MAD 6407: Midterm. February 22, 2019

First Name: .... **Last Name: ....**

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 5 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
M	10	
100%	50	

(* M is the Matlab component)

Problem 1. (10 points) Consider the fixed point problem $x = f(x)$ where $f(x) = e^{-(1+x)}$.

(a) Find the largest open interval in $\mathbb{R}$ where $f(x)$ is a contraction.

(b) Suppose a fixed-point iteration $x_{k+1} = f(x_k)$ is run starting from $x_0 = -10$. Will the iteration converge or diverge? Explain.

Problem 2. (10 points) Consider using Newton's method to find $x \in \mathbb{R}^n$ such that $F(x) = 0$, for $F : D \subseteq R^n \to \mathbb{R}^n$, where D is open and convex and F is continuously differentiable on D.

(a) Write down the Newton iteration for finding x_{k+1} from x_k.

(b) Suppose $\bar{x}$ satisfies $F(\bar{x}) = 0$; and $J(x)$, the Jacobian of F evaluated at x satisfies $\|J^{-1}(x)\| \leq \mu$ for some number $\mu > 0$ for all x in a neighborhood $N \subseteq D$ of $\bar{x}$. Show for $x_k \in N$ that

$$\|x_{k+1} - \bar{x}\| \leq \mu \|\bar{x} - x_k\| \cdot \max_{0 \leq t \leq 1} \|J(x_k + t(\bar{x} - x_k)) - J(x_k)\|.$$

Problem 3. (10 points) Consider the data points $(x_1, y_1) = (0, -1)$, $(x_2, y_2) = (1, -2)$, $(x_3, y_3) = (2, 3)$.

(a) Construct the Lagrange form of the interpolating polynomial through (x_1, y_1), (x_2, y_2) and (x_3, y_3). Leave your answer in the form of a linear combination of basis functions (but write out explicitly what the basis functions and coefficients are).

(b) Construct the Newton form of the interpolating polynomial through (x_1, y_1), (x_2, y_2) and (x_3, y_3). Leave your answer in the form of a linear combination of basis functions (but write out explicitly what the basis functions and coefficients are).

(c) Let $(x_4, y_4) = (3, -1)$. Construct the interpolating polynomial through (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4). You may use any form of the interpolating polynomial that you like.

Problem 4. (10 points) Consider the function

$$t(x) = \begin{cases} t_1(x) = (x+1)^3, & -1 \leq x \leq 0 \\ t_2(x) = (1-x)^3, & 0 \leq x \leq 1 \end{cases}$$

(a) Is $t(x)$ a cubic spline on $-1 \leq x \leq 1$? If so, what kind?

(b) Find a cubic spline of the form

$$v(x) = \begin{cases} v_1(x) = (x+1)^3 + a_1 x + b_1, & -1 \leq x \leq 0 \\ v_2(x) = (1-x)^3 + a_2 x + b_2, & 0 \leq x \leq 1 \end{cases}$$

that interpolates the endpoint data $v(-1) = 0,\ v(1) = 1$.

(c) Suppose you were asked to find a cubic spline $v(x)$ of the form given above in **(b)** that interpolates the data $v(-1) = 0$, $v(0) = 5$, $v(1) = 1$. Find such a spline or explain why there is no solution.

Problem 5. (10 points) Consider the interval $[a, b]$ with the partition $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. Suppose $s(x)$ is the natural cubic spline that interpolates the data $\{(x_i, y_i)\}_{i=1}^{n+1}$, and that $g \in C^2[a, b]$ interpolates the same data. Show that

$$\int_a^b (s''(x))^2 \,\mathrm{d}\,x \leq \int_a^b (g''(x))^2 \,\mathrm{d}\,x.$$