmid q_i$ for each $i \in [k-1]$. Since p divides none of the q_i's, we have the result. ☐

Problem 2.13. Give a proof that for all positive integers n, $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Proof. The left-hand-side counts all the NE lattice paths $(0,0) \to (n,n)$. The term $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$ counts the number of NE lattice paths $(0,0) \to (n,n)$ going through $(n-k,k)$. Since each NE lattice paths goes through one of these points on the diagonal, we have the result. ☐

Problem 2.14. Prove that for all integers n, the equality

$$\sum_{2|k\geq 0} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}$$

holds.

Proof.

$$(2+1)^n + (2-1)^n = \sum_{k\geq 0} \binom{n}{k} 2^{n-k} + \sum_{k \geq 0} \binom{n}{k} 2^{n-k} (-1)^k = 2 \sum_{2|k\geq 0} \binom{n}{k} 2^k.$$

☐

Problem 2.15. How many strings of length n can be formed using the alphabet $\{A, B, C, D, E\}$ if

(a) the letter A appears an odd number of times.

(b) The letters A and B are both used an odd number of times.

Proof. (a) We can pick an odd number of positions to place A, then place a word with the rest of letters on the rest of the positions. This yields

$$\begin{aligned}
\sum_{k \text{ odd}} \binom{n}{k} 4^{n-k} &= \frac{1}{2} \left(\sum_{k=0}^{n} \binom{n}{k} 4^{n-k} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{n-k} \right) \\
&= \frac{1}{2}((1+4)^n - (-1+4)^n) \\
&= \frac{5^n - 3^n}{2}.
\end{aligned}$$

(b) Similarly, we can pick an odd number of positions to place B, then put a word with an odd number of A's (and no B's) on the remaining positions. So the number of ways to put have a word with an odd number of A's and no B's is given by

$$\sum_{k \text{ odd}} \binom{n}{k} 3^{n-k} = \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} - \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k}\right)$$
$$= \frac{1}{2}((1+3)^n - (-1+3)^n)$$
$$= \frac{4^n - 2^n}{2}.$$

So the desired number is then

$$\sum_{k \text{ odd}} \binom{n}{k} \left(\frac{4^{n-k} - 2^{n-k}}{2}\right) = \frac{1}{2}\left(\sum_{k \text{ odd}} \binom{n}{k} 4^{n-k} - \sum_{k \text{ odd}} \binom{n}{k} 2^{n-k}\right)$$
$$= \frac{1}{2}\left((1+4)^n - (-1+4)^n - (1+2)^n + (-1+2)^n\right)$$
$$= \frac{5^n - 2 \cdot 3^n + 1}{2}.$$

□

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3 Partitions

Reference: *A Walk Through Combinatorics* - Bóna, Chapter 5.

3.1 Compositions

Definition. A *weak composition* of n is a sequence $(a_1, a_2, \ldots, a_k)$ of non-negative integers such that $a_1 + a_2 + \cdots + a_k = n$. A *composition* of n is a sequence $(a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

Theorem 3.1. *For all $n, k \in \mathbb{Z}^+$, the number of weak compositions of n into k parts is*

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

That is, the number of linear orderings of n stars and $k-1$ bars.

Theorem 3.2. *For all $n, k \in \mathbb{Z}^+$, the number of compositions of n into k parts is*

$$\binom{n-1}{k-1}.$$

That is, the number of ways of placing $k-1$ bars into the $n-1$ spaces between n stars.

Theorem 3.3. *Related to above, the number of k element subsets of an n-set with repetition is given by*

$$\left(\!\binom{n}{k}\!\right) := \binom{n+k-1}{n-1} = \binom{n+k-1}{k},$$

called n multichoose k.

That is, the number of linear orderings of k stars and $n-1$ bars.

Problems

Problem 3.4. Find the number of compositions of n into an odd number of parts.

Proof. The answer is 2^{n-2}, i.e. the number of compositions of $n-1$. That is, map the composition $(a_1, \ldots, a_k)$ of $n-1$ to $(a_1, \ldots, a_k+1)$ if k is odd, and to $(a_1, \ldots, a_k, 1)$ if k is even. The inverse map is clear since we need only look at if the last entry of the composition is 1 or not. □

Problem 3.5. Find the number of all compositions of n into parts that are odd.

Proof. Denote the number by $f(n)$. Then it is clear that $f(1) = 1$ and $f(2) = 1$, and we will show that

$$f(n) = f(n-1) + f(n-2)$$

for all $n \geq 3$, and hence $f(n)$ is the Fibonacci number. That is, take a composition $(a_1, \ldots, a_k)$ of $n-1$ into odd parts, map it to $(a_1, \ldots, a_k, 1)$, and it is clear we obtain every desired composition ending in 1. Then take the composition $(b_1, \ldots, b_j)$ of $n-2$ into odd parts, and map that to $(b_1, \ldots, b_j+2)$, and it is clear we obtain every desired composition not ending in 1 in this way. This gives the result. □

Problem 3.6. Let $\bar{c}(m, n)$ denote the number of compositions of n with largest part at most m. Find $\sum_{n\geq 0} \bar{c}(m, n)x^n$.

Proof. We have

$$\begin{aligned}
\sum_{n\geq 0} \bar{c}(m,n)x^n &= \sum_{k\geq 0}(x + x^2 + \cdots + x^m)^k \\
&= \sum_{k\geq 0}\left(\frac{x - x^{m+1}}{1-x}\right)^k \\
&= \frac{1-x}{1-2x+x^{m+1}}.
\end{aligned}$$

□

3.2 Set Partitions

Definition. The number of set partitions of $[n]$ into k parts is denoted $S(n, k)$, called the *Stirling number of the second kind*.

Theorem 3.7. *The Stirling numbers of the second kind satisfy the recurrence*

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k).$$

Theorem 3.8. *For all $x \in \mathbb{R}$ and all nonnegative integers n,*

$$x^n = \sum_{k=0}^{n} S(n, k)(x)_k.$$

For $x \in \mathbb{Z}^+$, both sides count x-colorings of $[n]$, i.e. both sides are polynomials that agree at infinitely many points.

Definition. The number of all set partitions of $[n]$ is denoted $B(n) = \sum_{k \geq 1} S(n,k)$, called the n^{th} Bell Number.

Theorem 3.9. *The Bell numbers satisfy the recurrence*

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i).$$

Problems

Problem 3.10. Let $F(n)$ be the number of partitions of $[n]$ that contain no singleton blocks. Prove that if $n \geq 1$, then $F(n) + F(n+1) = B(n)$, where $B(n)$ is the n^{th} Bell number.

Proof. Take any partition of $[n]$ (enumerated by $B(n)$). If it has no singletons, then map it to itself, accounting for all of the partitions enumerated by $F(n)$. If it has singletons, put all of the singletons in a block with $n+1$, accounting for all the terms counted by $F(n+1)$. The inverse map is clear. □

Problem 3.11. How many partitions does $[n]$ have in which no block contains two consecutive integers?

Proof. We see that it is $B(n-1)$, the $(n-1)^{st}$ Bell number. That is, take any partition of $[n]$ in which no block contains consecutive integers, take the block with n in it, delete n, then add the remaining integers of the block to the block containing the integer one greater than it. The inverse map is not terribly difficult to see. □

3.3 Integer Partitions

Theorem 3.12. *Let $p(n)$ denote the number of integer partitions of n. Then*

$$\sum_{n \geq 0} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}.$$

Definition. A *Ferrers shape* of a partition $\lambda = \{x_1, x_2, \ldots, x_k\}$ of n is a set of n boxes with horizontal and vertical sides so that the i^{th} row has x_i boxes and all rows start at the same vertical line. The *conjugate* partitions of λ is the partition corresponding to the Ferrers shape obtained by reflecting the Ferrers shape of λ over its diagonal.

Theorem 3.13. *It is immediate from the conjugation map that the number of partitions of n into at most k parts is equal to that of the partitions of n into parts not larger that k.*

Problems

Problem 3.14. Let $q(n)$ be the number of paritions of n in which each part is at least 2. Show $q(n) = p(n) - p(n-1)$ for all positive integers $n \geq 2$.

Proof. We have a bijection from the set of partitions of $n-1$ to the set of partitions of n with smallest part 1: add the part 1 to each partition of $n-1$. □

Problem 3.15. Let $p_d(n)$ and $p_{odd}(n)$ denote the numbers of partitions of n into distinct and odd parts, respectively. Prove that $p_d(n) = p_{odd(n)}$.

Proof.

$$\begin{aligned}\sum_{n\geq 0} p_d(n)x^n &= (1+x)(1+x^2)(1+x^3)\dots \\ &= \frac{(1+x)(1-x)}{1-x}\cdot\frac{(1+x^2)(1-x^2)}{1-x^2}\cdot\frac{(1+x^3)(1-x^3)}{1-x^3}\cdot\dots \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\dots \\ &= \sum_{n\geq 0} p_{odd}(n)x^n.\end{aligned}$$

□

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4 Permutations

Reference: *A Walk Through Combinatorics* - Bóna, Chapters 6 and 14, and *Combinatorics of Permutations* - Bóna.

4.1 Linear Orders

Theorem 4.1. *The number of n-permutations with $k-1$ descents, (equivalently, k ascending runs), is denoted $A(n,k)$. These numbers are called Eulerian numbers, and they satisfy the recurrence:*

$$A(n, k+1) = (k+1)A(n-1, k+1) + (n-k)A(n-1, k).$$

Theorem 4.2. *The number of alternating (i.e. down-up) n-permutations is denoted E_n, called the n^{th} Euler number. Noting that an $(n+1)$-alternating permutation can be written as $L(n+1)R$ where both R and L^r are reverse alternating, we obtain the recurrence*

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k},$$

and this yields the generating function

$$\sum_{n\geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z.$$

Definition. The number i is called a *weak excedance* of $w \in S_n$ if $w_i \geq i$ and an *excedance* if $w_i > i$.

Proposition 4.3. *The number of $w \in S_n$ with k weak excedances equals the is equal to the number of $w \in S_n$ with $n - k$ excedances.*

To see this, take the reverse complement: $w_i \geq i \iff n + 1 - w_i \leq n + 1 - i$.

Problems

Problem 4.4. How many permutations $a_1 a_2 \dots a_n \in S_n$ satisfy the property: if $2 \leq j \leq n$, then $|a_i - a_j| = 1$ for some $1 \leq i < j$? Equivalently, for all k, the set $\{a_1, a_2, \dots, a_k\}$ consists of consecutive integers in some order.

Proof. We see that we can build every such permutation by choosing arbitrarily a first element, then picking the next element to be one more or one less than all of the previous elements continually. This is then counted by

$$\sum_{k=1}^{n} \binom{n-1}{k-1},$$

where k can be thought of as the first element of the permutation, then the $k - 1$ elements chosen from the remaining positions are where we pick the 'lesser element' option. □

4.2 Inversions

Definition. The *inversion table* of $w \in S_n$ is a vector $I(w) := (a_1, a_2, \dots, a_n) \in [n-1] \times [n-2] \times \cdots \times [0]$, where a_i is the number of entries j in w to the left of i satisfying $j > i$. The *code* of $w \in S_n$ is given by $\text{code}(w) := I(w^{-1})$, that is, $\text{code}(w) := (c_1, \dots, c_n)$ is a vector where c_i is equal to the number of elements w_j such that $i < j$ and $w_i > w_j$.

Proposition 4.5. *The maps* $\text{code}, I : S_n \to [n-1] \times [n-2] \times \cdots \times [0]$ *are bijections.*

Theorem 4.6. *Let* $\text{inv}(p)$ *denote the number of inversions of* $p \in S_n$, *then*

$$\sum_{p \in S_n} q^{\text{inv}(p)} = (1+q)(1+q+q^2)\dots(1+q+\cdots+q^{n-1}).$$

Note. This generating function is sometimes denoted $(n)_q!$, the q-analog of $n!$. The following permutation has the same generating function, and is hence called equidistributed with inversions.

Definition. The *major index* of $w \in S_n$ is the sum of the elements of its descent set $D(w)$. That is, $\text{maj}(w) = \sum_{i \in D(w)} i$.

Definition. Here we list some q-analogs related to the generating function for inversions. These are called q-analogs because setting $q = 1$ will give the original object. As we just saw we have

$$(n)_q! = (1)_q(2)_q(3)_q \dots (n)_q = (1)(1+q)(1+q+q^2)\dots(1+q+\cdots+q^{n-1}),$$

the q-analog of n factorial. Then we have *Gaussian coefficients*, also called q-binomial coefficients, given by

$$\begin{aligned}\binom{n}{k}_q &= \frac{(n)_q!}{(n-k)_q!(k)_q!}\\ &= \frac{(n)_q(n-1)_q\dots(n-k+1)_q}{(k)_q(k-1)_q\dots(2)_q(1)_q}\\ &= \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})},\end{aligned}$$

where the last line is obtained by using the identity $q^m - 1 = (q-1)(q^{m-1} + \cdots + q + 1)$, and multiplying the numerator and denominator of the second line by $q \cdot q^2 \cdot q^3 \dots q^{k-1}$.

Theorem 4.7. *If q is a prime power, then $\binom{n}{k}_q$ is the number of all k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$.*

Proof. We first see that $(q^n-1)(q^n-q)\dots(q^n-q^{k-1})$ counts the number of ordered sequences of k linearly independent vectors of the vector space. That is, we choose a non-zero vector in $q^n - 1$ ways for the first vector, then a second vector not in the span of that vector in $q^n - q$ ways, then a third vector not in the span of the first two in $q^n - q^2$ ways, etc.

Second, we see that $(q^k-1)(q^k-q)\dots(q^k-q^{k-1})$ counts the number of ordered bases of a k-dimensional vector space over $\mathbb{F}_q$. This is shown in the same way as was done in the last paragraph. Hence the term $(q^n-1)(q^n-q)\dots(q^n-q^{k-1})$ counts every k-dimensional subspace exactly $(q^k-1)(q^k-q)\dots(q^k-q^{k-1})$ times, yielding the result. □

Theorem 4.8. *For all $n, k \geq 1$, we have*

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q.$$

Proof. Denoting our vector space $\mathbb{F}_q^m$, let $\pi : \mathbb{F}_q^m \to \mathbb{F}_q^{m-1}$ be a projection with one-dimensional nullspace N. Suppose V is a k-dimensional subspace, (what is enumerated by the left-hand side). If $N \not\subseteq V$, then $\pi(V)$ is k-dimensional, and hence is counted by $q^k\binom{n-1}{k}_q$ since... If $N \subseteq V$, then $\pi(V)$ is $(k-1)$-dimensional, and hence is counted by $\binom{n-1}{k-1}_q$. □

Problems

Problem 4.9. Let $n \geq 3$. How many permutations of length n are there whose number of inversions is divisible by 3?

Proof. We have that

$$\sum_{p\in S_n} q^{\mathrm{inv}(p)} = (1+q)(1+q+q^2)\dots(1+q+\cdots+q^{n-1}) = (1+q+q^2)p(x),$$

where $p(x)$ is some polynomial. Then it is clear by the right hand side that for the number of permutations with number of inversions in the same congruence class modulo 3 is $p(1)$, and therefore the answer is $1/3$. □

Problem 4.10. Let Π_n be the set of permutations π of $[n]$ such that $\pi(i+1) \leq \pi(i)+1$ for $i \in [n-1]$. Find $|\Pi_n|$.

Proof. We note that every such permutation has n in the first position or after $n-1$. This gives $|T_n| = 2 \cdot |T_{n-1}| = 2^{n-1}$. □

Problem 4.11. Let G_n be the graph whose vertex set is the set of all permutations of $[n]$ (in one line notation) with two vertices adjacent if they differ by switching two consecutive entries in the permutation. What is the chromatic number of G_n.

Proof. If $n = 1$ it is clearly one. So suppose $n \geq 2$. We see that since G_n has at least one edge, we have $\chi(G_n) \geq 2$. We show $\chi(G_n) = 2$. That is, color each of the vertices with an even number of inversions with red and each of the vertices with an odd number of vertices blue. Then since interchanging two consecutive entries of a permutation changes the number of inversions by $+1$ or -1, we have that this is a proper coloring. This gives the result. □

4.3 Cycles

Definition. A permutation $w \in S_n$ is *odd* if it has an odd number of inversions, or *even* if it has an even number of inversions.

Proposition 4.12. *For $w \in S_n$, then w is even if and only if the number of even length cycles of w is even.*

Definition. Let $w \in S_n$ have exactly a_i cycles of length i. Then the *type* of w is given by the vector $(a_1, a_2, \ldots, a_n)$.

