On a question of Hamkins and Löwe

(PRELIMINARY)

William J. Mitchell

June 19, 2008

Abstract

Hamkins and Löwe asked whether there can be a model N of set theory with the property that $N \equiv N[H]$ whenever H is a generic collapse of a cardinal of N onto ω. We obtain a lower bound, a cardinal κ with a κ^+-repeat point, for the consistency of such a model. We do not know how to construct such a model, under any assumption.

We do construct, from a cardinal κ with $o(\kappa) = \kappa^+$, a model N which satisfies the desired condition when H is the collapse of any successor cardinal. We also give a much weaker lower bound for this property.

Joel Hamkins and Benedikt Löwe have asked, in connection with results reported in [1], whether there can be a model N of ZFC set theory such that $N[H] \equiv N$ whenever H is the generic collapse of any cardinal onto ω. This note gives some partial results related to this question.

In the positive direction we have the following partial result:

Theorem 1. *Suppose there is a cardinal κ with $o(\kappa) = \kappa^+$. Then there is, in a generic extension, a model N of ZFC with the property that $N[H] \equiv N$ whenever H is a generic collapse of some successor cardinal λ of N onto ω.*

The following result gives a lower bound, much weaker than the hypothesis of Theorem 1, for consistency strength of the conclusion of that theorem:

Theorem 2. *Suppose that $V \equiv V[H]$ for any cardinal λ and any generic $H \subset$ $\mathrm{Coll}(\omega, \lambda^+)$. Then there is an inner model in which $\{\, \lambda : o(\lambda) > \alpha \,\}$ is stationary for all ordinals α.*

The conclusion of Theorem 2 does not imply the existence of a model with a cardinals κ such that $o(\kappa) = \kappa$, and so is much weaker than the hypothesis of Theorem 1. In addition if λ is a regular cardinal, regular, or if $\lambda = \Omega$, the class of all ordinals, then the proof of Theorem 2 does not imply that $o(\lambda) > 1$.

To state our lower bound for the full property which Hamkins and Löweasked for, we need a definition:

Definition 3. We define the notion of an *α-repeat point* by recursion on α: A measure U on a cardinal κ is an α-repeat point if for every $\alpha' < \alpha$ and every set $x \in U$ there is a α'-repeat point $U' \lhd U$ with $x \in U'$.

In particular any measure is a 0-repeat point, and a measure U is a 1-repeat point if and only if it is a weak repeat point.

The lower bound for the Hamkins-Löwe property is somewhat easier to state for a model of Morse-Kelley, rather than Zermelo-Fraenkel, set theory. If it were stated for a model of Zermelo-Frankel set theory, then the measures on the class Ω of ordinals would measure only definable subclasses of Ω.

Theorem 4. *Suppose that κ is an inaccessible cardinal, and that V is a model of Morse-Kelley set theory with the property that $V[H] \equiv V$ whenever $H \subset \mathrm{Coll}(\omega, \lambda)$ is generic for any cardinal $\lambda < \kappa$. Then there is an inner model of Morse-Kelly set theory in which there is a Ω^+-repeat point on Ω.*

The conclusion of this theorem is weaker than the existence of a cardinal such that $o(\kappa) = \kappa^{++}$; indeed if $o(\kappa) = \kappa^{++}$ and $\langle U(\kappa, \beta) : \beta < \kappa^{++} \rangle$ enumerates the measures on κ in K then for each $\alpha < \kappa^{++}$ there is a closed and unbounded set of $\beta < \kappa^{++}$ such that $U(\kappa, \beta)$ is an α-repeat point.

1 The Upper Bound

Let κ be a cardinal with $o(\kappa) = \kappa^+$. We may assume without loss of generality that $V = L[\mathcal{U}]$ where $\mathcal{U}$ is a coherent sequence of ultrafilters and κ is the least cardinal satisfying $o^{\mathcal{U}}(\kappa) = \kappa^+$. Our model will be $N = V_\kappa[C, F][H_0]$, where C is a closed unbounded subset of κ which is Radin generic, in the sense of [4]; F is a function with domain C such that if $\nu \in C$ then $F(\nu)$ is a generic subset of the collapse $\mathrm{Coll}(\nu^+, \min(C \setminus \nu + 1)$; and $H_0 \subseteq \mathrm{Coll}(\omega, \min(C))$ is generic.

