MAD 6406: Midterm. October 2, 2019

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature:..

Directions: Submit solutions to any 4 of the following 6 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
100%	40	

Problem 1. (10 points) let P be a projector.

- **(a)** Find all eigenvalues of P.
- **(b)** Show $\text{Null}\,(P) = \text{Col}\,(I - P)$
- **(c)** Show $\text{Null}\,(I - P) = \text{Col}\,(P)$.

Problem 2. (10 points) Let $A \in \mathbb{C}^{m\times n}$.

- **(a)** Show the matrix 2-norm is invariant under unitary transformation: $\|AV\|_2 = \|A\|$ and $\|UA\|_2 = \|A\|$ for unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.
- **(b)** Prove or give a counterexample: $\|A\|_2 \leq \sqrt{\|A\|_\infty \|A\|_1}$. If you prove this, make sure to justify each nontrivial step.
- **(c)** Prove or give a counterexample: $\|A\|_F \leq \|A\|_2$, where $\|A\|_F$ is the Frobenius norm of A. If you prove this, make sure to justify each nontrivial step.

Problem 3. (10 points) Let $A = U\Sigma V^*$ be the singular value decomposition (SVD) of $A \in \mathbb{C}^{m\times n}$ with $\text{rank}\,(A) = p \leq n \leq m$.

- **(a)** Show $\{u_1, u_2, \ldots, u_p\}$, is a basis for $\text{Col}\,(A)$, where $u_1, \ldots, u_p$ are the first p columns of U.
- **(b)** Show $\{u_{p+1}, u_{p+2}, \ldots, u_m\}$ is a basis for $\text{Null}\,(A^*)$.
- **(c)** Show $\|A\|_2 = \sigma_1$, the first singular value of A.

Problem 4. (10 points)

- **(a)** Prove that every square matrix A has a Schur decomposition.
- **(b)** Prove that if square matrix A is both normal and upper triangular then it is diagonal.

Problem 5. (10 points)

- **(a)** Let $w \in \mathbb{C}^n$. Determine all eigenvalues of the Householder reflector $H(w)$, including their multiplicities. Justify your answer.
- **(b)** Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Problem 6. (10 points) Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- **(a)** Find both full and economy singular value decompositions of A.
- **(b)** Find both full and economy QR decompositions of B.