Proposition 4.13. *The number of permutations in S_n of cycle type $(a_1, a_2, \ldots, a_n)$ is*

$$\frac{n!}{a_1!a_2!\ldots a_n!1^{a_1}2^{a_2}\ldots n^{a_n}}.$$

Proposition 4.14. *Two elements of S_n are conjugates in S_n if and only if they are of the came cycle types. That is, for $\sigma = (a_{1_1}a_{1_2}\ldots a_{1_k})\ldots(a_{m_1}a_{m_2}\ldots a_{m_j}) \in S_n$, for any $\tau \in S_n$ we have*

$$\tau\sigma\tau^{-1} = (\tau(a_{1_1})\tau(a_{1_2})\ldots\tau(a_{1_k}))\ldots(\tau(a_{m_1})\tau(a_{m_2})\ldots\tau(a_{m_j})).$$

Definition. The number of permutations $w \in S_n$ with k cycles is called the *signless Stirling number of the first kind*, and is denoted $c(n,k)$.

Proposition 4.15. *Set $c(n,0) = 0$ if $n \geq 1$ and $c(0,0) = 1$. Then the numbers $c(n,k)$ satisfy the recurrence*

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).$$

Theorem 4.16. *For all $n \in \mathbb{Z}^+$, the equality*

$$z(z+1)\ldots(z+n-1) = \sum_{k=0}^{n} c(n,k)z^k,$$

holds.

We can see this by showing that the coefficients of the polynomials on the left side satisfy the same recurrence as the $c(n,k)$. What happens when we evaluate at $z = -1$?

Theorem 4.17 (Transition Lemma). *Let $w \in S_n$ be written in canonical cycle notation, (i.e. largest element first in each cycle, order cycles in increasing order by largest elements), and let $f(w)$ be the permutation obtained by omitting all parentheses. Then the map $f : S_n \to S_n$ is a bijection.*

Definition. A permuation $w \in S_n$ is called a *derangement* if it has no fixed-points, and let set of them be denoted D_n.

Proposition 4.18. *Some important formulas regarding derangements:*

i. $D(z) := \sum_{n\geq 1} D_n \frac{z^n}{n!} = \frac{e^{-z}}{1-z}$;

ii. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right)$;

iii. $D_n = (n-1)(D_{n-1} + D_{n-2})$;

iv. $D_n = n \cdot D_{n-1} + (-1)^n$.

Proof. First, **i.** is solved by simple application of exponential formula for generating functions.

Second, **ii.** is solved using inclusion-exclusion:

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 1! = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right).$$

Third, **iii.** is clear by seeing that $(n-1)D_{n-1}$ counts the number of derangements of length n in which n is in a cycle of length ≥ 3, and $(n-1)D_{n-2}$ counts those with n in a cycle of length 2.

Finally, **iv.** follows from **ii.**:

$$\begin{aligned} D_{n-1} &= (n-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-1}}{n-1!}\right) \\ \implies n \cdot D_{n-1} + (-1)^n &= (n)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-1}}{n-1!}\right) + (-1)^n \\ \implies n \cdot D_{n-1} + (-1)^n &= D_n. \end{aligned}$$

□

Problems

Problem 4.19. Prove that the average number of cycles of a randomly chosen $[n]$-permutation is the n^{th} Harmonic number $H(n)$.

Proof. The average number of k-cycles is $\binom{n}{k}\frac{(k-1)!(n-k)!}{n!} = \frac{1}{k}$, and then use linearity of expectation. Also can take derivative of Theorem 4.16 and evaluate at $z = 1$. □

Problem 4.20. How many permutations $w \in S_n$ have the same number of cycles as weak excedances.

Proof. It is easy to see that every cycle admits at least one weak excedance, and the cycle $(x_1\ x_2 \ldots x_k)$ admits exactly one weak excedance if $x_1 > x_2 > \cdots > x_k$. Thus the number of permutations with as many cycles as weak excedances is $B(n)$, the n^{th} Bell number. □

Problem 4.21. How many permutations of length n are there in which the cycle containing the entry 7 is of length exactly k.

Proof. For $n < 7$, the answer is clearly 0, so suppose $n \geq 7$. This is the same question as asking how many permutations have n in a cycle of length k. By the transition lemma, the number of permutations with n in a cycle of length k is equal to the number of permutations with n in position $n - k + 1$. That is, there are $(n-1)!$ permutations. □

Problem 4.22. How many permutations $p = p_1 p_2 \ldots p_n$ of length n are there in which $p_i \neq i + 1$ if $i \in [n-1]$? For instance, for n=3, there are three such permutations: 123, 321, and 312.

Proof. We see that the answer is $D_n + D_{n-1}$. That is, mapping a derangement $a_1 a_2 \ldots a_n$ of length n to $a_2 a_3 \ldots a_n a_1$, we achieve every desired permutation such that 1 is not in the last position. Now the number of desired permutations with 1 in the last position is then D_{n-1}: take a derangement $b_1 b_2 \ldots b_{n-1}$, and map this to $(b_1+1)(b_2+1)\ldots(b_{n-1}+1)1$. □

Problem 4.23. Prove that the number of permutations of length $n+1$ with exactly two cycles is equal to the number of all cycles of all n-permutations.

Proof. We first count the number of all cycles of all n-permutations. We see that the number of all k-cycles of all n-permutations is given by

$$\binom{n}{k}(k-1)!(n-k)!.$$

Thus the number of all cycles is obtained by summing over all k as follows:

$$\sum_{k=1}^{n}\binom{n}{k}(k-1)!(n-k)!.$$

We see that this sum also counts all permutations of length $n+1$ with exactly two cycles since we choose must k, $0 \leq k \leq n-1$, elements from $[n]$ to be in a cycle without $n+1$, form a cycle on those elements, then form a cycle on the remaining elements with $n+1$. This gives the result. □

Problem 4.24. Pick a random cycle from all $n!$ permutations of length n. On average, what is the length of the selected cycle.

Proof. Let $X(C)$ be the random variable indicating the length of the cycles C. We find the expectation of X. We know from above that the average number of cycles in an n-permutation is $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Thus there are $n! \cdot H_n$ total cycles in all n-permutations. We also know that for each $k \in [n]$ that the total number of k-cycles is

$$\binom{n}{k}(k-1)!(n-k)! = \frac{n!}{k}.$$

So by linearity of expectation we have

$$\mathbb{E}(X) = \sum_{k=1}^{n} k \cdot \frac{n!/k}{n! \cdot H_n} = \sum_{k=1}^{n} \frac{1}{H_n} = \frac{n}{H_n}.$$

□

Problem 4.25. Let A_n be the number of all even derangements of length n, and let B_n be the number of all odd derangements of length n. Find an exact formula for $A_n - B_n$.

Proof. Using the cycle type indicator $\sum_{n\geq 0}\sum_{w\in S_n} t^{type(w)}\frac{x^n}{n!} = \exp(t_1\frac{x}{1} + t_2\frac{x^2}{2} + t_3\frac{x^3}{3} + \dots)$, it is clear that the desired exponential generating function for the sequence $A_n - B_n$ is given by

$$\exp\left(-\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right).$$

We see that $-\log(1+x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$, and thus $\log(1+x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, and so we have our desired generating function:

$$\exp(\log(1+x) - x) = (1+x)e^{-x} = (1+x)\left(\sum_{n\geq 0}(-1)^n\frac{x^n}{n!}\right).$$

Simple coefficient extraction yields

$$A_n - B_n = (-1)^n + n(-1)^{n-1} = (-1)^{n-1}(n-1).$$

□

4.4 Pattern Avoidance

Definition. Let $q = q_1 \dots q_k \in S_k$ be a permutation and let $k \leq n$. We say that the permutation $p = p_1 \dots p_n \in S_n$ *contains* q as a pattern is there are k entries $p_{i_1} \dots p_{i_k}$ in p so that $i_1 < \dots < i_k$ and $p_{i_a} < p_{i_b}$ is and only if $q_a < q_b$. If p does not contain q, then we say that p *avoids* q. We let $\mathrm{Av}_n(q)$ denote the number of permutations in S_n that avoid q.

Proposition 4.26. *It is clear that* $\mathrm{Av}_n(132) = \mathrm{Av}_n(231) = \mathrm{Av}_n(312) = \mathrm{Av}_n(213)$ *and that* $\mathrm{Av}_n(123) = \mathrm{Av}_n(321)$ *via the operations of complement and reverse on* S_n.

Theorem 4.27. *For all* $q \in S_3$ *and* $n \geq 3$, *we have* $\mathrm{Av}_n(q) = C_n = \binom{2n}{n}/(n+1)$, *i.e. the* n^{th} *Catalan number.*

Proof. By the proposition, all we need to show is that $\mathrm{Av}_n(132) = C_n$ and $\mathrm{Av}_n(321) = C_n$. Recall that the number of lattices paths on an $n \times n$ grid from one corner to another only on one side of the diagonal is C_n.

First, we see that a permutation $w \in S_n$ avoids 132 if and only if its diagram is a Ferrers shape contained in the staircase shape $[n-1] \times \dots \times [1] \times [0]$. That is, permutations avoiding 132 are in bijection with NE lattice paths from $(0,0)$ to (n,n) above the diagonal.

Second, a permutation $w \in S_n$ avoids 321 if and only if after removing its right-to-left maxima, we are left with another decreasing subsequence. That is, given a NE lattice path from $(0,0)$ to (n,n) that stays below the diagonal, we obtain the permutation matrix of a 321-avoiding permutation by filling in the inner corners above the lattice path (corresponding to right-to-left maxima), and the rest of the entries of the permutation are determined since they must form a decreasing subsequence. It is clear we can obtain each such permutation in that way. ☐

Problems

Problem 4.28. Let X be defined on the set of all 132-avoiding permutations of length n so that $X(p)$ is the number of of right-to-left maxima of p. Compute $\mathbb{E}(X)$.

Proof. Let X_i be the indicator of the event that the i^{th} position of p has a right-to-left maximum. This must mean that all of the entries in $p_{i+1}\dots p_n$ are exactly those in $[n-i]$ since we must avoid 132. Hence the sub-permuations $p_1\dots p_i$ and $p_{i+1}\dots p_n$ are 132-avoiders, and recalling that $\mathrm{Av}_n(132) = C_n$, we have that $\mathbb{E}(X_i) = \frac{C_iC_{n-i}}{C_n}$, and thus

$$\mathbb{E}(X) = \sum_{i=1}^{n} \frac{C_iC_{n-i}}{C_n} = \frac{C_{n+1}-C_n}{C_n} = \frac{3n}{n+2},$$

using the recurrence relation $\sum_{i=0}^{n} C_iC_{n-i} = C_{n+1}$. ☐

Problem 4.29. Find a formula for each of the following:

1. $\mathrm{Av}_n(132, 123)$,
2. $\mathrm{Av}_n(132, 231)$,
3. $\mathrm{Av}_n(132, 321)$,
4. $\mathrm{Av}_n(231, 312)$.

Proof. 1. We see that $\mathrm{Av}_n(132, 123) = 2^{n-1}$. We prove this by induction on n, with the case $n = 1$ being trivial. Taking all permutations of length $n-1$ avoiding these patterns, if we add 1 to each entry and then place 1 in the last or next to last spot, then we obtain 2 unique n-permutations avoiding 132 and 123. To see that we obtain all such permutations in this way, we see that 1 must be in the last two positions because any two entries after 1 would yield one of the forbidden patterns. Hence $\mathrm{Av}_n(132, 123) = 2 \cdot \mathrm{Av}_{n-1}(132, 123) = 2^{n-1}$.

2. Similarly, we have that $\mathrm{Av}_n(132, 231) = 2^{n-1}$ by induction. That is, if n isn't in the first or last position of an n-permutation, then any entries before and after n would lead to a forbidden pattern. Hence we obtain all the desired n-permutations by placing n at the beginning or end of an $(n-1)$-permutation avoiding these two patterns. Hence $\mathrm{Av}_n(132, 231) = 2 \cdot \mathrm{Av}_{n-1}(132, 231) = 2^{n-1}$.

3. To avoid 132 every entry before n must be larger than every entry after n, and to avoid 321, every entry after n must be in increasing order. Thus if n is not in the last position, it also must be that all of the entries before n must also be in increasing order, since any descent before n with an entry after n would then form a 321 pattern. Hence we have $n-1$ permutations of the form $(n-i+1)(n-i+2)\dots n12\dots(n-i)$. The n-permutations avoiding these patterns with n in the last position are simply those enumerated by $\mathrm{Av}_{n-1}(132, 321)$, obtained by simply appending n to the end, and thus $\mathrm{Av}_n(132, 321) = (n-1) + \mathrm{Av}_{n-1}(132, 321) = 1 + \binom{n}{2}$.

4. We note these are called layered permutations. That is, to avoid 312, everything after a descending run must be larger than anything in that run. Also, to avoid 231, If some entry is larger than an entry before it, then everything smaller than that entry is to the left of that smaller entry. So the first descending run must include each of the smallest elements of $[n]$, then the next descending run must consistent of the next smallest elements of $[n]$, etc. So the number of layered permutations is equivalent to the number of compositions of n, i.e. $\text{Av}_n(312, 231) = 2^{n-1}$. □

4.5 Robinson-Schensted-Knuth Correspondence

Theorem 4.30 (RSK Correspondence). *Let S be the set of all pairs (P, Q) of Standard Young Tableaux on $[n]$ with the same Ferrers shape. Then the map* $\text{rsk} : S_n \to S$ *is a bijection.*

Proof. Here we include an example of the RSK insertion algorithm, with the inverse clear and hence proving the theorem. So we find P and Q for $\text{rsk}(52314) = (P, Q)$. We insert the entries of 52314 from left to right:

$$5: \quad P = \begin{array}{|c|}\hline 5 \\ \hline\end{array} \qquad\qquad Q = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$2: \quad P = \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline\end{array} \qquad\qquad Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

$$3: \quad P = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$1: \quad P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad Q = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$4: \quad P = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

That is, we insert in the P-tableau with the next entry of the permutation, starting at the first row, following the rules:

i. If it is larger than each of the entries in that row, then insert that number at the end of the row.

ii. If it is not larger than all of the entries, place the number in the box of the smallest number larger than it, then bump the entry that was in that box to the next row, resuming the algorithm at **i**.

Once this P-tableau insertion terminates, we have a Ferrers shape augmented by one box, and so we augment the Q-tableau to the same shape, placing the next unused integer in that box. □

Proposition 4.31. *Here we list some interesting relationships between $\pi \in S_n$ and $\text{rsk}(\pi) = (P(\pi), Q(\pi))$. Let λ be the Ferrers shape of P (or Q).*

a. *For any $w \in S_n, P(\pi)^T = P(\pi^r)$.*

b. *For any $w \in S_n, \text{rsk}(\pi^{-1}) = (Q(\pi), P(\pi))$*

c. *If the longest increasing subsequence of π has length k, then λ has k columns.*

d. *A permutation $\pi \in S_n$ is an involution if and only if $P(\pi) = Q(\pi)$.*

e. *Define $i \in [n]$ to be a descent of a standard Young tableau if $i + 1$ appears in a row strictly above i. Then for $\pi \in S_n$ and $i \in [n-1]$, i is a descent of π if and only if it is a descent of $P(\pi)$.*

5 Inclusion-Exclusion

Reference: *A Walk Through Combinatorics* - Bóna, Chapter 7 and *Enumerative Combinatorics* - Stanley, Chapter 2.

Theorem 5.1 (Principle of Inclusion-Exclusion)**.** *For the collection $A_1, \ldots, A_n$ of finite sets, we have*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\leq i_1\leq\cdots\leq i_k\leq n}|A_{i_1}\cap\cdots\cap A_{i_k}|\right).$$

Proof. Suppose x is in exactly r of the A_i's. Then x is counted exactly

$$\sum_{k=1}^{r}(-1)^{k-1}\binom{r}{k} = 1 - \sum_{k=0}^{r}\binom{r}{k}(-1)^k = 1-(1-1)^r = \begin{cases}1 & \text{if } r>0,\\ 0 & \text{if } r=0.\end{cases}$$

times. This gives the result. □

Theorem 5.2. $g(S) = \sum_{T\subseteq S} f(T) \iff f(S) = \sum_{T\subseteq S}(-1)^{|S-T|}g(T)$.

Note. Need to find some good examples to put in here.

Problems

Problem 5.3. How many ways are there to set n married couples at a straight table so that no woman sits next to her husband?

