MAD 6406: Final exam. Due December 9, 2020, 3pm

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 6 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
100%	40	

Problem 1. (10 points)

(a) Show the matrix 2-norm is invariant under unitary transformation: For $A \in \mathbb{C}^{m \times n}$ it holds that $\|AV\|_2 = \|A\|$ and $\|UA\|_2 = \|A\|$ for unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$.

(b) Show the Frobenius norm is invariant under unitary transformation (as above this requires showing $\|UA\|_F = \|A\|$ and $\|AV\|_F = \|A\|$).

Problem 2. (10 points) Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m \times n}$ with $\operatorname{rank}(A) = p \le n \le m$.

(a) Show $\operatorname{Col}(A) = \operatorname{Span}\{u_1, u_2, \ldots, u_p\}$, where $u_1, \ldots, u_p$ are the first p columns of U.

(b) Show $\operatorname{Null}(A^*) = \operatorname{Span}\{u_{p+1}, u_{p+2}, \ldots, u_m\}$.

Problem 3. (10 points) Prove or provide a counterexample to the following statements

(a) Any square matrix A has a decomposition Q^*TQ where Q is unitary and T is triangular.

(b) The spectral radius is equal to the matrix 2-norm for any normal matrix A.

Problem 4. (10 points) Prove that every Hermitian positive definite matrix A has a Cholesky decomposition.

Problem 5. **(a)** Prove Gerschgorin's disk theorem: Let $r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$. Let D_i be the disk in $\mathbb{C}$ with center a_{ii} and radius r_i. If λ is an eigenvalue of A, then $\lambda \in \bigcup_i D_i$; in other words, λ is in at least one of the disks D_i.

(b) Consider the matrix A given by

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 1 & 25 & 4 & 6 \\ 2 & 4 & 20 & 8 \\ 3 & 6 & 8 & 25 \end{pmatrix}$$

What does Gerschgorin's disk theorem say about the location of each of the eigenvalues of A? Be specific.

Problem 6. **(a)** Let $v = (1,0,1,0,1)^T$ and $x = (1,1,2,3,5)^T$. Find $w \in \mathbb{C}^5$ and $c \in \mathbb{C}$ such that $x = cv + w$ and $w^*v = 0$. Is there any other vector w and/or scalar c that will work? Explain.

(b) Compute the Cholesky decomposition of the matrix in **Problem 5** or explain why it does not exist.