MAD 6406: Final exam. December 4, 2019

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 6 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
100%	40	

Problem 1. (10 points) Prove that every Hermitian positive definite matrix A has a Cholesky decomposition.

Problem 2. (10 points)

(a) Show for $A \in C^{m \times n}$ and $x \in \mathbb{C}^n$, $x \neq 0$, that

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0,$$

where σ_1 and σ_n are the largest and smallest singular values of A. (If you want to use the fact that $\|A\|_2 = \sigma_1$, then you need to show this as well).

(b) Show $\mathrm{cond}(A)_2 = \sigma_1/\sigma_n$

Problem 3. (10 points) Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

(a) If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

Problem 4. (10 points)

(a) Suppose A is $n \times n$ and nonsingular, and exact data b and solution x satisfy $Ax = b$. Suppose data pertubation Δb and solution perturbation Δx further satisfy $A(x + \Delta x) = (b + \Delta b)$. Show

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathrm{cond}(A)\frac{\|\Delta b\|}{\|b\|}.$$

(b) Prove Gerschgorin's disk theorem: Let $r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$. Let D_i be the disk in $\mathbb{C}$ with center a_{ii} and radius r_i. If λ is an eigenvalue of A, then $\lambda \in \bigcup_i D_i$; in other words, λ is in at least one of the disks D_i.

Problem 5. Consider the matrix A given by

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 1 & 25 & 4 & 6 \\ 2 & 4 & 20 & 8 \\ 3 & 6 & 8 & 25 \end{pmatrix}$$

(a) What does Gerschgorin's disk theorem say about the location of each of the eigenvalues of A? Be specific.

(b) Suppose the eigenvalues of A are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Describe an algorithm that could be used to determine λ_4.

Problem 6. **(a)** Let $v = (1, 0, 1, 0, 1)^T$ and $x = (1, 1, 2, 3, 5)^T$. Find $w \in \mathbb{C}^5$ and $c \in \mathbb{C}$ such that $x = cv + w$ and $w^*v = 0$. Is there any other vector w and/or scalar c that will work? Explain.

(b) Compute the Cholesky decomposition of the matrix in **Problem 5** or explain why it does not exist.