Proof. We count permutations of $M = \{1^2, 2^2, \ldots, n^2\}$ so that no two consecutive integers are equal. We use inclusion-exclusion to count the number of such permutations in which at least one pair of consecutive entries are equal. We see that the number of permutations of M such that each of the entries of $[i]$ appear in consecutive positions is equal to the number of permutations of $\{1, 2, \ldots, i, (i+1)^2, \ldots, n^2\}$, (simply input a k next to k for each $k \in [i]$). It is clear the number of such permutations is then $(2n-i)!2^{-(n-i)}$. So the number of permutations of M with at least one pair of consecutive entries being equal is

$$\binom{n}{1}(2n-1)!\cdot 2^{-n+1} - \binom{n}{2}(2n-2)!\cdot 2^{-n+2} + \dots (-1)^{n-1}\binom{n}{n}(2n-n)!\cdot 2^{-n+n}.$$

Thus the desired number is

$$(2n)!\cdot 2^{-n} - \binom{n}{1}(2n-1)!\cdot 2^{-n+1} + \cdots + (-1)^n\binom{n}{n}(2n-n)!\cdot 2^{-n+n},$$

and in summation notation:

$$\sum_{i=0}^{n}\binom{n}{i}(2n-i)!\cdot 2^{-n+i}\cdot(-1)^i.$$

□

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6 Generating Functions

Reference: *A Walk Through Combinatorics* - Bóna, Chapter 8 and *Enumerative Combinatorics* - Stanley, Chapters 4, 5 and 6.

6.1 Ordinary Generating Functions

Theorem 6.1. *Let a_n and b_n count the number of ways to build some structures on an n-element set, with $A(x)$ and $B(x)$ their ordinary generating functions.*

- Product Formula: *Let c_n count the number of ways to separate $[n]$ into intervals $S = \{1,\dots,i\}$ and $T = \{i+1,\dots,n\}$, build a structure of type a on S and of type b on T, with $C(x)$ their ordinary generating function. Then*

$$C(x) = A(x)B(x) = \sum_{n\geq 0}\sum_{i=0}^{n} a_i b_{n-i} x^n.$$

- Composition Formula: *Suppose $a_0 = 0, b_0 = 1$. Let g_n count the number of ways to split $[n]$ into an unspecified number of intervals, build a structure of type a on each interval and a structure of type b on the set of intervals, with $G(X)$ their ordinary generating function. Then*

$$G(X) = B(A(X)) = \sum_{k\geq 0} b_k A(x)^k.$$

Problems

Problem 6.2. Let ℓ_n be the total number of leaves in all unlabeled plane binary trees.

(a) Find the generating function $L(z) = \sum_{n\geq 1}\ell_n z^n$ in closed form.

(b) Find an explicit formula for the numbers ℓ_n.

Proof. (a) Noting that cutting off the root of such a trees yields a possibly entry right and left subtree of the same type, we see that

$$L(z) = z + 2zL(z)T(z),$$

where $T(z)$ is the ordinary generating function of the number of the number of such trees on n vertices. We know that these are enumerated be the Catalan numbers, and hence

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

but we can also find this formula by using the quadratic formula on the functional equation

$$T(z) = 1 + zT(z)^2.$$

So we have

$$L(z) = z + 2zL(z)\frac{1 - \sqrt{1 - 4z}}{2z} = x + L(z)(1 - \sqrt{1 - 4z}),$$

which implies

$$L(z) = \frac{z}{\sqrt{1 - 4z}} = z(1 - 4z)^{-\frac{1}{2}}.$$

(b) We now use the binomial theorem to extract coefficients. So we have

$$L(z) = z \sum_{n \geq 0} \binom{-\frac{1}{2}}{n} (-4)^n z^n = \sum_{n \geq 0} \frac{(2n-1)!!}{n!} 2^n z^{n+1}$$

Hence we have

$$\ell_n = \frac{(2n-3)!!2^{n-1}}{(n-1)!}$$

□

Problem 6.3. Find a generating function for the number of positive integer solutions to $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = n$.

Proof. By simple application of the product rule for ordinary generating functions, the generating function is clear:

$$\prod_{i=1}^{5} \frac{x^i}{1 - x^i} = (x + x^2 + \dots)(x^2 + x^4 + \dots)(x^3 + x^6 + \dots)(x^4 + x^8 + \dots)(x^5 + x^{10} + \dots).$$

□

6.2 Exponential Generating Functions

Theorem 6.4. *Let a_n and b_n count the number of ways to build some structures on an n-element set, with $A(x)$ and $B(x)$ their exponential generating functions.*

- Product Formula: *Let c_n count the number of ways to separate $[n]$ into sets S and T such that $S \cup T = [n], S \cap T = \varnothing$, build a structure of type a on S and of type b on T, with $C(x)$ their exponential generating function. Then*

$$C(x) = A(x)B(x) = \sum_{n \geq 0} \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \frac{x^n}{n!}.$$

- Composition Formula: *Suppose $a_0 = 0, b_0 = 1$. Let g_n count the number of ways to partition $[n]$ into any number of non-empty subsets, build a structure of type a on each interval and a structure of type b on the set of subsets, with $G(X)$ their exponential generating function. Then*

$$G(X) = B(A(X)) = \sum_{n \geq 0} b_n \frac{A(x)^n}{n!}.$$

Note. Very useful: $\frac{d}{dx}(\sum_{n\geq 0} a_n \frac{x^n}{n!}) = \sum_{n \geq 0} a_{n+1} \frac{x^n}{n!}$.

Problems

Problem 6.5. Let $t(n)$ be the number of of ways to color the integers $[n]$ using only the colors red, blue and yellow so that each color is used an odd number of times. Find a closed form for $t(n)$.

Proof. Letting $T(x) = \sum_{n \geq 0} t(n) \frac{x^n}{n!}$, we see that if $H(x) = \sum_{n \text{ odd}} \frac{x^n}{n!} = \frac{1}{2}(e^x - e^{-x})$, then $T(x) = H(x)^3$. Extracting coefficients, we have $t(n) = 0$ for even n and $\frac{1}{4}(3^n - 3)$ for n odd. □

Problem 6.6. Let h_n be the number of all permutations of length n in which all cycles are of even length. So $h_n = 0$ if n is odd. Find the exponential generating function of the numbers h_n.

Proof. Letting $A(x) = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots$, we see that $A(x) = \log \frac{1}{\sqrt{1-x^2}}$, and hence we have

$$H(x) = e^{A(x)} = \frac{1}{\sqrt{1-x^2}}.$$

□

Problem 6.7. A binary total partition of $[n]$ is a sequence of partitions achieved by repeatedly breaking up each non-singleton block into exactly two block. It can be viewed at as unordered binary trees with leaves labeled by the elements of $[n]$. Find the exponential generating function $B(x) = \sum_{n \geq 1} b(n) \frac{x^n}{n!}$, where $b(n)$ is the number of binary total partitions.

Proof. Some though yields:

$$\frac{1}{2} B(x)^2 = B(x) - x.$$

That is, the product rule on the left-hand side takes advantage of the fact that a binary total partition of $[n]$ is the same as taking a partition (S, T) of $[n]$ and then taking a total binary partition of each of those sets. Using the quadratic formula and extracting coefficients we see that $b(n) = (2n-3)!!$. □

Problem 6.8. Define total partitions similarly to the previous exercise, except non-singleton blocks can be broken up into any number of blocks greater than or equal to 2. These can be depicted by non-plane trees with leaves labeled with $[n]$. Letting $t(n)$ enumerate these objects, find a functional equation for the exponential generating function $T(x) = \sum_{n\geq 1} t(n)\frac{x^n}{n!}$.

Proof. Some thought yields:

$$\exp T(x) - T(x) - 1 = T(x) - x.$$

We see that the left-hand side equals $T(x)^2/2! + T(x)^3/3! + T(x)^4/4! + \dots$. The term $T(x)^k/k!$ corresponds to the binary total partition where the first partition is into k parts, i.e. where the corresponding tree has a root with k children. We subtract x on the right-hand side because the left-hand side clearly doesn't count the binary total partition of [1]. □

Problem 6.9. Decreasing binary trees are trees in which every vertex can have left and right children, on the vertex set $[n]$ and decreasing down the tree, (note they are in bijection with S_n). Let h_n denote the number of all leaves on all decreasing binary trees on $[n]$. Let $H(x) = \sum_{n\geq 1} h_n \frac{x^n}{n!}$. Find a functional equation for $H(x)$.

Proof. Some thought yields:

$$H'(x) = 2H(x)(1-x)^{-1} + 1$$

It is clear that cutting off the root of a DBT we obtain a left and right subtree. So the product $H(x)(1-x)^{-1}$ accounts for the number of leaves in all of the left subtrees, (since $(1-x)^{-1}$ is the exponential generating function of the number of DBT's), and $(1-x)^{-1}H(x)$ accounts for the leaves in all right subtrees. We add one to account for the trees whose root is a leaf. □

Problem 6.10. Let t_n be the number of ways to choose a permutation of length n, and then to color a subset of its even cycles red. Find a closed formula for the exponential generating function of the numbers t_n.

Proof. Letting $A(x) = x + 2\frac{x^2}{2} + \frac{x^3}{3} + 2\frac{x^4}{4} + \dots$, that is, the exponential generating function of ways to put a cycle on the set $[n]$ and color it red or not if it is even, then it is clear we want the function $e^{A(x)}$. Now we need only find $A(x)$. It is known that $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$, and then also that $\frac{x}{1-x^2} = x + x^3 + x^5 + \dots$ implies $-\log\sqrt{1-x^2} = \frac{x^2}{2} + \frac{x^4}{4} + \dots$, (simple u-substitution). Thus

$$A(x) = -(\log(1-x) + \log\sqrt{1-x^2}) = \log\left(\frac{1}{(1-x)\sqrt{1-x^2}}\right),$$

and our desired generating function is then

$$e^{A(x)} = \frac{1}{(1-x)\sqrt{1-x^2}}.$$

□

Problem 6.11. Choose a permutation p of length n uniformly at random. Let a_n be the probability that p has no fixed points and no cycles of length three. Compute $\lim_{n\to\infty} a_n$.

Proof. We find the exponential generating function of such permutations. If p_n is the number of such permutations of length n, and $P(x)$ their exponential generating function, it is clear that

$$P(x) = \sum_{n \geq 0} p_n \frac{x^n}{n!} = \sum_{n \geq 0} a_n x^n,$$

so we need only find the exponential growth rate of the coefficients of $P(x)$. We see that

$$P(x) = \exp\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^5}{5} + \dots\right) = \exp\left(-\log(1-x) - \frac{x^3}{3} - x\right).$$

We see that the smallest in modulus singularity of P is $x = 1$, and thus $\lim_{n\to\infty} a_n^{1/n} = 1 \implies \lim_{n\to\infty} a_n = 1$. ☐

6.3 Rational Generating Functions

Theorem 6.12. *Let $\alpha_1, \dots, \alpha_d$ be a fixed sequence of complex numbers with $\alpha_d \neq 0$. Then the following are equivalent for the function $f : \mathbb{N} \to \mathbb{C}$.*

i. We have

$$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{Q(x)},$$

where $Q(x) = 1 + \alpha_1 x + \dots + \alpha_d x^d$ and $P(x)$ is a polynomial of degree less than d.

ii. For all $n \geq 0$,

$$f(n+d) + \alpha_1 f(n+d-1) + \dots + \alpha_d f(n) = 0.$$

Note. We now outline the *Transfer Matrix Method*. Let $D = (V, E, \phi)$ be a digraph and $w : E \to \mathbb{C}$ a weight function. If $\Gamma = e_1 \dots e_n$ is a walk, then define $w(\Gamma) = w(e_1) \dots w(e_n)$. For $i, j \in [p]$ and $n \in \mathbb{N}$, since D is finite we can define

$$A_{ij}(n) = \sum_{\Gamma} w(\Gamma),$$

where the sum is over all walks of length n from v_i to v_j. In particular, $A_{ij}(0) = \delta_{ij}$. If $w(e) = 1$ for all $e \in E$. then $A_{ij}(n)$ is the number of walk of length n from v_i to v_j. So now define $A = (A_{ij})$ to be the adjacency matrix of D:

$$A_{ij} = \sum w(e)$$

where the sum is over all edges from v_i to v_j.

Theorem 6.13. *We have $(A^n)_{ij} = A_{ij}(n)$.*

Theorem 6.14. *We have $F_{ij}(\lambda) = \sum_{n\geq 0} A_{ij}(n)\lambda^n = \frac{(-1)^{i+j}\det(I-\lambda A : j,i)}{\det(I-\lambda A)}$, where $(I - \lambda A : j, i)$ is the matrix $I - \lambda A$ with the j^{th} row and i^{th} column deleted.*

Corollary 6.15. *Let $c_D(n) := \sum_\Gamma w(\Gamma)$, where the sum is over all closed walks of length n in D, (note $c_D(1)$ equals the trace of A). Let $Q(\lambda) = \det(I - \lambda A)$. Then*

$$\sum_{n\geq 1} c_D(n)\lambda^n = -\frac{\lambda Q'(\lambda)}{Q(\lambda)}.$$

That is $c_D(n)$ equals the trace of A^n, so with $\lambda_1, \ldots, \lambda_q$ the nonzero eigenvalues of A, we have

$$c_D(n) = \text{tr } A^n = \lambda_1^n + \cdots + \lambda_q^n.$$

Problems

Problem 6.16. Let B be the set of words in the alphabet $\{a, b, c\}$ so that the number of a's is even and the number of c's is odd. If b_n is the number of words in B with n letters, find

$$B(x) = \sum_{n\geq 0} b_n x^n.$$

Proof. We find a recurrence relation for b_n. We note that b_n also equals the number of such words with odd a's and even c's by interchanging all the a's and c's. The total number of such words starting with b is then b_{n-1}, and same with the number of words ending in b. Further the number of words starting and ending in b is b_{n-2}. Thus the number of desired words starting or ending in b is $2b_{n-1} - b_{n-2}$ by inclusion-exclusion. Now we find the number of all words starting and ending in a or c. The number of words awa and cwc is then $2b_{n-2}$, and the number of words awc and cwa is also $2b_{n-2}$, (note for these words are the ones in which a's and c's must be swapped). This gives the recurrence $b_n = 2b_{n-1} + 3b_{n-2}$ for all $n \geq 2$, hence multiplying by x^n and summing over all n yields

$$\sum_{n\geq 2} b_n x^n = 2\sum_{n\geq 1} b_n x^{n+1} + 3\sum_{n\geq 0} b_n x^{n+2}.$$

Noting that $b_0 = 0$ and $b_1 = 1$, (the word c), we have that

$$B(x) - x = 2xB(x) + 3x^2 B(x) \implies B(x) - 2xB(x) - 3x^2 B(x) = x,$$

and hence

$$B(x) = \frac{x}{1 - 2x - 3x^2}.$$

□

Problem 6.17. Let $f(n)$ be the number of words of length n over the alphabet $\{a, b, c\}$ that start and end with the same letter and that contain none of the following factors: ac, ba, or ca. Find a closed form for $f(n)$.

Proof. Omitting the corresponding digraph D, we have the adjacency matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

If $c_D(n)$ denotes the number of closed walks of length n in D, then it is clear that $f(n) = c_D(n-1)$. So we compute the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \implies -\lambda^3 + 3\lambda^2 - 2\lambda &= 0 \\ \implies -\lambda(\lambda-2)(\lambda-1) &= 0 \\ \implies \lambda &= 0, 1, 2. \end{aligned}$$

Thus we have that

$$f(n) = c_D(n-1) = 1^{n-1} + 2^{n-1} = 1 + 2^{n-1}.$$

□

6.4 The Lagrange Inversion Formula

Theorem 6.18 (The Lagrange Inversion Formula). *Let $F(x) = a_1x + a_2x^2 + \cdots \in xK[[x]]$ where $a_1 \neq 0$, and let $k, n \in \mathbb{Z}$. Then*

$$n[x^n]F^{\langle -1 \rangle}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n.$$

Note. Equivalently, suppose $G(x) \in K[[x]]$ with $G(0) \neq 0$, and let $f(x)$ be defined by

$$f(x) = xG(f(x)).$$

Then

$$n[x^n]f(x)^k = k[x^{n-k}]G(x)^n.$$

These are equivalent since the statement that $f(x) = F^{\langle -1 \rangle}(x)$ is easily seen to mean the same as $f(x) = xG(f(x))$ where $G(x) = \frac{x}{F(x)}$.

Problems

Problem 6.19. Let $D(z)$ denote the generating function for Dyck paths. Derive a functional equational for $D(z)$. Then use the Lagrange inversion formula to obtain a formula for $[z^n]D(z)$.

