MAD 6406: Final exam. December 10, 2018

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 6 of the following 8 problems, and clearly indicate on the front page which 6 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	**Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
100%	60	

Problem 1. (10 points)

(a) Show the matrix 2-norm is invariant under unitary transformation: For $A \in \mathbb{C}^{m\times n}$ it holds that $\|AV\|_2 = \|A\|$ and $\|UA\|_2 = \|A\|$ for unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

(b) Show the Frobenius norm is invariant under unitary transformation (as above this requires showing $\|UA\|_F = \|A\|$ and $\|AV\|_F = \|A\|$).

(c) Show the Frobenius norm is not induced by any vector norm.

Problem 2. (10 points) Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = p \leq n \leq m$.

(a) Show $\operatorname{Col}(A) = \operatorname{Span}\{u_1, u_2, \ldots, u_p\}$, where $u_1, \ldots, u_p$ are the first p columns of U.

(b) Show $\operatorname{Null}(A^*) = \operatorname{Span}\{u_{p+1}, u_{p+2}, \ldots, u_m\}$.

(c) Without using the SVD or any other matrix decomposition, show $\operatorname{Col}(A)$ is orthogonal to $\operatorname{Null}(A^*)$.

Problem 3. (10 points) Let $A \in \mathbb{C}^{m\times n}$, with $m \geq n$ and $\operatorname{rank}(A) = p = n \geq 3$.

(a) Using the classical Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.

(b) Show the vector q_3 found in part **(a)** is orthogonal to both q_1 and q_2.

(c) Write an expression for the first Householder reflector H_1, used to find the QR decomposition of A. Show H_1 is both unitary and Hermetian.

Problem 4. (10 points) let P be a projector.

(a) Find all eigenvalues of P.

(b) Show $\operatorname{Null}(P) = \operatorname{Col}(I - P)$

(c) Show $\operatorname{Null}(I - P) = \operatorname{Col}(P)$.

Problem 5. (10 points)

(a) Show that if $A \in \mathbb{C}^{m\times m}$, A has a Schur decomposition.

(b) Show that if a matrix $A \in \mathbb{C}^{m\times m}$ is normal, then A has a full set of m independent eigenvectors.

Problem 6. (10 points)

(a) Show for $A \in C^{m\times n}$ and $x \in \mathbb{C}^n$, $x \in 0$, that

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0,$$

where σ_1 and σ_n are the largest and smallest singular values of A. (If you want to use the fact that $\|A\|_2 = \sigma_1$, then you need to show this as well).

(b) Show $\operatorname{cond}(A)_2 = \sigma_1/\sigma_n$

Problem 7. (10 points) Let $\|\cdot\|$ be a subordinate (induced) matrix norm. If A is $n \times n$ invertible and and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show

(a) $A + E$ is nonsinguar

(b)

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

Problem 8. Consider the matrix A given by

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 1 & 25 & 4 & 6 \\ 2 & 4 & 20 & 8 \\ 3 & 6 & 8 & 25 \end{pmatrix}$$

(a) What does Gerschgorin's disk theorem say about the location of each of the eigenvalues of A? Be specific.

(b) Use the fact that A is symmetric positive definite (SPD) to improve your answer above about the location of the eigenvalues of A.

(c) Suppose the eigenvalues of A are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Describe an algorithm that could be used to determine λ_4.