The next two lemmas state the properties of this forcing needed for the proof of Theorem 1:

Lemma 5. *The only cardinals of V which are collapsed in $V[C, F]$ are the cardinals δ of V such that $\lambda^+ < \delta \leq \min(C \setminus \lambda^+)$ for some $\lambda \in C$. Furthermore the only regular cardinals of V which are not collapsed but are singular in $V[C, F]$ are the limit members of C.*

The next lemma is the key property of this forcing:

Lemma 6. *Let σ be any sentence of set theory with parameters from V. Then there is a cardinal $\xi_\sigma < \kappa$ such that whenever λ and λ' are successor cardinals in the interval $\xi_\sigma < \lambda', \lambda < \kappa$, and $H \subset \mathrm{Coll}(\omega, \lambda)$ and $H' \subset \mathrm{Coll}(\omega, \lambda')$ are generic, then $V[C, F][H] \models \sigma$ if and only if $V[C, F][H'] \models \sigma$.*

Proof of Theorem 1 from Lemmas 5 and 6. Let $\langle \sigma_n : n < \omega \rangle$ enumerate the sentences of set theory without parameters. For each $n < \omega$ let ξ_n be the cardinal ξ_σ given by Lemma 6, when σ is the sentence (with parameter κ) "$V_\kappa \models \sigma_n$". Now let λ_0 be any successor member of C greater than $\sup_{n<\omega} \xi_n$, and let N be $V_\kappa[C, F][H_0]$ where $H_0 \subset \mathrm{Coll}(\omega, \lambda_0)$ is generic. Then N is as required, since for any $n < \omega$, any successor cardinal λ of N, and any generic

$H \subseteq \mathrm{Coll}(\omega, \lambda)$,

$$N \models \sigma_n \iff V[C,F][H_0] \models (V_\kappa \models \sigma_n)$$
$$\iff V[C,F][H_0][H] \models (V_\kappa \models \sigma_n) \iff N[H] \models \sigma_n.$$

Here the second equivalence uses the fact that the forcing notion $\mathrm{Coll}(\omega, \lambda_0) \times \mathrm{Coll}(\omega, \lambda)$ is equivalent to $\mathrm{Coll}(\omega, \lambda)$. $\square$

1.1 The forcing

We begin by defining the forcing which we will be using. We will assume the universe V is equal to $L[\mathcal{U}]$ for a coherent sequence $\mathcal{U}$, and that there is a cardinal κ such that $o(\kappa) = \kappa^+$. The assumption that $o(\kappa) = \kappa^+$ will be needed only to show that κ remains inaccessible. Throughout the definition of the forcing and exposition of its properties (other than the regularity of κ), we allow κ to be any cardinal.

Definition 7. A condition of the Radin forcing R_κ is a pair (a, A) where $a \subset A \subset \kappa$, a is finite, and $A \cap \lambda \in U$ whenever $\lambda \in A \cup \{\kappa\}$ and U is a measure on λ.

The forcing order on R_κ is given by $(a', A') \leq (a, A)$ if $a' \supseteq a$ and $A' \subseteq A$. The order $\leq^*$ is defined by $(a', A') \leq^* (a, A)$ if $a' = a$ and $A' \subseteq A$.

This is equivalent to the forcing defined in [2]. If $G \subseteq R^*_\kappa$ is generic then we write $C = C(G) = \bigcup\{\, a^p : p \in G \,\}$, and we say that C is *Radin generic.* If κ is measurable then C is closed and unbounded in κ, but it should be noted that we do not assume that κ is measurable. If κ is not measurable then C will be bounded in κ and may be empty.

We use a modification of the forcing R_κ which simultaneously generates the collapse function F. We write $\mathrm{Coll}(\alpha, \beta)$ for the usual forcing to collapse β onto α: conditions in $\mathrm{Coll}(\alpha, \beta)$ are functions $\sigma \colon x \to \beta$ with $x \in [\alpha]^{<\alpha}$. We rely on the observation that $\mathrm{Coll}(\alpha, \lambda) \subset \mathrm{Coll}(\alpha, \lambda')$ whenever $\alpha < \lambda < \lambda'$.

Definition 8. The conditions in the forcing R^*_κ are triples (a, A, f) such that $(a, A) \in R_\kappa$, f is a function with $\mathrm{domain}(f) = A$, and $f(\nu) \in \mathrm{Coll}(\nu^+, \min(A \setminus \nu^+))$ for each $\nu \in A$.

The forcing ordering $\leq$ on R^*_κ is defined by $(a', A', f') \leq (a, A, f)$ if $(a', A') \leq (a, A)$ in R_κ and $f'(\nu) \leq f(\nu)$ in $\mathrm{Coll}(\nu, \cdot)$ for each $\nu \in A'$. The direct order $\leq^*$ is defined by $(a', A', f') \leq^* (a, A, f)$ if $(a', A', f') \leq (a, A, f)$ and $a' = a$.