Proof. The functional equation is given by $D(z) = 1 + zD(z)^2$. To apply LIF, we want $D(0) = 0$, so we actually want to use the functional equation $D(z) = z(1 + D(z))^2$. Then solving for z we obtain

$$z = \frac{D(z)}{(1 + D(z))^2},$$

and hence

$$D(z) = \left(\frac{z}{(1+z)^2}\right)^{\langle -1 \rangle}.$$

Thus the LIF yields

$$[z^n]D(z) = \frac{1}{n}[z^{n-1}]\left(\frac{z}{\frac{z}{(1+z)^2}}\right)^n = \frac{1}{n}[z^{n-1}](1+z)^{2n}.$$

Using binomial theorem we obtain

$$[z^n]D(z) = \frac{1}{n}[z^{n-1}]\sum_{k\geq 0}\binom{2n}{k}z^k = \frac{\binom{2n}{n-1}}{n} = \frac{\binom{2n}{n}}{n+1}.$$

□

Problem 6.20. Let t_n be the number of all unlabeled rooted plane trees on n vertices in which every vertex has an even number of successors. Let $T(x) = \sum_{n\geq 1} t_n x^n$. Find a simple functional equation for $T(x)$, then find a closed formula for t_n using the Lagrange inversion formula.

Proof. We first see that

$$T(x) = \frac{x}{1 - T(x)^2},$$

since by removing the root of such a tree we obtain an order sequence of even length of such trees. Solving for x we then see that

$$x = T(x) - T(x)^3 \implies T(x) = (x - x^3)^{\langle -1\rangle},$$

so by LIF we have

$$[x^n]T(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{x-x^3}\right)^n = \frac{1}{n}[x^{n-1}]\left(1-x^2\right)^{-n}.$$

Using the binomial theorem we obtain

$$[x^n]T(x) = \frac{1}{n}[x^{n-1}]\sum_{k\geq 0}\binom{-n}{k}(-1)^k x^{2k} = \frac{1}{n}[x^{n-1}]\sum_{k\geq 0}\binom{n+k-1}{k}x^{2k}.$$

So for n even we have $t_n = 0$, and for n odd we have

$$t_n = \frac{1}{n}\binom{n + (\frac{n-1}{2}) - 1}{(\frac{n-1}{2})}.$$

□

Problem 6.21. Using the Lagrange inversion formula, show that the number of rooted trees on $[n]$ is n^{n-1}.

Proof. Letting $T(x)$ be the exponential generating function of the number rooted trees on $[n]$. Then it is clear that removing the root of such a tree results in an unordered sequence of such labeled trees. Some though yields:

$$T(x) = xe^{T(x)} \implies x = T(x)e^{-T(x)} \implies T(x) = (xe^{-x})^{\langle -1\rangle}.$$

Applying LIF we have

$$[x^n]T(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n = \frac{1}{n}[x^{n-1}]e^{xn}.$$

We then see that

$$\frac{1}{n}[x^{n-1}]\sum_{k\geq 0} n^k\frac{x^k}{k!} = \frac{n^{n-1}}{n!}.$$

□

6.5 Algebraic Generating Functions

Definition. A formal power series $f(x) \in K[[x]]$ is said to be *algebraic* if there exist polynomials $p_0(x), \ldots, p_d(x) \in K[x]$, not all 0, such that

$$p_0(x) + p_1(x)f(x) + \cdots + p_d(x)f(x)^d = 0.$$

Note. An algebraic series $f(x)$ is rational if and only if it is algebraic of degree $d = 1$.

Theorem 6.22. *The following objects each have the same algebraic generating functions. Let $S \subseteq \mathbb{Z}^+$ and $n, m \in \mathbb{Z}^+$. There are nice bijections between the following sets.*

i. Plane S-trees (rooted trees in which each vertex has a number of children in S) with n vertices and m endpoints.

ii. Sequences $i_1 i_2 \ldots i_{n-1}$ where each $i_j + 1 \in S$ or $i_j = -1$ such that there are a total of $m-1$ values of j for which $i_j = -1$, and such that $i_1 + \cdots + i_j \geq 0$ for all j and $i_1 + \cdots + i_{n-1} = 0$.

iii. Paths P in the plane from $(0,0)$ to $(n-1, 0)$ using steps $(1,k)$ where $k+1 \in S$ or $k = -1$, with a total of $m-1$ steps of the form $(1,-1)$, such that P never passes below the x-axis.

iv. Paths P in the plane from $(0,0)$ to $(m-1, m-1)$ using steps $(k,0)$ or $(0,1)$ with $k+1 \in S$, with a total of $n-1$ steps, such that P never passes above the line $x = y$.

Proof. (*i* $\leftrightarrow$ *ii*) Take a plane S-tree, order its vertices by a depth first search (read, left, right), then in this order record the number of children of the vertex minus one.

(*ii* $\leftrightarrow$ *iii*) Take such a sequence $i_1 \ldots i_{n-1}$, then map it to the path starting at $(0,0)$ with the j^{th} step being $(1, i_j)$.

(*ii* $\leftrightarrow$ *iv*) Take such a sequence $i_1 \ldots i_{n-1}$, then map it to the path starting at $(0,0)$ with the j^{th} step being $(k, 0)$ if $i_j = k > 0$, or $(0,1)$ if $i_j = -1$. □

Corollary 6.23. *The following equivalences are obtained by letting $S = \{2\}$ in the the last theorem. The Catalan number $C_n = \binom{2n}{n}/(n+1)$ counts the following:*

i. Plane binary trees with $n+1$ endpoints (i.e. 2n+1 vertices).

ii. Sequences $i_1 \ldots i_{2n}$ of 1's and -1's in which all initial sums are non-negative and $i_1 + \ldots i_{2n} = 0$. These sequences are called ballot sequences.

iii. Paths P in the plane from $(0,0)$ to $(2n, 0)$ with steps $(1,1)$ and $(1,-1)$ that never passes below the x-axis. Such paths are called Dyck paths.

iv. Paths P in the plane from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$ that never passes above the diagonal $x = y$.

6.6 D-Finite Generating Functions

Definition. Let $u \in K[[x]]$, then we say that u is *d-finite* if it satisfies one of the following equivalent conditions:

i. The vector space over $K(x)$ spanned by $\{u, u', u'', \ldots\}$ is finite dimensional.

ii. There exists polynomials $p_0, \ldots, p_d \in K[x]$ with $p_d(x) \neq 0$ such that

$$p_d u_p^{(d)}{}_{d-1} u^{(d-1)} + \ldots p_1 u' + p_0 u = 0.$$

Definition. A function $f : \mathbb{N} \to K$ is called *P-recursive* if there exist polynomials $p_0, \ldots, p_e \in K[n]$ with $p_e \neq 0$ such that

$$p_e(n) f(n+e) + p_{e-1} f(n+e-1) + \cdots + p_0(n) f(n) = 0$$

for all $n \in \mathbb{N}$.

Proposition 6.24. *Let $u = \sum_{n \geq 0} f(n) x^n \in K[[x]]$. Then u is d-finite if and only if f is P-recursive.*

Problems

Problem 6.25. Let $f(n) = 1! + 2! + \cdots + n!$. Is $f(n)$ a polynomially recursive sequence?

Proof. Yes, it is P-recursive. We have that

$$\begin{aligned}(n+1)f(n) &= (n+1) \cdot 1! + (n+1) \cdot 2! + \cdots + (n+1)(n-1)! + (n+1)! \\ &= (n(1! + 2! + \cdots + (n-2)!) + n!) + (1! + 2! + \cdots + (n-1)! + (n+1)!) \\ &= nf(n-2) + f(n+1).\end{aligned}$$

Hence we obtain

$$nf(n-2) - (n+1)f(n) + f(n+1) = 0$$

for all $n \geq 3$. This gives the result. ☐

6.7 Recognizing Rational, Algebraic and D-Finite Series

Proposition 6.26 (Stirling's Approximation)**.** *The following is useful for finding the growth rates of sequences:*

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Proposition 6.27. *For some constants C and α, the coefficients of a power series $f(x)$ grow at*

- $C\frac{\alpha^n}{n^d}$ *for $d \in \mathbb{Z}^+$ if f is rational,*
- $C\frac{\alpha^n}{n^{d+\frac{1}{2}}}$ *for $d \in \mathbb{Z}^{\geq 0}$ if f is algebraic.*

Proposition 6.28. *The following table shows which operations preserve the classification of each type of function. The Y indicating 'yes' and N indicating 'not necessarily.' Note that $A \star B$ denotes the Hadamard product. The one significant combination not included in this table is that 'rational $\star$ algebraic' is always algebraic.*

	cA	$A+B$	AB	$1/A$	$A \star B$	$A \circ B$	A'	$\int A$
Rational	Y	Y	Y	Y	Y	Y	Y	N
Algebraic	Y	Y	Y	Y	N	Y	Y	N
D-Finite	Y	Y	Y	N	Y	N	Y	Y

Problems

Problem 6.29. Let $F(x) = (\log \frac{1}{1-x})^k$. Is F algebraic? Is F d-finite?

Proof. It is not algebraic. We note that the coefficients of $\log \frac{1}{1-x}$ are $\frac{1}{n}$, and hence their growth rate forbids that they are algebraic. Therefore, since algebraic functions are closed under multiplicative inverses, F cannot be algebraic.

It is d-finite. This follows from induction on k, where the base case of $k = 1$ is clear because its coefficients are clearly P-recursive. Taking derivatives we see that

$$F_k'(x) = \frac{kF_{k-1}(x)}{1-x},$$

which is d-finite because $F_{k-1}(x)$ is d-finite by the inductive hypothesis and $1/(1-x)$ is rational, and hence d-finite. Thus $F_k(x)$ must also be d-finite by integrating. □

6.8 Rational and Algebraic Languages

Note. We give the simplest results on rational and algebraic languages.

Theorem 6.30. *A language L is rational (i.e. regular) if and only if it is accepted by a finite automaton.*

Theorem 6.31. *A language L is algebraic (i.e. context-free) if and only if L^+ (L without 1) is a component of a proper algebraic system.*

Theorem 6.32. *Rational languages are algebraic.*

Example 6.33. The Dyck language D is the subset of $\{x, y\}^*$ such that $w_1 \dots w_n \in D$ if and only if the number of x's is greater than or equal to the number of y's in $w_1 \dots w_i$ for all $i \in [n]$, and the number of x's is equal to the number of y's in $w_1 \dots w_n$. We have that D is the solution to

$$z = 1 + xzyz,$$

and that $D^+ = D - 1$ is the solution to

$$z' = x(z'+1)y(z'+1).$$

Hence D is algebraic.

Problems

Problem 6.34. Prove that the language $\mathcal{L} = \{ww : w \in \{a,b\}^*\}$ is not regular.

Proof. We can show that this requires an infinite automaton. □

Problem 6.35. Let $X = \{x, y, u\}$ and let $L \subseteq X^*$ be the language of all words in which there are as many letters x as letters y, and in every initial segment there are at least as many letters x as letters y. Is L an algebraic language?

Proof. We first see that L solves the system

$$z = 1 + xzyz + uz.$$

That is, each non-empty word in L must start with an x or u since starting with y would break the condition that each initial segment has at least as many x's as y's. It is clear that every word $w \in L$ starting with u can be decomposed into $w = uw'$, where $w' \in L$, hence the uz term accounts for these words exactly.

Furthermore, each word $w \in L$ starting with x can be decomposed uniquely into $w = xw_1yw_2$, where $w_1, w_2 \in L$ and xw_1y is the first initial segment of w in which the number of x's is equal to the number of y's. Hence the $xzyz$ term accounts for exactly all the words of L starting with x.

It follows quickly that L^+ solves the component of the proper algebraic system

$$z' = x(z' + 1)y(z' + 1) + u(z' + 1),$$

and therefore L is algebraic. □

7 Graph Theory

Reference: *A Walk Through Combinatorics* - Bóna, Chapters 9-13.

7.1 Basic Concepts and Traversibility

Definition. Let $G = (V, E)$ be a graph.

i. A sequence of edges $e_1e_2 \dots e_k$ is called a *walk* if we can take a continuous walk in our graph, first through e_1, then e_2, etc.

ii. A *trail* is a walk in which each of the edges are distinct.

iii. A *path* is a trail in which no vertex is visited twice.

iv. A *cycle* is a closed trail that does not visit any vertex twice.

v. An *Eulerian trail* is a trail in which all of the edges of G are used. An *Eulerian circuit* or *tour* is an Eulerian trail that is closed. A graph that has an Eulerian circuit is called *Eulerian*.

vi. A cycle that includes all vertices of a graph is called a *Hamiltonian cycle*, whereas a path that includes all vertices of a graph is called a *Hamiltonian path*.

Theorem 7.1 (Handshaking Lemma). *For any graph $G = (V, E)$ we have*

$$\sum_{v \in V} \deg v = 2 \cdot |E|.$$

It follows that the number of vertices of G with odd degree is even.

Theorem 7.2. *A connected graph G has an Eulerian circuit, (i.e. G is eulerian), if and only if every vertex has even degree. A connected graph G has an Eulerian trail from u to v if and only if the degrees of u and v are odd and the rest of the vertices have even degree.*

Theorem 7.3. *A directed graph D has an Eulerian circuit if and only if it is balanced and strongly connected.*

Theorem 7.4. *Let $n \geq 3$, let G be a simple graph on n vertices, and let us assume that all vertices in G are of degree at least $n/2$. Then G has a Hamiltonian cycle.*

Proof. G is clearly connected by these conditions. Let us assume G does not have Hamiltonian cycle and add as many edges to G as possible so that it doesn't have a Hamiltonian cycle but adding any edge will add one. Call this graph G'. Let P be a path of maximal length in G', and we see that P contains every vertex of G'. This is clear since if x and y aren't adjacent, then adding that edge would give a Hamiltonian cycle using that edge. Hence we can assume the path is $x = z_1, z_2, \ldots, z_n = y$. Together x and y have n neighbors, thus the pigeon-hole principle yields that there exists an i such that yz_{i-1} and xz_i are edges. Hence we have a Hamiltonian cycle $xz_2 \ldots z_{i-1}yz_{n-1} \ldots z_i$, a contradiction. ☐

Theorem 7.5. *All tournaments have a Hamiltonian path.*

Proof. We do induction on n, with base cases of $n = 1, 2$ clear. So take a tournament T on n vertices, ignore one vertex v so that T' is a tournament on $n - 1$ vertices, hence by the inductive hypothesis has a Hamiltonian path, call it $h_1h_2 \ldots h_{n-1}$. If h_iv and vh_{i+1} are arcs in T, then we are done. Otherwise, (via some omitted reasoning), vh_1 or $h_{n-1}v$ is an arc in T, and we are done. ☐

Theorem 7.6. *A tournament T has a Hamiltonian cycle if and only if it is strongly connected.*

Problems

Problem 7.7. Let T be a tournament that is not strongly connected. Prove that the vertex set of T can be partitioned into two blocks A and B so that all edges between A and B go from A to B.

Proof. Let v be a vertex such that there are some vertices u in which there is no path u to v. These vertices u must exist by our hypothesis. Let A be the set of vertices with a path to v (including v), and let B be the rest of the vertices. If there were an edge going from $b \in B$ to $a \in A$, then b would have a path to v and so would be in A, a contradiction. Hence all edges between A and B are from A to B. ☐

Problem 7.8. There are several people in a classroom; some of them know each other. It is true that if two people know the same number of people in the classroom, then there is nobody in the classroom both of these people know. Prove that there is someone in the classroom who knows exactly one other person in the classroom.

Proof. This is equivalent to saying that for each person v in the classroom, the person v knows no two people who know the same number of people. That is, correlating this classroom with the obvious graph model, we have that for each vertex v, the vertices in $N(v)$ have all distinct degrees. So consider a vertex u with maximal degree, say k. Then there are k vertices in $N(u)$ such that each one has degree in $[k]$ and no two have the same degree. Thus one of them must have degree one, i.e. there must be a person in the classroom who knows exactly one other person. □

Problem 7.9. Each vertex of a simple graph has degree k. Prove that G has a cycle of length at least $k+1$.

Proof. Let $v_1v_2\dots v_\ell$ be a maximal path in G. Suppose that v_1 is adjacent to some vertex u not in $\{v_1,\dots,v_\ell\}$, then $uv_1v_2\dots v_\ell$ is a longer path, a contradiction. Hence $N(v_1)\subseteq\{v_2,\dots,v_\ell\}$. Since $N(v_1)\geq k$, there exists some index $i\geq k+1$ such that $v_i\in N(n_1)$, and thus $v_1v_2\dots v_i$ is a cycle of length at least $k+1$. □

Problem 7.10. Show that a k-regular graph of girth 4 has at least $2k$ vertices, and a k-regular graph of girth 5 has at least k^2+1 vertices.

Proof. A graph of girth 4 is triangle-free, hence the neighborhoods of any two adjacent vertices must be disjoint. So let uv be an edge in such a graph. Then $N(u)\cup N(v)$ is the union of disjoint sets of vertices, and $|N(u)\cup N(v)|=2k$.

