MAD 6406: Midterm. October 12, 2018

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 6 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	**Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
100%	40	

Problem 1. (10 points) Let $A \in C^{m\times n}$ with $\text{rank}(A) = n \leq m$. Let $A = QR$ be the QR decomposition of A, and $A = Q_1R_1$ be the economy QR decomposition.

(a) Show $Q_1Q_1^*$ is an orthogonal projector.

(b) Show $Q_1Q_1^*y = y$ for any $y \in \text{Col}\,(A)$.

(c) Show $Q_1Q_1^*z = 0$ for any $z \in \text{Null}\,(A^*)$.

Problem 2. (10 points)

(a) Show the matrix 2-norm is invariant under unitary transformation: For $A \in \mathbb{C}^{m\times n}$ it holds that $\|AV\|_2 = \|A\|$ and $\|UA\|_2 = \|A\|$ for unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

(b) Show the Frobenius norm is invariant under unitary transformation.

(c) Show the Frobenius norm is not induced by any vector norm.

Problem 3. (10 points) Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $\text{rank}\,(A) = p \leq n \leq m$.

(a) Show $\text{Col}\,(A) = \text{Span}\,\{u_1, u_2, \ldots, u_p\}$, where $u_1, \ldots, u_p$ are the first p columns of U.

(b) Show $\text{Null}\,(A^*) = \text{Span}\,\{u_{p+1}, u_{p+2}, \ldots, u_m\}$.

(c) Without using the SVD or any other matrix decomposition, show $\text{Col}\,(A)$ is orthogonal to $\text{Null}\,(A^*)$.

Problem 4. (10 points)

(a) Let A be a Hermetian matrix. Show the eigenvalues of A are real.

(b) Let A be a Hermetian matrix. Show the eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(c) Let $\{v_1, \ldots v_n\}$ be a set of n mutually orthogonal vectors. Show $\{v_1, \ldots, v_n\}$ is a linearly independent set.

Problem 5. (10 points) Let $A \in \mathbb{C}^{m\times n}$, with $m \geq n$ and $\text{rank}\,(A) = p = n \geq 3$.

(a) Using the classical Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.

(b) Show the vector q_3 found in part **(a)** is orthogonal to both q_1 and q_2.

(c) Write an expression for the first Householder reflector H_1, used to find the QR decomposition of A. Show H_1 is both unitary and Hermetian.

Problem 6. (10 points) Let matrix $A \in \mathbb{C}^{m\times n}$, and let vector $b \in \mathbb{C}^m$ be written in terms of the decomposition $b = b_R + b_N$, where $b_R \in \text{Col}\,(A)$, and $b_N \in \text{Null}\,(A^*)$.

(a) Let r denote the residual vector $r = b - Ax$. Show that x solves the least-squares problem $\min \|b - Ax\|_2$ precisely when $Ax = b_R$ and $r = b_N$.

(b) Write down an expression for the least-squares solution x_{ls} in terms of the singular value decomposition (SVD) of A.

(c) Write down an expression for the least-squares solution x_{ls} in terms of the QR decomposition of A.