It will sometimes be convenient to write triples (a, A, f) which do not, strictly speaking, satisfy Definition 8. We will identify such a triple with the triple $(a, A', f{\restriction}A')$ where A' is defined resursively as the set of $\lambda \in A$ such that $A' \cap \lambda \in U$ for all measures U on λ and $f[A' \cap \lambda] \subseteq \lambda$, provided that the latter triple is a condition in R^*_κ.

We need to define some notation in order to describe the factorization properties of R^*_κ. If P is any forcing order and $p \in P$ then we write P/p for

$\{ p' \in P : p' \leq p \vee p \leq p' \}$. Thus a generic subset of P/p is a generic subset G of P with $p \in P$.

If $p = (a, A, f) \in R^*_\kappa$ and λ is a cardinal then we write $p{\restriction}\lambda = (a \cap \lambda, A \cap \lambda, f)$ and $p{\downharpoonright}\lambda = (a \setminus \lambda, A \setminus \lambda, f)$.

In the proof of Lemma 6 we make use of the following observation:

Proposition 9. *Suppose that $p = (a, A, f) \in R^*_\kappa$, $\lambda \in a$, and $o(\lambda) = 0$. Then $p{\restriction}\lambda \in R^*_\lambda$, $p{\downharpoonright}\lambda \in R^*_\kappa$, and*

$$R^*_\kappa/p \equiv R^*_\lambda/p{\restriction}\lambda \times R^*_\kappa/p{\downharpoonright}\lambda.$$

□

Here $\equiv$ means equivalence as forcing orders. For the present, the following definitions are more useful:

Definition 10. Suppose that $p \in R^*_\kappa$ and $\lambda < \kappa$. Then $p{\restriction_<}\lambda = p{\restriction}\gamma^+$ where $\gamma = 0$ if $a \cap \lambda = \emptyset$ and otherwise $\gamma = \sup\operatorname{range}(f(\sup(a \cap \lambda)))$, provided that $a \cap \lambda \neq \emptyset$.

We write $R^*_{<\lambda} = \{ p{\restriction_<}\lambda : p \in R^*_\kappa \}$.

If $q \in R^*_{<\lambda}$ and $p \in R^*_\kappa$ then we write $q \frown p = q \frown p{\downharpoonright}\gamma^+ = (a^q \cup a^p \setminus \gamma, A^q \cup A^p \setminus \gamma^+, f^q \cup f^p{\restriction}(A^p \setminus \gamma^+)$ where γ is as in the definition of ${\restriction_<}$.

Note that if $\mu = \sup(a^q)$ then a condition $q \in R^*_{<\lambda}$ could be regarded as a pair $(q{\restriction}\mu, F^q(\mu)) \in R^*_\mu \times \operatorname{Coll}(\mu^+, \cdot)$. This gives us the factorization

$$R^*_\kappa/p \equiv R^*_\mu/p{\restriction}\mu \times \operatorname{Coll}(\mu^+, \cdot) \times R^*_\kappa/p{\downharpoonright}\gamma^+.$$

Here $\mu \in a^p$, $\gamma = \sup\operatorname{range}(f^p(\mu))$, and the second use of $\times$ requires the caveat that the cardinal which is collapsed, denoted by '$\cdot$', is determined by a condition in $R^*_\kappa/p{\downharpoonright}\gamma^+$.

====================================

Lemma 11. *Suppose $\eta \leq \kappa$ and $\vec{p} = \langle p_\nu : \nu < \eta \rangle$ is a $\leq^*$-descending sequence such that $\nu < \nu'$ implies $p_{\nu'}{\restriction}\nu = p_\nu{\restriction}\nu$. Then there is a greatest lower bound $\bigwedge_{\nu<\eta} p_\eta$ of the sequence $\vec{p}$.* □

Lemma 12. *Suppose that p is a condition and σ is a sentence of the forcing language. Then there is $p' \leq^* p$ with the following property: Suppose $p'' \leq p'$ decides σ, $p''{\restriction}\xi$ is defined, $\xi > \max(q^{p''} \setminus a^p)$, and $p''{\downharpoonright}\xi \leq^* p'{\downharpoonright}\xi$. [[Also ξ inaccessible?]] Then $p''{\restriction}\xi \frown p'{\downharpoonright}\xi$ decides σ.*

Proof. Let $\langle q_\nu : \nu < \kappa \rangle$ enumerate $R^*_{<\kappa}$ so that $\xi < \kappa$, $\nu < \nu'$ and $q_{\nu'} \in R^*_{<\xi}$ implies $q_\nu \in R^*_{<\xi}$. Note that if p'' and ξ are is as above then $p''{\restriction}\xi = q_\nu$ for some $\nu < \xi$.