Now in a graph of girth 5, we have no triangles or four cycles, so if there is a path of length 2 or 3 between any two vertices, those vertices cannot be adjacent. So if we consider a vertex v in such a graph, then for each $x,y\in N(v)$, we have $N(x)\cap N(y)=v$. Therefore we have k vertices in the neighborhood of v, and $k-1$ distinct vertices in the neighborhood of each of those vertices. This gives $1+k+k(k-1)=k^2+1$ unique vertices. □

Problem 7.11. Prove bijectively that the number of graphs with vertex set $[n]$ for which all vertices have even degree is $2^{\binom{n-1}{2}}$.

Proof. We show a bijection with all graphs on vertex set $[n-1]$. That is, take a a graph G on vertex set $[n-1]$, then take all vertices of odd degree, of which we know there is an even number by the handshaking lemma, and add a vertex labeled n and connect them all to this vertex to obtain a graph G'. Then all vertices in G' have even degree. The inverse map is then clear, so we have a bijection and we are done. □

7.2 Trees

Definition. A graph G on is called a *tree* if it is minimally connected, or equivalently, is connected and acyclic.

Proposition 7.12. *Some easy facts about trees:*

i. A tree on n vertices has $n-1$ edges.

ii. Every tree with greater than one vertex has at least 2 leaves.

Theorem 7.13 (Cayley's Formula)**.** *For any positive integer n, the number of all trees with vertex set $[n]$ is n^{n-2}.*

Proof. (André Joyal) We will show that the number of doubly rooted trees (where the two roots can be the same vertex) on vertex set $[n]$ is n^n. We give an example of the bijection mapping the set of all functions from $[n]$ to $[n]$ to the set of all doubly rooted trees and it will be clear. Consider the function $f : [n] \to [n]; 1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 7, 7 \mapsto 8, 8 \mapsto 6$. The function f creates the cycles $(13), (5)$ and (678). This gives a permutation on the set $\{1 < 3 < 5 < 6 < 7 < 8\}$, so we map this to the path $f(1)f(3)f(5)f(6)f(7)f(8)$, so our start and end roots are $f(1) = 3$ and $f(8) = 6$. We then extend this to the desired doubly rooted tree in the natural way in the following figure. This makes the bijection clear. □

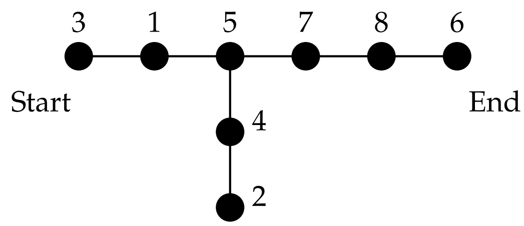

Figure 1: The doubly rooted tree corresponding to the function f.

Corollary 7.14. *The number of planted forest on vertex set $[n]$ is equal to $(n+1)^{n-1}$, i.e. the number of Cayley trees on vertex set $[n+1]$.*

Theorem 7.15. *Let G be a graph on vertex set $[n]$ and let A be its adjacency matrix. Then for $k \in \mathbb{Z}^+$, the entry $A^k_{i,j}$ is equal to the number of walks from i to j that are of length k.*

Proof. We do induction on k. It is clear for the base case $A_{i,j}$. Suppose it holds for some $k-1$ that $A^{k-1}_{i,j}$ is the number of walks of length $k-1$ from i to j for each pair of vertices i, j. Then from matrix multiplication we have

$$A^k_{i,j} = \sum_{m \in [n]} A^{k-1}_{i,m} A_{m,j},$$

which makes the result clear. □

Definition. For a directed graph D, define its *Laplacian matrix* $L(D)$ to be

$$L(D)_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j \text{ and there are } m_{ij} \text{ edges from } v_i \text{ to } v_j, \\ \text{outdeg}(v_i) - m_{ii} & \text{if } i = j. \end{cases}$$

Similarly, for an undirected graph G, define its *Laplacian Matrix* $L(G)$ to be

$$L(G)_{ij} = \begin{cases} -m_{ij} & \text{if } i \neq j \text{ and there are } m_{ij} \text{ edges between } v_i \text{ and } v_j, \\ \deg(v_i) - m_{ii} & \text{if } i = j. \end{cases}$$

Note that, if $\hat{G}$ is the digraph obtained from the undirected graph G by replacing each edge $v_iv_j \in E(G)$ with the arcs (v_i, v_j) and (v_j, v_i), then $L(G) = L(\hat{G})$.

Theorem 7.16. *For a loopless digraph D, let $L_0(D)$ be the matrix obtained by deleting the k^{th} row and column of $L(D)$, then the number of oriented spanning trees rooted at v_k, (pointing 'towards' the root), denoted $\tau(D, v_k)$, is given by*

$$\tau(D, v_k) = \det L_0(D).$$

If D is balanced and the eigenvalues of $L(D)$ are $\mu_1, \ldots, \mu_n = 0$, then for each $v \in V(D)$

$$\tau(D, v) = \frac{1}{n}\mu_1 \ldots \mu_{n-1}.$$

Theorem 7.17 (Matrix-Tree Theorem)**.** *Let G be a connected n-vertex graph, $L_0(G)$ the matrix obtained by deleting the k^{th} row and column of $L(G)$, and $\mu_1 \ldots \mu_n = 0$ the eigenvalues of $L(G)$. Then the number of spanning trees of G, known as the complexity of G and denoted $c(G)$, is given by*

$$c(G) = \det L_0(G),$$

or by

$$c(G) = \frac{1}{n}\mu_1 \ldots \mu_{n-1}.$$

Problems

Problem 7.18. Compute the number of spanning trees of the complete bipartite graph $K_{n,n}$.

Proof. Label the vertices of the two maximal independent sets of $K_{n,n}$ by $\{1, \ldots, n\}$ and $\{n + 1, \ldots, 2n\}$. Then it is clear that

$$L(K_{n,n}) = \left(\begin{array}{c|c} nI_n & -1 \\ \hline -1 & nI_n \end{array}\right)$$

is the Laplacian matrix of $K_{n,n}$. So $L(k_{n,n}) = nI_{2n} - A$ where

$$A = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right).$$

We see that A has rank 2 and trace 0, so A has at most 2 nonzero eigenvalues that sum to 0. It is easy to see that n is an eigenvalue, and thus the other eigenvalue is $-n$.

Therefore the eigenvalues of $L(K_{n,n})$ are $2n, n, n, \ldots, n, 0$. By the matrix-tree theorem it follows that

$$c(K_{n,n}) = \frac{1}{2n} \cdot 2n \cdot n^{2n-2} = n^{2n-2}.$$

□

Problem 7.19. Let A be the graph obtained from K_n by deleting an edge. Find a closed formula for the number of spanning trees of A.

Proof. Suppose that the deleted edge was v_1v_2. Then we have that

$$L(A) = \left(\begin{array}{cc|cccc} n-2 & 0 & & & & \\ 0 & n-2 & & -1's & & \\ \hline & & (n-1) & -1 & \dots & -1 \\ & & -1 & (n-1) & \dots & -1 \\ & -1's & \vdots & & \ddots & \vdots \\ & & -1 & -1 & \dots & (n-1) \end{array}\right)$$

Then we have that $L(A) = nI_n - B$ where

$$B = \left(\begin{array}{cc|c} 2 & 0 & \\ 0 & 2 & 1's \\ \hline \multicolumn{2}{c|}{1's} & 1's \end{array}\right).$$

We have that B has trace $n+2$ and rank 2, so has at most two nonzero eigenvectors summing to $n+2$. It is easy to see that $(1, 1, \dots, 1)$ is an eigenvector with eigenvalue n, hence the other eigenvalue of B is 2.

Thus the eigenvalues of $L(A)$ are $0, n-2, n, \dots, n$. By the matrix-tree theorem it follows that

$$c(A) = \frac{1}{n}(n-2)n^{n-2} = (n-2)n^{n-3}.$$

□

Problem 7.20. Let $f : [n] \to [n]$ be a function. Draw the diagram of f by drawing an arrow from i to j if $f(i) = j$. Let us say that f is acyclic if the diagram of f does not contain any directed cycles with two or more vertices. Find the number of acyclic functions $f : [n] \to [n]$.

Proof. We can draw each such function as a planted forest, with the root set being the fixed points of the function. The inverse map is then also clear, hence the answer is the number of planted forests on vertex set $[n]$, with is $(n+1)^{n-1}$. □

Problem 7.21. There are n parking spots $1, 2, \dots, n$ on a one way street. Cars $1, 2, \dots, n$ arrive in this order. Each car i has a favorite spot $f(i)$. If the spot is free, the car will take it, if not it goes to the next spot. Again, if that spot is free, the car will take it, if not, the car goes to the next spot. If a car had to leave even the last spot and did not find the space, then its parking attempt has been unsuccessful. If, at the end of this procedure, all cars have a parking spot, we say that f is a parking function on $[n]$. Prove that the number of parking functions on $[n]$ is $(n+1)^{n-1}$.

Proof. We consider a circular arrangement of $n+1$ parking spots, and instead, consider all $(n+1)^n$ ways to assign the n cars their favorite spots. Since the spots are in a circle, each car drives up to their favorite spot and and continues around the circle until they find a free spot. For every such assignment, it is clear that every spot will be filled except for one. It is then clear that each of the assignments that results in spot $n+1$ being empty corresponds to a successful parking function. Each of the assignments can then be seen to be equivalent to one of the successful parking assignments by a cyclic rotation of the spots. There are $n+1$ assignments in each equivalence class, and hence there are $(n+1)^{n-1}$ parking functions on $[n]$. □

Problem 7.22. How many parking functions are there on $[n]$ without like consecutive elements? That is, we want to enumerate all parking functions on $[n]$ such that there is no $i \in [n-1]$ so that $f(i) = f(i+1)$.

Proof. We consider the same setup as in the previous solution. It is clear we can assign any spot to 1 in $n+1$ ways, and then each of the following spots can only be assigned in one of n ways. Then there is a total of $(n+1)n^{n-1}$ assignments, and each successful parking assignment is equivalent to $n+1$ of them via cyclic rotations. Hence we have n^{n-1} parking functions without consecutive elements. □

Problem 7.23. Prove that if G is a simple graph on $[n]$, then at least one of G and its complement is connected. Show an example when they are both connected.

Proof. Suppose G is disconnected with connected components $C_1, C_2, \ldots, C_k$. Then in $\bar{G}$ each vertex of one component is connected to every vertex of every other component. So suppose $v \in C_i$ and we will find a path to every other vertex in $\bar{G}$. Suppose $u \in C_j$ where $i \neq j$, then vu is a path. If $i = j$, then by taking $w \in C_k$ where $k \neq i$ we have a path vwu. Hence $\bar{G}$ is connected.

Both the pentagon and its complement are connected. □

Problem 7.24. We say $F_1, F_2, \ldots, F_k$ is a refining sequence if for all $i \in [k]$, F_i is a rooted forest on $[n]$ having i components, and F_i contains F_{i+1}. Now fix F_k.

a. Find the number $N^*(F_k)$ of refining sequences ending in F_k.

b. Find the number $N(F_k)$ of rooted trees containing F_k.

c. Deduce Cayley's Formula.

Proof. a. So how many options are there for F_{k-1}? We pick any vertex of F_k, then append one of the roots of one of the components to it. There are n vertices and $k-1$ options for the other component. So there are $n(k-1)$ options. Continuing in this way, we get $N^*(F_k) = n^{k-1}(k-1)!$.

b. It isn't difficult to see that $N^*(F_k) = N(F_k) \cdot (k-1)!$, which yields

$$N(F_k) = \frac{n^{k-1}(k-1)!}{(k-1)!} = n^{k-1}.$$

c. We then want $N(F_n)$, the number of rooted trees containing the empty graph on $[n]$. From (b), it is then clear that the number of rooted trees on $[n]$ is n^{n-1}, and hence the number of (unrooted) trees on $[n]$ is n^{n-2}. □

Problem 7.25. Let G be a connected graph, and let T_1 and T_2 be two of its spanning trees. Prove that T_1 can be transformed into T_2 through a sequence of intermediate trees, each arising from the previous one by removing and adding an edge.

Proof. First choose any edge e in T_1 that is not in T_2 and delete it. Then $T_1 - \{e\}$ is disconnected with two components A and B. Since T_2 is a spanning tree, there must be an edge f between A and B, and thus $T_1 - \{e\} + \{f\}$ is one edge closer to becoming T_2. Since these trees are finite, this iterated process will get end with T_2. □

Problem 7.26. Let μ be the largest eigenvalue of the adjacency matrix of a graph G and Δ the maximum degree of G. Prove that $\mu \leq \Delta$.

Proof. Let A be the adjacency matrix of G and $\mathbf{x} = (x_1, \ldots, x_n)^T$ be the eigenvector satisfying

$$A\mathbf{x} = \mu\mathbf{x}.$$

Suppose that x_i is the largest coordinate in absolute value of **x**, then we see that

$$|\mu x_i| = \left|\sum_{k=1}^{n} A_{i,k} x_k\right| \leq \left|x_i \sum_{k=1}^{n} A_{i,k}\right| \leq |x_i \Delta|.$$

Hence $|\mu| \leq |\Delta| \implies \mu \leq \Delta$, since Δ must be nonnegative. □

Problem 7.27. Consider the set T_n of non-rooted trees with $n \geq 3$ labeled leaves for which each interior vertex has degree 3.

(a) Prove that each tree in T_n has exactly $2n - 3$ edges.

(b) Prove that the number of trees T_n is $(2n - 5)!!$.

Proof. (a) We do induction on n, with the base case $n = 3$ being given by the star with three labeled leaves. Now take $T \in T_n$, delete the leaf n, the edge adjacent to it, then remove the vertex at the other end of the leaf so that the two vertices that were adjacent to it are now adjacent. Then this is a tree in T_{n-1}, and hence has $2(n-1) - 3$ edges. Thus the tree T has $2n - 3$ edges.

(b) Take any tree $T \in T_{n-1}$, then by choosing any edge in T in any of $2n - 5$ ways, we can add a vertex in the middle of this edge and add a leaf labeled n to this vertex. This makes it clear that $|T_n| = (2n - 5) \cdot |T_{n-1}| = (2n - 5)!!$. This gives the result. □

7.3 Coloring and Matching

Definition. The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest integer k such that G is k-(vertex)-colorable, in which case we say G is k-*chromatic*. We list some important definitions and notation:

- The set of all vertices assigned the same color in a proper coloring of a graph G is called a *color class*.
- *Clique number* - $\omega(G)$ is the order of the largest clique of G.
- *Independence number* - $\alpha(G)$ is the order of the largest independent set of G.
- *Maximum degree* - $\Delta(G)$ is the largest degree of any vertex in G.
- *Minimum degree* - $\delta(G)$ is the smallest degree of any vertex in G.

Proposition 7.28. *For any graph G with order n,* $\chi(G) \geq \omega(G)$ *and* $\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$.

Theorem 7.29. *For any graph G,* $\chi(G) \leq \Delta(G) + 1$.

Definition. A *greedy coloring* of a graph G is done by ordering the vertices of G, then coloring the vertices in that order with the smallest possible positive integer.

Theorem 7.30 (Brooks's Theorem)**.** *For every connected graph G that is not complete or an odd cycle, $\chi(G) \leq \Delta(G)$.*

Definition. The *chromatic index* of a graph G, denoted $\chi'(G)$, is the smallest integer k such that G is *k-edge-colorable*. An edge set is *independent* if no two edges in the set share an endpoint. The *edge independence number* of G, denoted $\alpha'(G)$, is then the order of the largest independent edge set of G.

Proposition 7.31. *For any graph G with size m, $\chi'(G) \geq \frac{m}{\alpha'(G)}$ and $\chi'(G) \geq \Delta(G)$.*

Definition. A 2-colorable graph is called *bipartite*. Equivalently, a graph is bipartite if its vertex set can be partitioned into disjoint sets A and B such that every edge is adjacent to one vertex of A and one vertex of B.

Theorem 7.32. *A graph G is bipartite if and only if it does not contain a cycle of an odd length.*

Theorem 7.33. *Let G be a simple bipartite graph on n vertices. Then G has at most $n^2/4$ edges if n is even, and at most $(n^2-1)/4$ edges, if n is odd.*

Proof. Clearly the bipartite graph with the most edges on n vertices will be $K_{a,b}$ for some $a, b \in \mathbb{Z}^+$ summing to n. We have that $K_{a,b}$ has $ab = a(n-a)$ edges, and so we maximize the number of edges via calculus. □

Definition. Let G be any graph, and let S be a set of edges in G so that no two edges in G have a vertex in common. Then we say that S is a *matching* in G. If each vertex in G is covered be an edge in S, then we call S a *perfect matching*. Let $G = (X, Y)$ be a bipartite graph. If S is a matching in G that covers all vertices of X, then we say that S is a *perfect matching of X into Y*.

