# MAD 6407: Final exam. April 30, 2019

**First Name: .....................................** **Last Name: .....................................**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: ........................................... UFID: .............................................

**Directions: Submit solutions to any 5 of the following 6 problems, and clearly indicate on the front page which 5 you would like graded.**

**No books, no notes, no tablets, no calculators, no computers, no phones!**

**Write your solutions clearly and legibly for full credit.**

**Good luck!**

| **#** | **Points** | **Score** |
|---|---|---|
| **1** | 10 | |
| **2** | 10 | |
| **3** | 10 | |
| **4** | 10 | |
| **5** | 10 | |
| **6** | 10 | |
| **100%** | 50 | |

**Problem 1.** (10 points)

**(a)** Let $g(x) = \sqrt{1+x^2}$. Show that the Newton iteration for finding a zero of $g'(x)$ converges to zero for $|x_0| < 1$ and diverges for $|x_0| > 1$.

**(b)** Consider using Newton's method to find $x \in \mathbb{R}^n$ such that $F(x) = 0$, for $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, where $F$ is continuously differentiable on $D$. Write down the Newton iteration for finding $x_{k+1}$ from $x_k$.

**Problem 2.** (10 points)

**(a)** Define the maximum norm $\|f\|_{C[a,b]}$ for a function $f \in C[a,b]$.

**(b)** Prove for any $f \in C[a,b]$ and integer $n \geq 0$ that the best uniform approximation of $f$ in $P_n$ is unique.

**Problem 3.** (10 points) Find the values of $\alpha$ and $\beta$ that minimize

$$\int_{-1}^{1} (x^2 - (\alpha x - \beta))^2 \, \mathrm{d}\, x$$

**Problem 4.** (10 points)

**(a)** Consider the inner product on $C[0,\infty)$ given by

$$(f,g) = \int_0^\infty f(t)g(t)e^{-t} \, \mathrm{d}\, t.$$

Find three orthonormal polynomials $\pi_0, \pi_1, \pi_2$ on $[0,\infty)$ with respect to the given inner product such that the degree of $\pi_n$ is equal to $n$, $n = 0, 1, 2$.

**(b)** Find the nodes $t_1$ and $t_2$ and weights $w_1$ and $w_2$ which yield the weighted Gaussian Quadrature formula

$$\int_0^\infty f(t)e^{-t} \, \mathrm{d}\, t \approx w_1 f(t_1) + w_2 f(t_2)$$

with degree of exactness $m = 3$. **You should find the nodes exactly, and may leave the weights $w_1, w_2$ in integral form**.

**Problem 5.** (10 points) Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$ with respect to inner product $(f,g) = \int_a^b f(x)g(x)w(x) \, \mathrm{d}\, x$. Show that $\phi_k$ has $k$ distinct roots $\{x_j^{(k)}\}_{j=1}^k$ with $x_j^{(k)} \in [a,b]$, $j = 1, \ldots, k$.

**Problem 6.** (10 points) Let $x_0 = a$, $x_1 = a + h$ and $x_2 = b = a + 2h$, and let $f \in C^2[a,b]$.

**(a)** Construct the central difference approximation to $f'(x_1)$ based on the $\mathcal{P}_2$ interpolant of $f$ on $[a,b]$ with interpolation points $x_0, x_1, x_2$ (you should explicitly show how the central difference approximation is derived from the interpolant).

**(b)** Recall that the central difference approximation satisfies $|f'(x_1) - p_2'(x_1)| \leq \frac{Mh^2}{6}$, where $M$ is a constant that depends on $f$. Suppose the data is noisy and the approximation is based on the values $f_i$ where $f_i - f(x_i) = \epsilon_i$ with $|\epsilon_i| < \epsilon$, $i = 0, 1, 2$, for a given value of $\epsilon$. What is the best accuracy with which $f'(x_1)$ can be approximated? For what value of $h$ is it attained?