Now define a $\leq^*$-decreasing sequence $\langle p_\nu : \nu < \kappa \rangle$ so that if $\nu < \nu'$ then $p_{\nu'}{\restriction}\nu = p_\nu{\restriction}\nu$ and if there is some $p' \leq^* p_\nu{\downharpoonright}\nu$ such that $q_\nu \frown p'$ decides σ (or, for the second paragraph, decides σ_α for some $\alpha < \nu$) then $q_\nu \frown p_\nu{\downharpoonright}\nu$ does so. □

Definition 13. [[Move this up somewhere]] If $G \subseteq R^*_\kappa$ is generic then we write $C = C(G)$ for $\bigcup\{\, a^p : p \in G \,\}$ and $F = F(G)$ for the function with domain C such that $F(\lambda) = \bigcup\{\, f^p(\lambda) : p \in G \,\}$.

Then $V[G] = V[C, F]$.

*** [[This too?]] Define: If $p \in R^*_\lambda$ and $p' \in R^*_{\lambda,\kappa}$ then $p \frown p' = (a^p \cup a^{p'}, A^p \cup A^{p'}, f^p \cup f^{p'}) \in R^*_\kappa$.

Lemma 14. *1. (Factorization) Suppose that $p = (a, A, f) \in R^*_\kappa$, $\lambda \in a$, and $o(\lambda) = 0$. Then $p \restriction \lambda \in R^*_\lambda$ and $p \downharpoonright \lambda \in R^*_\kappa$, and R^*_κ / p is equivalent to $R^*_\lambda / p \restriction \lambda \times R^*_\kappa / p \downharpoonright \lambda$.*

If $\lambda \in C$ then $V[C, F] = V[C \cap \lambda, F \restriction \lambda][C \setminus \lambda, F \downharpoonright \lambda]$ where $(C \cap \lambda, F \restriction \lambda) \times (C \setminus \lambda, F \downharpoonright \lambda]$ is a generic subset of $R^*_\lambda \times R^*_{\lambda,\kappa}$.

2. (Prikry property) Suppose that $p \in R^*_\kappa$ and σ is a sentence of the forcing language. Then there is $p' \leq^* p$ such that $p' \parallel \sigma$.

3. (Presaturation) The forcing R^*_κ is κ^+-presaturated. Furthermore if p is a condition, and $\lambda \in a^p$ then for any name $\dot{x}$ denoting a subset of λ there is a condition $p' \leq^(p$ such that for each $\nu < \lambda$ any any condition $q \leq p'$, if $q \parallel \nu \in \dot{x}$ then $q \restriction \lambda \frown p' \downharpoonright \lambda \parallel \nu \in \dot{x}$.

4. $V[G] = V[C][F]$.

Proof of presaturation, Clause 3. First, to prove that R^*_κ is presaturated, suppose that $\langle\, D_\nu : \nu < \kappa \,\rangle$ is a sequence of dense subset of κ and $p \in R^*_\kappa$. Define a $\leq^*$-decreasing sequence $\langle\, p_\xi : \xi < \kappa \,\rangle$ of conditions, with $p_0 \leq^* p$ so that if $q \in R^*_{<\xi}$ and $\nu < \xi$ then either $q \frown p_\xi \downharpoonright \xi \in D_\nu$ or else there is no $p' \leq^* p_\xi$ such that $q \frown p' \downharpoonright \xi \in D_\nu$.

Now let p' be the diagonal intersection of the conditions p_ξ. If $q \leq p'$ is in D_ν for some ν then pick $\xi < \lambda$ so $a^q \subseteq \xi$ and $q \restriction \xi \in R^*_{<\xi}$. Then $q \downharpoonright \xi \leq^* p_\xi \downharpoonright \xi$, so by the choice of p_ξ we have $q \restriction \xi \frown p_\xi \downharpoonright \xi \in D_\nu$. Since $p' \leq^* p_\xi$ it follows that $q \restriction \xi \frown p' \downharpoonright \xi \in D_\nu$.