Theorem 7.34 (Philip Hall's Theorem)**.** *Let $G = (X, Y)$ be a bipartite graph. Then X has a perfect matching into Y if and only if for all $T \subseteq X$, the inequality $|T| \leq |N(T)|$ holds, where $N(T) \subseteq Y$ is the neighborhood of T.*

Problems

Problem 7.35. Let G be a bipartite graph with partitioning sets of equal size that does not have a perfect matching. Let A be the adjacency matrix of G. Is it true that $\det A = 0$?

Proof. It is true. We can order the vertices of G so that

$$A = \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right).$$

The determinant of A is then non-zero only if there is a way to pick n entries of A so that no column of row is selected from twice, and such that each of those entries are non-zero. However those entries would mean there are vertex disjoint edges cover each vertex in G, so G would have a perfect matching. □

Problem 7.36. Let G be any simple graph with labeled vertices, and let $p(k)$ be the number of ways to properly k-color G. Prove that p is a polynomial function of k. What is the degree of that polynomial?

Proof. Suppose G has n vertices, and let $p_1, p_2, \ldots, p_n$ be the number of ways to properly color G with *exactly* $1, 2, \ldots, n$ colors, respectively. Then it is clear that

$$p(k) = \sum_{i=1}^{n} p_i \binom{k}{i},$$

since we have to pick which colors we will use. Since the p_i's are constants and the $\binom{k}{i}$ are polynomials, we see that $p(k)$ is a polynomial of degree n. □

Problem 7.37. Let G be a regular bipartite graph. Prove that G has a perfect matching.

Proof. Suppose G does not have a perfect matching. Then there exists some vertex set T within one of the color classes such that $T > N(T)$. Suppose G is regular of degree d, then it is clear that there are $|T|d$ edges between T and $N(T)$. However, by the pigeon-hole principle this would imply that $N(T)$ has at least one vertex of degree greater d, a contradiction. □

Problem 7.38. A medium-size city has three high schools, each of them attended by n students, Each student knows exactly $n+1$ students who attend a high school different from his. Prove that we can choose three students, one from each school, so that they each know the other two.

Proof. Let the three schools be A, B and C. Choose the student v, without loss of generality in school A, such that he knows the most people k from one other school, without loss of generality let it be B. That is, v knows $k \geq (n+1)/2$ students from school B and $n+1-k$ from school C.

Let X and Y be the sets of students v knows in B and C, respectively. Consider a student u in Y. If u knows any students in X then we are done, so assume he does not. So u knows at most $n-k$ students in school B, and therefore must know $n+1-(n-k) = k+1$ students in A, contradicting our choice of v. Hence u must know a student in X, yielding a triangle. □

Problem 7.39. Fix two positive integers n and k so that $k < n/2$. Let $G = (X, Y)$ be the bipartite graph in which the vertices of X are the k-element subsets of $[n]$ and Y are the $(k+1)$-element subsets of $[n]$, and there is an edge between $x \in X$ and $y \in Y$ if and only if $x \subset y$. Prove that X has a perfect matching into Y by (a) using Philip Hall's Theorem, and (b) by finding a perfect matching of X to Y.

Proof. (a) Let $S \subseteq X$, we want to show that $|S| \leq |N(S)|$. Let E be the set of edges incident to the set X, and let E' be the set of edges incident to the set $N(S)$. Then certainly $|E| \leq |E'|$ since $E \subseteq E'$. We see further that $|E| = |S|(n-k)$ and $|E'| = |N(S)|(k+1)$, and hence we obtain

$$|N(S)| = \frac{|E'|}{k+1} \geq |S|\frac{n-k}{k+1} \geq |S|.$$

(b) This is actually quite a difficult construction. □

Problem 7.40. König's Theorem states that the size of maximum independent edge set (or matching) is equal to the size of a minimum vertex cover of edges in a bipartite graph.

(a) Deduce Philip Hall's Theorem from König's Theorem.

(b) Deduce König's Theorem from Philip Hall's Theorem.

Proof. (a) Let $G = (X, Y)$ be a bipartite graph satisfying $|S| \leq |N(S)|$ for all $S \subseteq X$. Suppose for a contradiction that G does not have a matching of X into Y. So a minimum vertex cover C, equivalent in size to a maximum matching, satisfies $|C| < |X|$. Let $X' = X \cap C$ and $Y' = Y \cap C$. Then $|X'| + |Y'| = |C| < |X|$, which implies $|Y'| < |X| - |X'| = |X \setminus X'|$. Since C is a vertex cover, if $x \in X \setminus X'$, then it may not be connected to any of $Y \setminus Y'$, because that would mean there is an edge with neither vertex in C, and hence C doesn't cover the edges of G. Therefore, we have that

$$|N(X \setminus X')| \leq |Y'| < |X \setminus X'|,$$

a contradiction. This proves the result.

(b) Assume König's Theorem is false and show this contradicts Philip Hall's Theorem? □

7.4 Planarity

Definition. A graph G is a *planar graph* if G can be drawn in the plane without any two of its edges crossing. A graph G that is already drawn in the plane is called a *plane graph*. When the vertices and edges of a plane graph are removed, the resulting connected pieces are called *regions* of G.

Theorem 7.41 (Euler's Theorem). *For every connected plane graph of order n, size m and having r regions, we have*

$$n - m + r = 2.$$

Proof. Induction on m. In inductive step, we either delete an edge from a cycle, or we have a tree. □

Theorem 7.42. *If G is a planar graph of order $n \geq 3$ and size m, then*

$$m \leq 3n - 6.$$

Proof. As G is planar, each of its faces has at least three edges, and each edge is in at most two faces, hence $3r \leq 2m$, (the two sides under and over count edge-face pairs). That is, $r \leq \frac{2m}{3}$. Comparing this with Euler's theorem, $m + 2 = n + r$, we obtain $m + 2 \leq \frac{2m}{3} + n$, which yields

$$\frac{m}{3} \leq n - 2 \implies m \leq 3n - 6.$$

□

Theorem 7.43. *All simple planar graphs have a vertex of degree at most 5.*

Proof. Assume all vertices of a graph have degree at least 6. Then the sum of all vertex degrees is at least $6n$, and hence the number of edges of the graph is at least $3n$, hence proving the graph can't be planar by the last theorem. □

Theorem 7.44. *If G is a planar bipartite graph of order $n \geq 3$ and size m, then*

$$m \leq 2n - 4.$$

Proof. As G is planar and bipartite, each edge is on at most two faces and each face has at least 4 edges, hence $4r \leq 2m \implies r \leq m/2$. Comparing with Euler's theorem $n - m + r = 2$, then

$$2 \leq n - m + \frac{m}{2} = n - \frac{m}{2} \implies m \leq 2n - 4.$$

□

Definition. A graph H is called a *subdivision* of a graph G if either $H \simeq G$ or H can be obtained from G by inserting vertices of degree 2 into some, all or none of the edges of G.

Theorem 7.45. *A graph G is planar if and only if G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Problems

Problem 7.46. Let P be a convex polyhedron whose faces are all either a-gons or b-gons, and whose vertices are each incident to three edges. Let p_a, p_b, and n respectively denote the number of a-gonal faces, b-gonal faces and vertices of P. Prove that $p_a(6-a) + p_b(6-b) = 12$.

Proof. We use the following system of equations:

(1) $n - m + r = 2$,

(2) $p_a + p_b = r$,

(3) $ap_a + bp_b = 2m$,

(4) $3n = 2m$.

Combining (1) and (4) we obtain $\frac{2}{3}m - m + r = r - \frac{m}{3} = 2$. Next we plug in (2) and obtain $p_a + p_b - \frac{m}{3} = 2$. Next we plug in (3) and obtain $p_a + p_b - \frac{ap_a + bp_b}{6} = 2$, and we have

$$6p_a + 6p_b - ap_a - bp_b = 12 \implies p_a(6-a) + p_b(6-b) = 12.$$

□

Problem 7.47. Let G be a simple graph of order n such that each vertex has degree at least $(n-1)/2$.

(a) Prove that the diameter of G is at most 2.

(b) Prove that G has a Hamiltonian path.

(c) Prove that G is not planar if $n > 10$.

Proof. (a) Let u and v be any two vertices of G. Then the shortest path between u and v is greater than 2 if and only if the closed neighborhood of each of these vertices are disjoint. The closed neighborhood of each of these vertices has at least $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ in each of them, hence $N[u] \cap N[v] \neq \varnothing$. Thus the diameter of G is at most 2.

(b) Add a new vertex v to G adjacent to every vertex. Then the new graph G' satisfies the hypotheses of Theorem 7.4 since every vertex will have degree at least $\frac{n+1}{2}$. Thus G' has a Hamiltonian cycle, hence deleting v will leave us with a Hamiltonian path.

(c) If $n \geq 12$, then every vertex has degree at least 6, contradicting the fact that planar graphs must have a vertex of degree at most 5. If $n = 11$, then we know that every vertex has degree at least 5, and therefore G has at least $\frac{5 \cdot 11}{2} = 27.5$ edges, and therefore at least 28 edges. However, by Theorem 7.42, we know that the number of edges of G is at most $m \leq 3n - 6 = 27$ edges if it is planar, a contradiction. ☐

Problem 7.48. (a) Prove that a planar graph of girth at least 6 has a vertex of degree at most 2.

(b) The Four Color Theorem states that any planar graph is 4-colorable. Grotzsch proved that any triangle-free planar graph is 3-colorable. Without using his result, prove that a planar graph of girth at least 6 is 3-colorable.

Proof. (a) In a plane graph of girth at least 6, each region has at least 6 edges. Therefore, considering region-edge pairs, we have that $6r \leq 2m \implies r \leq \frac{m}{3}$. Comparing with Euler's Theorem, we have that

$$n - m + r = 2 \implies n - m + \frac{m}{3} \geq 2 \implies n - 2 \geq \frac{2m}{3} \implies \frac{3n}{2} - 3 \geq m.$$

So suppose that each vertex has degree greater than or equal to three, then be the handshaking lemma, we have that $2m \geq 3n \implies m \geq \frac{3n}{2}$, which contradicts the above inequality.

(b) We do induction on n, with the base case clear. Now pick such a graph G on n vertices, and by part (a), there exists a vertex of degree 1 or 2. Deleting that vertex, the new graph is 3-colorable by the inductive hypothesis, and thus a 3-coloring can be extended to G since it has degree less than 3. ☐

Problem 7.49. Prove that for a simple graph G with $n \geq 11$ vertices, at most one of G or its complement $\bar{G}$ is planar.

Proof. Note if G is a graph on n vertices, one of G or $\bar{G}$ has at least $\binom{n}{2}/2$ edges. We also know that for a graph on n vertices to be planar, it must have less than or equal to $3n - 6$ edges. So we are able to prove this if we can show that the polynomial $f(n) = \binom{n}{2}/2 - 3n + 6$ is positive for all $n \geq 11$. We see that $f(11) = \binom{11}{2}/2 - 3 \cdot 11 + 6 = \frac{11 \cdot 5}{2} - 33 + 6 > 0$. Further, taking the derivative of f, we see that it is increasing on an interval including $n \geq 11$. This gives the result. ☐

7.5 Ramsey Theory

Definition. The *Ramsey number* $R(k, \ell)$ is the least positive integer n such that every graph on n vertices has a clique on k vertices or an anticlique on ℓ vertices.

Theorem 7.50 (Ramsey's Theorem). *For every two integers $k, \ell \geq 1$, the Ramsey number $R(k, \ell)$ exists.*

Theorem 7.51. *For every two integers $k, \ell \geq 2$, the Ramsey number $R(k, \ell)$ exists and satisfies*

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1).$$

Proof. Take any graph G on $R(k-1, \ell) + R(k, \ell - 1)$ vertices and take a vertex $v \in V(G)$. By the pigeon-hole principle, v is either adjacent to $R(k-1, \ell)$ vertices or it is not adjacent to $R(k, \ell - 1)$ vertices.

If v is adjacent to $R(k-1,\ell)$ vertices, those vertices either have a clique of $k-1$ vertices or an anticlique with ℓ vertices. In the latter case we are done, and in the former case we then have a clique on k vertices by adding the vertex v.

If v is not adjacent to $R(k,\ell-1)$ vertices, then the argument is very similar.

Note that since $R(k,2)=k$ and $R(2,\ell)=\ell$, induction proves Ramsey's theorem. □

Theorem 7.52. *For all $k \geq 2$, we have $R(k,k) > \sqrt{2}^k$.*

Proof. Let G be an n-vertex graph and let S be a k-vertex subset of $V(G)$. Let A_S denote the event that S is a clique or an anticlique, hence

$$\mathbb{P}[A_s] = \frac{2}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Now letting $\mathcal{S}$ denote the set of all k-vertex subsets of $V(G)$, we see that

$$\mathbb{P}\left[\bigcup_{S\in\mathcal{S}} A_S\right] \leq \sum_{S\in\mathcal{S}} \mathbb{P}[A_S] = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Noting that $(n)_k \leq n^k$, we have that

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq 2 \cdot \frac{n^k}{k!} \cdot \sqrt{2}^{-k(k-1)}.$$

Now letting $n = \sqrt{2}^k$, then

$$2 \cdot \frac{n^k}{k!} \cdot \sqrt{2}^{-k(k-1)} = \frac{2\sqrt{2}^{k^2}}{k!\sqrt{2}^{k(k-1)}} < 1.$$

This gives the result. □

Problems

Problem 7.53. We color each point of space either red, blue, green or yellow. Prove that there is a segment of unit length with monochromatic vertices.

Proof. Take a unit tetrahedron in space $ABCD$. If A is red, then if any of BCD are red or two of them are the same color, then we have a monochromatic segment of unit length. So suppose B, C and D are blue, green and yellow. Append another unit tetrahedron $BCDE$ to the other side of $ABCD$. If E is one of blue, green or yellow, we are again done, so it must be red.

Suppose A and E are at distance m from each other. Then by the above argument, if any two points in space at distance m are the same color then we have a segment of unit length with monochromatic vertices. So suppose all points at distance m are the same color. Then considering the sphere of radius m around any point, all points on that sphere must be monochromatic, and hence we can find two monochromatic points at unit distance on that sphere. □

7.6 Extremal Graph Theory

Theorem 7.54 (Mantel's Theorem)**.** *The maximum possible number of edges in a triangle-free graph of order n is $\lfloor n^2/4 \rfloor$.*

Proof. Since G does not contain a K_3, every neighborhood of $N(v)$ must be an independent set. So let $A \subseteq V(G)$ be an independent set of maximum size, so $\deg v \leq |A|$ for all $v \in V$. Every edge of G must be incident to at least one vertex in $V \setminus A$, so we have

$$|E(G)| = \sum_{v \in V \setminus A} \deg v \leq |A| \cdot |V \setminus A|.$$

Setting $x = |A|$, we seek to maximize the quadratic $x(n-x)$, which gives the result. ☐

Theorem 7.55 (Turán's Theorem)**.** *For all integers $n \geq r \geq 1$, among all graphs of order n that do not contain an $(r+1)$-clique, there exists precisely one with the maximum number of edges, namely the Turán graph $T(n,r) = K_{n_1,\ldots,n_r}$ where*

$$n = n_1 + \cdots + n_r, \quad n_1 \geq \cdots \geq n_r \geq 1, \quad \text{and} \quad n_1 - n_r \leq 1.$$

8 Partially Ordered Sets

Reference: *A Walk Through Combinatorics* - Bóna, Chapter 16 and *Enumerative Combinatorics* - Stanley, Chapter 3.

8.1 Basic Concepts

Definition. A *partially ordered set* P is a set with a binary relation $\leq$ satisfying

- *Reflexivity:* $t \leq t$ for all $t \in P$;
- *Antisymmetry:* if $s \leq t$ and $t \leq s$, then $t = s$;
- *Transitivity:* if $s \leq t$ and $t \leq u$, then $s \leq u$.

Example 8.1. Some easy examples:
Boolean: B_n is the set $2^{[n]} = \mathbb{P}([n])$ ordered by inclusion.
Divisor: D_n is the set of all divisors of n, with $i \leq j$ iff $i \mid j$.
Partition: Π_n is the set of all partitions of $[n]$ ordered by refinement.

Definition. An *interval* $[s,t]$ of a poset P is a subposet of P given by $\{u \in P : s \leq u \leq t\}$. The poset P is called *locally finite* is every interval is finite.

An element of P is called $\hat{0}$ if for all $t \in P$ we have $\hat{0} \leq t$. An element of P is called $\hat{1}$ if for all $t \in P$ we have $t \leq \hat{1}$.

A *chain* is a poset in which all elements are comparable. The length of a chain C is given by $\ell(C) = |C| - 1$, i.e. the number of edges in the chain. If every maximal chain of P has length n, then we say P is *graded* with rank n.

