MAD 6406: Final exam. December 15, 2021

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 5 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
100%	40	

Problem 1. (10 points)

(a) Show the matrix 2-norm is invariant under unitary transformation: For $A \in \mathbb{C}^{m\times n}$ it holds that $\|AV\|_2 = \|A\|$ and $\|UA\|_2 = \|A\|$ for unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

(b) Show the Frobenius norm is invariant under unitary transformation (as above this requires showing $\|UA\|_F = \|A\|$ and $\|AV\|_F = \|A\|$).

Problem 2. (10 points) Prove or provide a counterexample to the following statements

(a) Any square matrix A has a decomposition Q^*TQ where Q is unitary and T is triangular.

(b) The spectral radius is equal to the matrix 2-norm for any square matrix A.

Problem 3. (10 points)

(a) Show for full rank $A \in C^{m\times n}$ and $x \in \mathbb{C}^n$, $x \neq 0$, that

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0,$$

where σ_1 and σ_n are the largest and smallest singular values of A. (If you want to use the fact that $\|A\|_2 = \sigma_1$, then you need to show this as well).

(b) Show $\text{cond}(A)_2 = \sigma_1/\sigma_n$

Problem 4. (10 points) For $x, y > 0$, consider computing $f(x,y) = \sqrt{y + x^2} - \sqrt{y}$ in floating-point arithmetic with machine precision ϵ_m.

(a) Explain the difficulties in computing $f(x,y)$, if $x^2 \ll y$. What are the absolute and relative errors if $x^2/y < \epsilon_m$, if $f(x,y)$ is computed directly from the form given above?

(b) Suppose $x^2/y < \epsilon_m$. Describe a way to compute $f(x,y)$ with more accuracy in this situation.

Problem 5. (10 points) Compute the Cholesky decomposition of the following matrix, or explain why it does not exist.

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 5 & 4 & 2 \\ 2 & 4 & 14 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix}.$$