This completes the proof of presaturation. For the remainder of clause 3 we modify the argument above, using the Prikry property: we define $\langle\, p_\xi : \xi < \lambda \,\rangle$ so that if $q \in R^*_{<l}$ and $\nu < \xi$ then p_ξ whether there is $q' \in \dot{G}$ such that $q' \restriction \xi = q$ and q' decides $\nu \in \dot{x}$ and, if so, which way q' decides $\nu \in \dot{x}$. Again, set p' equal to the diagonal intersection of the conditions p_ξ. Now suppose that $q \leq \pi'$ decides $\nu \in \dot{x}$, and fix $\xi < \lambda$ such that $a^q \cap \lambda \subseteq \xi$, $q \restriction \xi \in R^*_{<\xi}$, and $\nu < \xi$. By the choice of p_ξ the condition $q \restriction \xi \frown p_\xi \downharpoonright \xi$, and hence $q \restriction \xi \frown p' \downharpoonright \xi$, decides $\nu \in \dot{x}$ the same way as q does. $\square$

Proof of Prikry property. [[For the case $a^p \neq \emptyset$, use the argument for presaturation. First, $\mathrm{Coll}(\lambda^+, \min(C \setminus \lambda) \times R^*_\kappa / p \downharpoonright \lambda$

$p_\xi \restriction \lambda$ decides if there is a direct extension of $p_{<\xi} \downharpoonright \lambda$ deciding σ, and, if so, which way. If the answer is yes, then $p_\xi \downharpoonright \lambda$ is this extension. If not, then using the Prikry property to get $p_\xi \downharpoonright \lambda$ to force that no $q' \leq p_\xi$ decides σ with $q' \restriction \xi = q$.]]

Fix a sentence σ, and a condition $p = (\emptyset, A, f)$, find $(\emptyset, A', f') \leq^* p$ so $p \parallel \sigma$. [[*** will deal later with $a^p \neq \emptyset$.***]]

First, by Lemma 12 we may assume that if q is any condition with $q{\downharpoonleft}\xi \leq^* p_\xi$ which decides σ, then $q{\upharpoonright}\xi \frown p{\downharpoonleft}\xi$ does so.

Now for any $\xi < \kappa$ and $q \in R^*_{<\xi}$, let A_q be the set of $\lambda \in A \setminus \xi$ such that for some $r_\lambda \leq^* (\{\lambda\}, A^p, f^p){\downharpoonleft}\xi$ the condition $q \frown r_\lambda{\downharpoonleft}\xi \parallel \sigma$.

We claim that if q is such that $A_q \in U$ for some measure U on κ, then $q \frown p{\downharpoonleft}\xi \parallel \sigma$. Towards proving this claim, let U be $\triangleleft$-minimal such that $A_q \in U$. By shrinking A^q if necessary, and using $\neg\sigma$ for σ if necessary, we may assume that $q \frown r_\lambda{\downharpoonleft}\xi \Vdash \sigma$ for each $\lambda \in A_q$ From the choice of p we may also assume that $r_\lambda{\downharpoonleft}\eta = p{\downharpoonleft}\eta$ where η is least such that $\eta = \sup(\mathrm{range}(f^{r_\lambda}(\lambda)))$. Set $B_\lambda = A^{r_\lambda} \cap \lambda$ and $g_\lambda = f^{r_\lambda}{\upharpoonright}(B_\lambda)$. By shrinking A_q again, we can assume that there is B and g so that if $\lambda \in A_q$ then $B_\lambda = B \cap \lambda$ and $g_\lambda = g{\upharpoonright}\lambda$. By the minimality of U, we can also assume that $A_q \cap B = \emptyset$, and it follows that if $\lambda \in A_q \cup B$ then there is no measure U' on λ such that $A_q \cap \lambda \in U'$.

Now we can find a condition $p' = (\emptyset, B \cup A_q \cup B', f^{p'}) \leq^* p$ where for each $\lambda \in B'$ there is a measure U' on λ with $A_q \cap \lambda \in U'$. and with $f^{p'}$ defined by $f^{p'}{\upharpoonright}B = g$, $f^{p'}{\upharpoonright}B' = f^p{\upharpoonright}B'$, and $f^{p'}(\lambda) = f^{r_\lambda}(\lambda)$ for $\lambda \in A_q$.

I claim that $q \frown p'{\downharpoonleft}\xi \Vdash \sigma$. To see this, it is enough to show that below each condition $p'' \leq q \frown p'{\downharpoonleft}\xi$ there is a condition p''' forcing σ First, if there is $\lambda \in a^{p''} \cap A_q$ such $a^{p''} \cap \lambda \setminus \xi \subseteq B$ then by replacing $A^{p''} \cap (\lambda \setminus \xi)$ with $A^{p''} \cap B \cap (\lambda \setminus \xi)$ we obtain a condition p''' below both p'' and r_λ, so that $p''' \Vdash \sigma$. Otherwise, if there is no $\lambda \in A_q \cap a^{p''} \setminus \xi$, let λ' be the least member of $a^{p''} \cap (B' \setminus \xi)$ is there is one, and κ otherwise. Then $A_q \cap A^{p''} \cap (\xi, \lambda') \neq \emptyset$, so if we pick λ in this set then the condition $(a^{p''} \cup \{\lambda\}, A^{p''}, f^{p''}) \leq p''$ falls into the previous case.