Note. If P is graded of rank n, then there exists a rank function $\rho : P \to [n]$ such that $\rho(s) = 0$ if s is minimal and $\rho(t) = \rho(s) + 1$ if t covers s.

Definition. An *antichain* is a subset A of a poset P in which no two distinct elements are comparable. An *order ideal* of P is a subset I of P such that if $t \in I$ and $s \le t$, then $s \in I$.

Note. There is a one-to-one correspondence between antichains and order ideals of finite posets. Namely, if A is the set of maximal elements of an order ideal I, then $I = \{s \in P : s \le t \in A\}$. The set of all order ideals of P, ordered by inclusion, forms a poset denoted $J(P)$.

Problems

Problem 8.2. Let $A_2(n)$ denote the number of 2-element antichains of the boolean algebra B_n. Find $A_2(n)$.

Proof. We count the number of ways to pick a subset of $2^{[n]}$, then pick another subset such that neither subset is contained in the other. This will clearly give $2 \cdot A_2(n)$. So we have

$$\begin{aligned}
2 \cdot A_2(n) &= \sum_{k=0}^{n} \binom{k}{n} \left(\sum_{i=0}^{k-1} \binom{k}{i} \right) \left(\sum_{k=1}^{n-k} \binom{n-k}{j} \right) \\
&= \sum_{k=0}^{n} \binom{k}{n} (2^k - 1)(2^{n-k} - 1) \\
&= \sum_{k=0}^{n} \binom{k}{n} (2^n - 2^k - 2^{n-k} + 1) \\
&= 2^n \sum_{k=0}^{n} \binom{k}{n} - \sum_{k=0}^{n} \binom{k}{n} 2^k - \sum_{k=0}^{n} \binom{k}{n} 2^{n-k} + \sum_{k=0}^{n} \binom{k}{n} \\
&= 2^n \cdot 2^n - (1+2)^n - (2+1)^n + 2^n \\
&= 4^n - 2 \cdot 3^n + 2^n.
\end{aligned}$$

Hence

$$A_2(n) = \frac{1}{2}(4^n - 2 \cdot 3^n + 2^n).$$

□

Problem 8.3. How many maximal chains does Π_n have?

Proof. The number of maximal chains is clearly the number of ways to take the minimal element $\hat{0} = \{1\}, \{2\}, \ldots, \{n\}$, and continually merge two blocks until we obtain $\hat{1} = [n]$. This is clearly done in

$$\binom{n}{2}\binom{n-1}{2} \cdots \binom{2}{2} = \frac{n(n-1)}{2} \cdot \frac{(n-1)(n-2)}{2} \cdots \frac{3 \cdot 2}{2} \cdot \frac{2}{2} = \frac{n!(n-1)!}{2^{n-1}}.$$

□

Problem 8.4. The famous Dilworth Theorem for a finite poset P states that the minimum number of chains in any partition of P into chains is equal to the maximum number of elements in an antichain of P.

(a) Prove that if P has size at least $rs+1$, then there is either a chain of size r of an antichain of size s.

(b) There are many proofs of the following Erdős-Szekeres Theorem: For any sequence $a_1a_2\ldots a_{n^2+1}$ of integers, there is a subsequence of length $n+1$ that is monotone. Prove the Erdős-Szekeres Theorem using Dilworth's Theorem.

Proof. (a) Suppose that P contains no antichain of size s, then a partition of P into the minimum number of chains contains less than s chains. Suppose none of these chains contain r elements, then P contains less than rs elements, a contradiction.

(b) Define a poset P on the set $\{a_1, a_2, \ldots, a_{n^2+1}\}$ given by $a_i \le a_j$ if and only if $i \le j$ and $a_i \le a_j$ in the real number sense. It is then clear that a chain in this posets corresponds to a monotone increasing subsequence, and an antichain corresponds to a monotone decreasing subsequence. By part (a), P must then contain a chain or antichain of size n, yielding the result. □

8.2 Lattices

Definition. For s, t in a poset P, an *upper bound* u of s and t is an element such that $s \le u$ and $t \le u$. A *least upper bound* or *join* u of s and t is an upper bound such that for every upper bound v of s and t we have $u \le v$. *Lower bounds* and *meets* defined similarly.

Note. Joins and meets are clearly unique, and are denoted $s \vee t$ and $s \wedge t$, respectively.

Definition. A *lattice* is a poset for which every pair of elements has a meet and a join. If every pair of elements of P has a meet (resp. join), then we say that P is a *meet-semilattice* (resp. *join-semilattice*).

Proposition 8.5. *If P is a finite meet-semilattice with $\hat{1}$, then P is a lattice.*

Proof. We let $S = \{u \in P : u \ge s, u \ge t\}$, which is not empty because P has $\hat{1}$. Then $s \vee t = \bigwedge_{u \in S} u$. □

Proposition 8.6. *Let L be a finite lattice. The following are equivalent:*

i. L is graded and its rank function ρ satisfies

$$\rho(s) + \rho(t) \ge \rho(s \vee t) + \rho(s \wedge t) \quad \forall s, t \in L,$$

ii. If s and t both cover $s \wedge t$, then $s \vee t$ covers both s and t.

Definition. Finite lattices satisfying either of the above conditions are called *finite upper semimodular lattices*.

Definition. A finite lattice L is *modular* if it is graded, and its rank function ρ satisfies

$$\rho(s) + \rho(t) = \rho(s \vee t) + \rho(s \wedge t),$$

for all $s, t \in L$, (i.e. both upper and lower semimodular).

Note. A finite lattice L if modular if and only if for all $s, t, u \in L$ with $s \leq u$,

$$s \vee (t \wedge u) = (s \vee t) \wedge u).$$

This shows distributive lattices are modular.

Definition. A *distibutive lattice* L is a lattice satisfying the distributive laws:

$$s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u), \text{ and } s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u).$$

Definition. An element s of a lattice L is *join-irreducible* if $s \neq \hat{0}$ and one cannot write $s = t \vee u$ where $t < s$ and $u < s$. That is, s covers exactly one element.

Theorem 8.7 (Fundamental Theorem of Finite Distributive Lattices)**.** *Let L be a finite distributive lattice. Then there exists a poset P for which $L \cong J(P)$.*

Proof. Let P be the subposet of join-irreducibles of L. Then we see that $L \cong J(P)$. Define $\varphi : L \to J(P)$ by $t \mapsto I_t = \{s \in P : s \leq t\}$. φ is order preserving and so is its inverse. Since φ is meet-preserving and $J(P)$ is a lattice, φ is injective.

Now let $I \in J(P)$ and $t = \bigvee\{s \in I\}$, then we show $I = I_t$. Clearly $I \subseteq I_t$. Now suppose $u \in I_t$, then we have

$$\bigvee\{s \in I\} = \bigvee\{s \in I_t\} \implies \bigvee\{s \wedge u : s \in I\} = \bigvee\{s \wedge u : s \in I_t\}.$$

The right-hand side is u, so we have $\bigvee\{s \wedge u : s \in I\} = u$. Since $u \in P$ is join-irreducible, we have $u \in I$, thus $I = I_t$. □

Problems

Problem 8.8. Let $(L, \leq)$ be a finite distributive lattice and Irr(L) be the set of join-irreducibles in L. For $t \in L$, define $K_t = \{p \in \text{Irr}(L) : p \leq t\}$. Show

$$t = \bigvee_{p \in K_t} p.$$

Proof. First, it is clear that $t \geq \bigvee_{p \in K_t} p$. If t is join-irreducible, then $t \in K_t$, and we are done. If not, then we can write $t = q \vee p$ for some two elements $q, p < t$. If they are join-irreducible, we are done. If not, we can write either of them as the join of two smaller elements. Since L is finite, there must be able to find a finite number of join-irreducibles $r_1, \ldots, r_k$ such that $t = r_1 \vee \cdots \vee r_k$. Since each or the r_i's must be in K_t, we are done. □

8.3 Incidence Algebras and Möbius Inversion Formula

Definition. Let P be locally finite and define Int(P) to be the set of closed intervals of P. Then we define the *incidence algebra* over the field K to be the set of functions $f : \text{Int}(P) \to K$ with multiplication

$$fg(s, u) = \sum_{s \leq t \leq u} f(s, t)g(t, u).$$

Note. By fixing a linear extension of P, $I(P)$ can be represented by upper triangular matrices, and this shows that $f \in I(P)$ is invertible if and only if $f(t,t) \neq 0$ for all $t \in P$. The identity is then $\delta(s,t)$ which is 1 if $s = t$ and 0 otherwise.

Definition. We define two elements of every $I(P)$. The *zeta function* is given by

$$\zeta(t,u) = 1 \ \ \forall t \leq u,$$

and the *Möbius function* μ is defined to be the inverse of the zeta function, i.e. $\zeta\mu = \delta$.

Note. Here we list some properties of both functions. The zeta function:

- $\zeta^k(s,u) = \sum_{s=s_0 \leq \cdots \leq s_k = u} 1$,
- $(\zeta - \delta)^k(s,u) = \sum_{s=s_0 < \cdots < s_k = u} 1$.

Both properties can be easily seen by induction. The Möbius function:

- $\mu(s,s) = 1$ for all $s \in P$,
- $\sum_{s \leq t \leq u} \mu(s,t) = 0$ which implies $\mu(s,u) = -\sum_{s \leq t < u} \mu(s,t)$ for all $s < u \in P$.

Theorem 8.9 (Möbius Inversion Formula)**.** *Let P be a poset for which every order ideal is finite. Define the some functions $f, g : P \to K$, where K is a field. Then*

$$g(t) = \sum_{s \leq t} f(s) \ \ \forall t \in P \iff f(t) = \sum_{s \leq t} g(s)\mu(s,t) \ \ \forall t \in P.$$

Theorem 8.10 (Hall's Theorem)**.** *Let P be a finite poset and let $\hat{P}$ be P with $\hat{0}$ and $\hat{1}$ added. Let c_i be the number of chains of length i between $\hat{0}$ and $\hat{1}$. Then*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - \ldots$$

Theorem 8.11 (Weisner's Theorem)**.** *Let L be a finite lattice with at least two elements, and let $\hat{1} \neq a \in L$. Then*

$$\sum_{t \wedge a = \hat{0}} \mu(t, \hat{1}) = 0.$$

Problems

Problem 8.12. Let D be the division poset, (the set of all positive integers ordered by divisibility). Find a formula for the Möbius function μ and prove that it is correct.

Proof. We show that

(a) $\mu(x,y) = (-1)^k$ if $\frac{y}{x} = p_1 p_2 \ldots p_k$ where $p_1, p_2, \ldots, p_k$ are distinct primes, and

(b) $\mu(x,y) = 0$ if $\frac{y}{x}$ is divisible by the square of a prime.

We first note that the intervals $[x, y]$ and $[1, \frac{y}{x}]$ are isomorphic as posets, so we prove our statement for intervals $[1, y]$, i.e. when $x = 1$.

To prove (a), note that if $y = p_1 \dots p_k$ is square-free, then $[1, y]$ is isomorphic to the boolean poset B_k. Hence it follows that $\mu(1, y) = (-1)^k$.

To prove (b), we do strong induction on y. If $y = 4$, the result is true. So we assume now that the statement is true for all integers smaller than y. Let $p_1, \dots, p_k$ be the distinct divisors of y, and we see that the interval $[1, p_1 \dots p_k]$ consists of exactly all the divisors of y that are square-free. Also, the rest of the elements in $[1, y]$ (that aren't y) are divisible by the square of a prime, so the inductive hypothesis holds. This yields

$$\mu(1, y) = -\sum_{z<y} \mu(1, z) = -\sum_{z \in [1, p_1 \dots p_k]} \mu(1, z) - \sum_{y > z \notin [1, p_1 \dots p_k]} \mu(1, z) = -0 - 0 = 0,$$

using the fact that for any non-singleton interval I that $\sum_{z \in I} \mu(1, z) = 0$. □

Problem 8.13. Let P be a finite poset with $\hat{0}$ and $\hat{1}$. Let $f \in I(p)$ be defined by $f(x, y) = 1$ if y covers x, and $f(x, y) = 0$ otherwise. Find a formula for the total number of maximal chains of P in terms of f.

Proof. It is clear that $\delta(\hat{0}, \hat{1})$ is the number of maximal chains of length 0, $f(\hat{0}, \hat{1})$ is the number of maximal chains of length 1, $f^2(\hat{0}, \hat{1})$ is the number of maximal chains of length 2, etc. So the number of maximal chains of P is

$$\delta(\hat{0}, \hat{1}) + f(\hat{0}, \hat{1}) + f^2(\hat{0}, \hat{1}) + \cdots = (\delta + f + f^2 + \dots)(\hat{0}, \hat{1}) = (\delta - f)^{-1}(\hat{0}, \hat{1}).$$

□

Problem 8.14. Let P be a poset with $\hat{0}$ and $\hat{1}$. Let $x \notin \{\hat{0}, \hat{1}\}$. Finally, let P_x be the poset obtained from P by removing x and leaving all other comparability relations unchanged. Prove that

$$\mu_{P_x}(\hat{0}, \hat{1}) = \mu_P(\hat{0}, \hat{1}) - \mu_P(\hat{0}, x)\mu_P(x, \hat{1}).$$

Proof. By Hall's theorem, we see that

$$\mu_P(\hat{0}, x)\mu_P(x, \hat{1}) = b_0 - b_1 + b_2 - \dots,$$

where b_i is the number of length i chains $\hat{0} = s_0 < s_1 < \cdots < s_i = \hat{1}$ in P such that x is in the chain. Noting that $\mu_P(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - \dots$ where c_i is the number of length i unrestricted chains in P, it is clear that

$$\mu_{P_x}(\hat{0}, \hat{1}) = (c_0 - b_0) - (c_1 - b_1) + (c_2 - b_2) - \dots,$$

yielding the result. □

Problem 8.15. Let P be a poset with a minimal element $\hat{0}$, and let x be an element of P that covers one single element y. Let us assume that $y \neq \hat{0}$. Prove that $\mu(\hat{0}, x) = 0$.

Proof. We see that

$$\mu(\hat{0}, x) = -\sum_{\hat{0} \le z < x} \mu(\hat{0}, z) = -\sum_{z \in [\hat{0}, y]} \mu(\hat{0}, z) = -\mu\zeta(\hat{0}, y) = -\delta(\hat{0}, y) = 0.$$

□

9 Horizons

Reference: *A Walk Through Combinatorics* - Bóna, Chapters 15, 17-20.

9.1 Probability

Definition. If $P(A \cap B) = P(A) \cdot P(B)$, then the events A and B are called *independent*. Otherwise, they are called *dependent*.

Definition. Let A and B be events from the sample space, and assume $P(B) > 0$. Let

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Then $P(A|B)$ is called a *conditional probability*, and is read the "probability of A given B".

Theorem 9.1 (Bayes' Theorem)**.** *Let A and B be mutually exclusive events so that $A \cup B = \Omega$. Let C be any event. Then*

$$P(C) = P(C|A) \cdot P(A) + P(C|B) \cdot P(B).$$

Definition. For a sample space Ω, a *random variable* is a function $X : \Omega \to \mathbb{R}$. If X has a finite range S, the number

$$\mathbb{E}(X) = \sum_{i \in S} i \cdot P(X = i)$$

is called the *expectation* of X on Ω.

Theorem 9.2 (Linearity of Expectation)**.** *Let $X, X_1, \ldots, X_k$ be random variables defined over the finite sample space Ω such that $X = X_1 + \cdots + X_k$. Then*

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_k).$$

Theorem 9.3. *Every graph of size m contains a bipartite graph of size at least $m/2$.*

Proof. For any graph G, choose a subset $S \subseteq V(G)$ uniformly at random, so that $\mathbb{P}[v \in S] = 1/2$. For any edge $e = uv \in E(G)$, define the event X_e that has value 1 if exactly one of u and v is in S and the other in $V \setminus S$, and 0 otherwise. By linearity of expectation we have

$$\mathbb{E}(X) = \sum_{e \in E} \mathbb{E}(X_e) = m/2.$$

Since this is the expected value of X, there exists a choice of $S \subseteq V$ such that $X \geq m/2$. This gives the result. □

Theorem 9.4. *For every positive integer n, there is a tournament of order n with at least $n!/2^{n-1}$ hamiltonian paths.*

Proof. For any tournament T on vertices $\{v_1, \ldots, v_n\}$, consider for any permutation $\pi = \pi_1 \ldots \pi_n \in S_n$ on the set of vertices giving an ordering $\pi(V) := v_{\pi_1} v_{\pi_2} \ldots v_{\pi_n}$. We consider the probability space where each arc has half probability to point in either direction, then the probability that

$\pi(V)$ is a hamiltonian path is $(1/2)^{n-1}$, since we need for each $i \in [n-1]$, the arc must be pointing from v_{π_i} to $v_{\pi_{i+1}}$.

