MAD 6407: Final exam. April 21, 2021
Upload solution to Canvas by 6:00 pm on 04/21

First Name: .... **Last Name:** ....

"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Signature: .... UFID: ....

Directions: Submit solutions to any 4 of the following 5 problems, and clearly indicate on the front page which 4 you would like graded.

You MAY use Matlab for this exam. You MAY NOT consult anyone (other than the course instructor) on any of the problems or solutions.

Write your solutions clearly and legibly for full credit.

Good luck!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
100%	40	

Problem 1. (10 points) Prove for any $f \in C[a,b]$ and integer $n \geq 0$, that the best uniform approximation of f in P_n is unique.

Problem 2. (10 points) Find $x_1, x_2 \in [a,b]$, and $c_1, c_2 \in \mathbb{R}$, that maximize the degree of exactness for the quadrature formula

$$\int_a^b f(x)\,\mathrm{d}\,x \approx c_1 f(x_1) + c_2 f(x_2).$$

Problem 3. (10 points) Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$ with respect to inner product $(f,g) = \int_a^b f(x)g(x)w(x)\,\mathrm{d}\,x$. Show that ϕ_k has k distinct roots $\{x_j^{(k)}\}_{j=1}^{k}$ with $x_j^{(k)} \in [a,b]$, $j = 1, \ldots, k$.

Problem 4. (10 points) Let $x_0 = a$, $x_1 = a + h$ and $x_2 = b = a + 2h$, and let $f \in C^2[a,b]$.

(a) Construct the central difference approximation to $f'(x_1)$ based on the $\mathcal{P}_2$ interpolant of f on $[a,b]$ with interpolation points x_0, x_1, x_2 (you should explicitly show how the central difference approximation is derived from the interpolant).

(b) Construct the approximation to $f''(x_1)$ based on the p_2 interpolant of f on $[a,b]$ with interpolation points x_0, x_1, x_2.

(c) Recall that the central difference approximation satisfies $|f'(x_1) - p_2'(x_1)| \leq \frac{Mh^2}{6}$, where M is a constant that depends on f. Suppose the data is noisy and the approximation is based on the values f_i where $f_i - f(x_i) = \epsilon_i$ with $|\epsilon_i| < \epsilon, \quad i = 0,1,2$, for a given value of ϵ. What is the best accuracy with which $f'(x_1)$ can be approximated? For what value of h is it attained?

Problem 5. (10 points) [Computational: Use Matlab, Python (or some other language if you prefer)]

The *Littlewood-Salem-Izumi constant* α_0 is the unique solution $\alpha_0 \in (0,1)$ of

$$\int_0^{3\pi/2} \frac{\cos(t)}{t^\alpha}\,\mathrm{d}\,t = 0.$$

Use Newton's method together with a composite Clenshaw-Curtis quadrature rule based on the $\mathcal{P}_1$ interpolant (recall, that means choose the interpolation points as the two Chebyshev points on each subinterval) to approximate α_0. Show your results using $n = \{10, 100, 1000\}$ subintervals.