This completes the proof that if there is U on κ such that $A_q \in U$ then there is $p' \leq^* p$ so that $q \frown p'{\downharpoonleft}\xi \parallel \sigma$. Now define $p' = (\emptyset, A^{p'}, f^p) \leq^* p$ by letting $A^{p'}$ be the intersection of A^p with the diagonal intersection of $\{\, \kappa \setminus A_q : q \in R^*_\xi \;\&\; q \frown p{\downharpoonleft}\xi \nparallel \sigma \,\}$. Let $q \leq p'$ be such that $q \parallel \sigma$ and $|a^q|$ is as small as possible. If $a^q = \emptyset$ then $q \leq^* p$, so that q is as required. Thus it will be sufficient to reach a contradiction from the assumption that $a^q \neq \emptyset$. To this end, let $\lambda = \max(a^q)$. Then $\lambda \in A_{q{\upharpoonright}\lambda}$, and 1 By the choice of $A^{p'}$ it follows that $A_{q{\upharpoonright}\lambda} \in U$ for some measure on κ. Fix $\xi < \lambda$ so $q{\upharpoonright}\lambda \in R^*_{<\xi}$. Then Then there is $q' \leq^* (q{\upharpoonright}\xi) \frown (p'{\downharpoonleft}\xi)$ so that $q' \parallel \sigma$. But $a^{q'} = a^q \setminus \{\lambda\}$, contradicting the minimality of a^q.

This completes the proof that if $a^p = \emptyset$ then there is $p' \leq^* p$ such that $p' \parallel \sigma$. We now combine the proof of the general case of clause 2 with the proof (assuming the truth of clause 2) of clause 3. □

Proof of Lemma 5. We need to show that the only cardinals collapse by R^*_κ are those in the half closed intervals $(\lambda^+, \lambda']$ where $\lambda \in C$ and $\lambda' = \min(C \setminus \lambda^+)$. For cardinals $\tau < \min(C)$ this follows from the Prikry property together with the closure of R^*_κ, and for cardinals $\tau > \kappa^+$ it follows from the fact that $|R^*_\kappa| = \kappa^+$. Thus it is sufficient to show that no cardinal λ^+ with $\lambda \in C \cup \{\kappa\}$ is collapsed.

[[Clause 3 is misstated (and is certainly not presaturation). Suppose $\Vdash \dot\tau \to \lambda \to \lambda^+$. Using the Prikry property, $\tau \in V[C{\restriction}\lambda, F{\restriction}\lambda, F(\lambda)]$. This is equal to $V[F(\lambda)][C{\restriction}\lambda, F{\restriction}\lambda]$ which is a generic extension via $\mathrm{Coll}(\lambda^+, \lambda') \times R^*_\lambda$. Working in $V[F(\lambda)]$, let p' be the diagonal intersection of a $\leq^*$-descending conditions $p_\xi < p$ such that for each $q \in R^*_\xi$ and $\nu < \xi$, either there is η such that $q \frown p_\xi{\downharpoonright}\xi \Vdash \dot\tau(\nu) = \eta$ or else there is no $q' \leq^* q \frown p_\xi{\downharpoonright}\xi$ with $q'{\restriction}\xi = q$ so that $q \frown p'{\downharpoonright}\xi$ decides the value of $\dot\tau(\nu)$. Then p' forces, for each $\nu < \lambda$, $\dot\tau(\nu) \in \{\,\eta : \exists\xi < \lambda \exists q \in R^*_\lambda\; q \frown p'{\downharpoonright}\xi \Vdash \dot\tau(\nu) = \eta\,\}$, which is a set of size at most λ.

[[This *is presaturation!* Also it can be stated for R^*_κ and then used for arbitrary R^*_λ. It can come before the proof of the Prikry property.]] $\square$

Now we are ready to prove Lemma 6 and thereby complete the proof of Theorem 1.

Proof of Lemma 6. We are given a sentence σ, with parameters from V, and a generic set $G \subseteq R^*_\kappa$, which in turn gives the generic model $V[C, F]$. We will work inside the model $V[C, F]$.

Note that the homogeneity of the collapse forcing implies that if $H \subseteq \mathrm{Coll}(\omega, \lambda)$ is generic then the truth of σ in $V[C, F][H]$ does not depend on the genetic set H, but only on the cardinal λ. The Prikry property implies that there is a dense set of conditions in R^*_κ of the form $q \frown p_0 \in R^*_\kappa$ where $q \in R^*_{<\lambda}$, $a^{p_0} = \emptyset$, and p_0 decides, for $H \in \mathrm{Coll}(\omega, \min(C))$, whether $V[\dot G][H] \models \sigma$. Fix such a condition in G, and let $\lambda_\sigma < \kappa$ be such that $q \in R^*_{\lambda_0}$.