Letting X_π be the indicator variable of this event, it is clear that $X = \sum_{\pi \in S_n} X_\pi$ is the variable indicating the number of hamiltonian paths in T. We obtain

$$\mathbb{E}(X) = \sum_{\pi \in S_n} \mathbb{E}(X_\pi) = n!(1/2)^{n-1}.$$

Hence there must exists a tournament T such that it has at least $n!(1/2)^{n-1}$ hamiltonian paths. □

Problems

Problem 9.5. Let $Y(\alpha)$ be the number of parts of a randomly selected composition α of n. Find $\mathbb{E}(Y)$.

Proof. Returning to the stars and bars model of compositions, we see that $Y(\alpha)$ is one plus the number of bars of the composition. A bar can placed in each of the $n-1$ spots between the n balls, and it is easily seen that the probability that this occurs for each spot in a randomly selected composition is $\frac{1}{2}$. Hence we have that

$$\mathbb{E}(Y) = 1 + \frac{n-1}{2} = \frac{n+1}{2}.$$

□

Problem 9.6. Sperner's Theorem states that if A is an antichain in $2^{[n]}$, the power set of $[n]$ ordered by containment, then

$$|A| \le \binom{n}{\lfloor n/2 \rfloor}.$$

Prove Sperner's Theorem.

Proof. Let A be an antichain and C a maximal chain in $2^{[n]}$. Then how many elements of A can we expect to be in C? Clearly ≤ 1. Let $S \subseteq [n]$ with $|S| = k$, then we have $P(S \in C) = \binom{n}{k}^{-1}$, and thus the expected number of elements of A in C is given by

$$\sum_{S \in A} \binom{n}{|S|}^{-1} = \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}},$$

where a_k is the number of elements of A containing k elements. Since $\binom{n}{\lfloor n/2 \rfloor} \ge \binom{n}{k}$ for all k, we see that

$$\frac{|A|}{\binom{n}{\lfloor n/2 \rfloor}} = \sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor n/2 \rfloor}} \le \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \le 1.$$

Finally we see that

$$|A| \le \binom{n}{\lfloor n/2 \rfloor}.$$

□

Problem 9.7. Among a population of $n+1$ people, a rumor is spread at random. One person tells the rumor to a second, who in turn repeats it to a third person, etc. What is the probability that the rumor will be told k times without being repeated to any person?

Proof. The first person told clearly can't violate the condition. Then the first person told can tell any of the $n-1$ people who haven't been told, then he can tell any of the $n-2$ people who haven't been told, etc. Since these events are independent, the probability of this event is then

$$\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-k+1}{n}=\frac{(n)_k}{n^k},$$

where $(n)_k$ is a falling factorial. □

9.2 Block Designs and Error Correcting Codes

Definition. Let S be a finite set of v elements called *vertices*. Let B be a collection of b non-empty subsets of S called *blocks*. Then the pair (S,B) is called a *block design*.

If a design (S,B) contains at least one block that does not contain all of the elements of S, then it is called *incomplete*. If each block consists of exactly k vertices, then the design is called *uniform*. If each vertex occurs in exactly r blocks, then the design is called *regular*. Finally, if each pair of vertices occurs together in exactly λ blocks, then (S,B) is called *balanced incomplete block design* or BIBD of parameters (b,v,r,k,λ).

Theorem 9.8. *If a (b,v,r,k,λ)-design exists, then*

$$bk=vr \quad \text{and} \quad r(k-1)=\lambda(v-1).$$

Proof. The equality $bk=vr$ clearly holds since both sides count pairs (w,X), where x is in the block X.

To prove the second equality, fix a vertex x. Then we can pick a block that contains x in r ways, and then another element in that block in $k-1$ ways, hence the term $r(k-1)$ counts the pairs (y,T) where T is a block containing x and y. We see that the right hand side counts the same thing. Pick a vertex that is not x in $v-1$ ways, then choose a common block in λ ways. □

Definition. Let S and T be two finite alphabets. A *code* c is an injective function $c:S\to T^*$. If $t\in T^*$ is in the range of c, then t is called a *codeword* for the code c, and we denote the set of codewords by C. It $T=\{0,1\}$ then the code is called *binary*.

Definition. The injection c can easily be extended to the set S^* of all finite sequences over S by setting

$$c(s_1s_2\dots s_n)=c(s_1)c(s_2)\dots c(s_n).$$

If this extended function $c:S^*\to T^*$ is injective then we say that c is *uniquely decodable*.

A code $c:S\to T^*$ is called *prefix-free* if there are no two codewords $c(x)$ and $c(y)$ such that $c(x)=c(y)q$ for some $q\in T^*$.

Theorem 9.9. *If c is prefix-free, then it is uniquely decodable.*

Proof. Let $c : S \to T^*$ be a prefix-free code, and let us say that

$$c(x_1x_2 \dots x_k) = c(y_1y_2 \dots y_m) = t_1t_2 \dots t_n.$$

We prove that $k = m$ and that $x_1x_2 \dots x_k = y_1y_2 \dots y_k$ with $x_i = y_i$ for each i using strong induction on n. For $n = 1$ the statement is obvious. We see that $c(x_1) = c(y_1)$ must hold, otherwise one would be a prefix of the other. As c is injective this implies that $x_1 = y_1$, and hence

$$c(x_2 \dots x_k) = c(y_2 \dots y_m) = t_h \dots t_n$$

for some $h > 1$. Since the word $t_h \dots t_n$ has less than n letters, it is uniquely decodable by the inductive hypothesis, and therefore $x_i = y_i$ for all i. □

Definition. Let v and w be two n-letter codewords. The *Hamming distance* of v and w, denoted $d(v, w)$ is the number of positions in which v and w differ. A code c is *e-error correcting* if the ball of radius e around any two of its codewords are disjoint, i.e. the distance between any two codewords is at least $2e + 1$. An *(n, m, d)-code* is a code that consists of m codewords of length n so that the Hamming distance of any two codewords is at least d.

Proposition 9.10. *Let c be an r-error correcting code over the binary alphabet in which all codewords are of length n. Then the number $|C|$ of codewords in c is at most*

$$\frac{2^n}{\sum_{i=0}^{r} \binom{n}{i}}.$$

Proof. We see that 2^n is the total number of binary words of length n, and that the size of a ball of radius r must be $\sum_{i=0}^{r} \binom{n}{i}$. That is, all the ways to pick less than r positions of a word and 'flip' them. Since each ball must be disjoint in an r-error correcting code, we have that

$$|C| \cdot \sum_{i=0}^{r} \binom{n}{i} \leq 2^n,$$

yielding the result. □

Definition. Let c be an r-error correcting code over B^n. We say that c is *perfect* if each word $v \in B^n$ belongs to a ball $B(w, r)$ for some codeword w. That is, equality holds in the statement of the previous proposition.

Problems

Problem 9.11. Prove that if a (v, k, λ) BIBD exists with block set $\mathcal{B}$ on a point set V, then the set $\mathcal{B}' = \{V \setminus B : B \in \mathcal{B}\}$ is a BIBD.

Proof. We see that $\mathcal{B}'$ is a $(v, b - k, b - 2r + \lambda)$ BIBD, where b is the number of blocks of $\mathcal{B}$ and r is the number of blocks each vertex is in. The first two values are clear, and the fact that $b - 2r + \lambda$ is the number of blocks that two elements are together in is a result of inclusion exclusion. □

Problem 9.12. A t-design is a design in which every t-element set of vertices appears together in exactly λ blocks. So BIBDs are 2-designs. Prove that if D is a t-design with parameters (b, v, r, k, λ), then

$$b\binom{k}{t} = \lambda\binom{v}{t}.$$

Proof. Both sides count the number of pairs (S, B), where S is a t-element subset of the block B. On the left-hand side we pick a block in b ways, then choose t vertices from the k it contains. Then the right hand side picks t vertices and then one of the blocks containing that set of vertices. □

Problem 9.13. Construct a BIBD with parameters (12,9,4,3,1).

Proof. Just try it:

- (123)(456)(789)
- (147)(258)(369)
- (159)(267)(348)
- (168)(249)(357).

□

Problem 9.14. Construct a $(7, 4, 2)$ BIBD.

Proof. That is $v = 7$ vertices, each block having $k = 4$ vertices, and every pair of vertices appearing together in a block twice, i.e. $\lambda = 2$. To find the number of blocks b and the number of blocks r each element appears in, we have that $bk = vr$ and $\lambda(v-1) = r(k-1)$, i.e. $4b = 7r$ and $2 \cdot 6 = r \cdot 3$. So we have $r = 4$ and $b = 7$. So the complement of such a design would be a $(v = 7, k = 3, \lambda = 1)$ design, so that $3b = 7r$ and $6 = 2r$, and therefore $r = 3$ and $b = 7$. This should be easier to find:

$$(123)(145)(167)(246)(347)(257)(356),$$

and thus the desired BIBD is as follows:

$$(4567)(2367)(2345)(1357)(1256)(1346)(1247).$$

□

Problem 9.15. Let $c : S \to \{0, 1\}^*$ be a prefix-free code in which b_i codewords have length i. Prove that $\sum_i \frac{b_i}{2^i} \leq 1$.

Proof. Let w be a binary word of length longer than each codeword of c. Then the probability that the codeword p of length i is a prefix of w is $\frac{1}{2^i}$. Let S_p be the event that the codeword p is a prefix of w, and define $S = \bigcup_p S_p$, the event that w has a prefix in C. Then since the events S_p are independent we have

$$P(S) = \sum_p P(S_p) = \sum_i \frac{b_i}{2^i},$$

which must be less than or equal to one since it is a probability. □

Problem 9.16. Show an example of a code that is not prefix-free but still uniquely decodable.

Proof. Consider the code $c(x) = 10$ and $c(y) = 100$. □

Problem 9.17. Let C be a binary code of length n and minimum distance $d \geq 2e+1$. Prove the following two bounds.

$$|C| \leq \frac{2^n}{\sum_{i=0}^{e}\binom{n}{i}} \quad \text{and} \quad |C| \leq 2^{n-d+1}.$$

Proof. Since any two codewords must have Hamming distance at least $d \geq 2e+1$ from each other, then the balls of any two codewords of radius e must be disjoint. It is clear that the ball of radius e of any codeword has exactly $\sum_{i=0}^{e}\binom{n}{i}$ words in it. Since there are 2^n total binary words of length n, we have

$$|C| \cdot \sum_{i=0}^{e}\binom{n}{i} \leq 2^n,$$

yielding the first bound.

Now for the second bound, for each codeword in C, if we delete the last $d-1$ entries of each of them, then each of them is still unique since they must have Hamming distance at least d. Thus it must be that $|C| \leq 2^{n-d+1}$. □

9.3 Unlabeled Structures

Definition. Let G be a group acting on a set S, and let $i \in S$. The the set

$$G_i = \{g \in G : g(i) = g\},$$

is called the *stabilizer* of i. The set

$$i^G = \{g(i) : g \in G\}$$

is called the orbit of i.

Theorem 9.18 (Orbit-Stabilizer Theorem)**.** *Let G be a finite group acting on a set S, and let $i \in S$. Then*

$$\frac{|G|}{|G_i|} = |i^G|.$$

Definition. Let G be a group acting on a set S, and let $g \in G$. Then define

$$F_g = \{i \in S : g(i) = i\},$$

i.e. the elements of S fixed by the action of g.

Theorem 9.19 (Burnside's Lemma)**.** *Let G be a group acting on a set S. Then the number of orbits of S under the action of G is equal to*

$$\frac{1}{|G|}\sum_{g \in G}|F_g|.$$

Note. The rest of this section is on generating functions of unlabeled trees, which will be covered in the section on generating functions. Thus we omit those notes here.

Problems

Problem 9.20. A non-plane 2-tree is a rooted tree in which each non-leaf vertex has exactly 2 children, Prove that all non-plane 2-trees have an odd number of vertices. Let d_n be the number of non-plane 2-trees on $2n+1$ vertices. Prove that $d_n = b_n$, where b_n is the number of non-plane 1-2 trees on n vertices.

Proof. The first claim follows by induction. Each such tree with more than one vertex has a two subtrees that are plane 2-trees and a root.

Let T be a non-plane 2-tree enumerated by d_n, then T has $n+1$ leaves, and thus the tree $f(T)$ obtained by removing all the leaves of T is a non-plane 1-2 tree on n vertices, enumerated by b_n. We show this is a bijection by showing it has an inverse. If we take a 1-2 plane tree on n vertices and, add two leaves to each on each of its leaves, and one leaf to each vertex with one child, then we have the we will have the original non-plane 2-tree, (it must be on $2n+1$ vertices because it is a 2-tree and has n non-leaves). □

Problem 9.21. Let $d_0 = 0$, let d_n be the number of all decreasing non-plane trees on vertex set $[n]$ if $n \geq 1$. Let $D(x)$ be the exponential generating function of these numbers.

(a) Let $D_k(x)$ be the exponential generating function for the sequence counting decreasing non-plane trees in which the has exactly k children. Prove that

$$D_k'(x) = \frac{D^k(x)}{k!}.$$

(b) Use part (a) to find a closed form for $D(x)$, and then for d_n.

Proof. (a) We see that removing the root of such a tree on n vertices, (which must be labeled n), we obtain an unordered sequence of k of these trees with labels $[n-1]$. It is clear that the coefficient of $x^n/n!$ in $D_k'(x)$ is the number of such trees on $[n+1]$, (taking the derivative shifts the index). Similarly, by the product rule of exponential generating functions, the coefficient of $x^n/n!$ in $\frac{D^k(x)}{k!}$ counts the number of such unordered sequences of k trees on $[n]$.

(b) Summing over all k, we see that $D'(x) = e^{D(x)}$, which gives $D'(x)e^{-D(x)} = 1$. Integrating we obtain $-e^{-D(x)} = x - 1$, since the coefficient of integration must be -1 to conform with the fact that $D(0) = 0$. Taking logarithms we obtain

$$D(x) = \log \frac{1}{1-x} = \sum_{n \geq 1} \frac{x^n}{n},$$

and hence $d_n = (n-1)!$. □

Problem 9.22. Let T be a rooted non-plane 2-tree with n leaves, i.e. every non-leaf vertex has two children. Let $sym(T)$ be the number of non-leaf vertices v of T such that the two children of v are roots of identical subtrees.

(a) How many automorphisms does T have?

(b) How many different ways are there to bijectively label the leaves of T with the numbers $1, 2, \ldots, n$?

(c) Find an explicit formula for $\sum_T \frac{1}{|Aut(T)|}$, where the sum is taken over all rooted non-plane 2-trees with n leaves.

Proof. (a) It is not difficult to see that $|Aut(T)| = 2^{sym(T)}$.

(b) It is then clear that there are $x = \frac{n!}{2^{sym(T)}}$ bijective labelings of the leaves of T. That is, embed T in the plane, label its leaves left to right with the elements of $[n]$ in one of $n!$ ways, then each one is equivalent to $2^{sym(T)}$ labelings.

(c) For a fixed tree T, we have that $\frac{1}{|Aut(T)|} = \frac{x}{n!}$, where x is the number of bijective labelings of the leaves of T found in part (b). The sum $\sum_T x$ is then the total number of labeled rooted non-plane 2-trees, which we know is $(2n-3)!!$, (these are the same as binary total partitions), and hence we have

$$\sum_T \frac{1}{|Aut(T)|} = \sum_T \frac{x}{n!} = \frac{(2n-3)!!}{n!}.$$

□

9.4 Combinatorial Algorithms

Note. I don't anticipate any questions on this section either. I include a couple of definitions that I should know anyways.

Definition. Let $f, g : \mathbb{Z}^+ \to \mathbb{R}$. We say that

$$f(n) = O(g(n))$$

if there exists a positive constant c such that $f(n) \leq cg(n)$ for all $n \in \mathbb{Z}^+$. We say that

$$f(n) = \Omega(g(n))$$

if there exists a positive constant c such that $f(n) \geq cg(n)$. We then say that

$$f(g) = \Theta(g(n))$$

if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Note. This section includes and discusses the sorting algorithms bubble sort and merge sort, and then some algorithms pertaining to minimum weight spanning trees and paths in graphs.

9.5 Computational Complexity

Definition. A *decision problem* is a "yes-no" problem question asked about a combinatorial object.

Definition. We say that a language L is in **P** if there exists a Turing machine T and a positive integer k so that T accepts L in $O(n^k)$ time, where n is the size of the input. That is, membership in L can be tested in polynomial time.

Definition. We say that a language L is in **NP** if membership in L can be verified in polynomial time, but not necessarily found in polynomial time.

Proposition 9.23. *We have* $\mathbf{P} \subseteq \mathbf{NP}$.

Definition. We say that a language L is in **coNP** if non-membership in L can be verified in polynomial time.