To see that this λ_σ is as required, fix any successor cardinal λ in the interval $\lambda_\sigma \leq \lambda < \kappa$. Then λ is the cardinal successor in V of a cardinal $\lambda^- \in C$, and $F(\lambda^-)$ is a generic subset of $\mathrm{Coll}(\lambda, \lambda')$ where $\lambda' = C \setminus \lambda$. Then we can write $V[C, F][H]$ as $V[C{\downharpoonright}\lambda', H{\downharpoonright}\lambda'][C \cap \lambda, F{\restriction}\lambda][F(\lambda^-)][H]$. The last three terms are generic for the forcing $R^*_{\lambda^-} \times \mathrm{Coll}(\lambda, \lambda') \times \mathrm{Coll}(\omega, \lambda)$. This is a forcing of cardinality λ' which collapses λ' to ω, and hence is equivalent to $\mathrm{Coll}(\omega, \lambda')$. Thus there is generic $H' \subset \mathrm{Coll}(\omega, \lambda')$ such that $V[C{\downharpoonright}\lambda', H{\downharpoonright}\lambda'][C \cap \lambda, F{\restriction}\lambda][F(\lambda^-)][H] = V[G{\downharpoonright}\lambda'][H']$. Now $p_0 \in G{\downharpoonright}\lambda'$, so σ is true in $V[G][H] = V[G{\downharpoonright}\lambda'][H']$ if and only if p_0 forces it to to be true. Since $\lambda > \lambda_\sigma$ was arbitrary, this completes the proof of lemma 6 and hence of Theorem 1. $\square$

5/13/08 — This does not work if the model is of the form $V[C][F]$ where more or less normal collapse maps $F(\eta)$ are generic over $V[C]$. The problem is that if we try to find H' as above, and the forcing for $F(\eta)$ is λ'^{+}-closed, then in $V[C][F][H]$ the set $C \cap \lambda'$ is coded up into $F(\eta)$. However this would not be true in a generic extension $V[C'][F'][H]$, where $\lambda' = \min(C')$.

5/13/08 — This can be modified to have successor members of C be the successor cardinals instead of their successors. That was my original model, but I switched in an unsuccessful attempt to allow collapses of singular cardinals as well as successor cardinals (following on Philip Welch's observation that if the core model exists with the weak covering lemma, then every successor cardinal below κ must be the successor in K of a cardinal of K.

5/13/08 — The problem with getting a model where any cardinal below κ can be collapsed with a method like this one is that since the limit members of C have a generic subset making them singular, the successor members of C must also have such a generic subset, at least in $V[G][H]$. It's fairly easy to see that H doesn't add such a sequence, at least over K and using the construction here.

It doesn't seem to be impossible that the forcing G could add such a sequence for each successor member λ of C, since λ is collapsed in the actual model. However it is not clear how one would arrange, say, that collapsing the ωth member of C would look like collapsing the ω_1th member of C.

We conclude this section of the paper with some remarks on possible strengthenings of Theorem 1. First, the hypothesis $o(\kappa) = \kappa^{+}$ is clearly more than is needed: it implies that κ is regular in $V[C]$, but all that is needed is that $V_\kappa[C] \models \text{ZFC}$. However, since the lower bound given by Theorem 2 below is qualitatively weaker than than the hypothesis of Theorem 1, it does not seem useful to look at such incremental improvements to the hypothesis of this proof of the theorem.

2 The lower bound

***** Joint with Philip Welch?????

This section gives the proof of Theorems 2 and 4, giving lower bounds for the consistency strength of the conclusion of Theorem 1 and of the actual question asked by Hamkins and Löwe. The proof of Theorem 2 is given in subsection 2.1. The proof of Theorem 4, given in subsection 2.2, will build on subsection 2.1.

We assume throughout this section that the core model K exists and satisfies the covering lemma. The failure of this assumption would imply much more than the conclusions of Theorems 2 and 4.

2.1 Collapsing successor cardinals only

For the proof of Theorem 2 we write S for the class of successor cardinals, and we assume that the conclusion of Theorem 1 holds: that is, $V[H] \equiv V$

whenever $\lambda \in S$ and $H \subset \mathrm{Coll}(\omega, \lambda)$ is generic. All of the results of this section remain valid if S is, instead, the set of all cardinals (or, indeed, any suitably definable proper class of cardinals). We will make use of this observation in Subsection 2.2, where we will take S to be the class of all cardinals and use the results of Subsection 2.1 using this alternate choice of class S.

Because of the homogeneity of the collapse forcing $\mathrm{Coll}(\omega, \lambda)$, the truth of all assertions we will make about the models $V[H]$, with $H \subset \mathrm{Coll}(\omega, \lambda)$, will depend only on the cardinal λ and not on the choice of H. Hence we will write $V^{\mathrm{Coll}(\omega,\lambda)}$ for a models $V[H]$ with H any generic subset of $\mathrm{Coll}(\omega, \lambda)$.

Since ${\lambda^+}^V$ is equal to ω_1 in any model $V^{\mathrm{Coll}(\omega,\lambda)}$, our assumption implies that for any $\lambda \in S$, any generic $H \subset \mathrm{Coll}(\omega, \lambda^-)$, and any formula ϕ of Morse-Kelly set theory with no parameters other than those stated,

$$V \models \phi(\omega_1, S) \quad \iff \quad V^{\mathrm{Coll}(\omega,\lambda)} \models \phi(\lambda, S \setminus \lambda). \tag{1}$$

Of course this statement implies that (1) holds if the formula ϕ has parameters which are definable (using the same formula) in the relevant models.

Through most of this section we will use (1) only for formulas ϕ in the language of Zermelo-Frankel set theory, without quantifiers over classes. Where formulas of Morse-Kelly set theory are used, this fact will be explicitly noted.

The following is our main lemma:

Lemma 15. *Let $\phi(\alpha, \lambda, A)$ be a formula of set theory, with the class variable A being used as a predicate, and let α be an ordinal. Further suppose that*

$$\forall\lambda \in S \forall\nu \le \lambda \left(\phi(\alpha, \lambda^+, S \setminus \lambda + 1) \iff V^{\mathrm{Coll}(\omega,\nu)} \models \phi(\alpha, \lambda^+, S \setminus \lambda + 1)\right. \tag{2}$$

Then for any ordinal α the class $\{\, \lambda \in S : \phi(\alpha, \lambda^+, S \setminus \lambda + 1) \,\}$ is a finite union of intervals of S.

Since the forcing $\mathrm{Coll}(\omega, \nu)$ does not change either K or $S \setminus \lambda + 1$, any formula with quantifiers restricted to these two classes will satisfy the condition (2).

Proof. Fix a formula ϕ as in the statement of the lemma. First, note that if ϕ and α satisfy (2), then they continue to satisfy (2) in $V^{\mathrm{Coll}(\omega,\delta)}$ for any cardinal δ. On the other hand, since the universal quantifier $\forall\lambda \in S$ is effectively replaced with $\forall\lambda \in (S \setminus \lambda + 1)$, it may be that α satisfies (2) in $V^{\mathrm{Coll}(\omega,\delta)}$ (and hence in $V^{\mathrm{Coll}(\omega,\delta')}$ for all $\delta' > \delta$) but not in V. Let α_0 be the least ordinal α such that, for some cardinal δ, Lemma 15 fails in $V^{\mathrm{Coll}(\omega,\delta)}$. By working in $V^{\mathrm{Coll}(\omega,\delta)}$ instead of in V we can assume that the lemma already fails for the α_0 in V. Now fix, as a witness the the failure of the lemma, some limit point μ of S such that $\phi(\alpha_0, \lambda^+, S \setminus \lambda + 1)$ holds for cofinally many $\lambda \in S \cap \mu$ and also fails for cofinally many $\lambda \in S \cap \mu$. Let $\sigma(\alpha)$ be the formula asserting that α is the least ordinal for which the statement of the lemma (using the fixed formula ϕ) fails. Then $\sigma(\alpha_0)$ holds, and hence defines α_0, in all of the models $V^{\mathrm{Coll}(\omega,\nu)}$ for $\nu < \mu$.

Now let $\phi'(\lambda^+, S \setminus \lambda + 1)$ be the formula

$$\exists\alpha\, (\sigma(\alpha) \wedge \phi(\alpha, \lambda^+, S \setminus \lambda + 1). \tag{3}$$

Then $\phi'(\lambda^+, S\setminus\lambda{+}1)$ is equivalent to $\phi(\alpha_0, \lambda^+, S\setminus\lambda)$, both in V and in $V^{\mathrm{Coll}(\omega,\nu)}$ for all $\nu < \mu$ and all $\lambda \in S \cap (\nu, \mu)$ Since ϕ' has no parameters (other than the class $S \setminus \lambda + 1$, which is definable from λ) the assumptions (2)implies that $\phi'(\lambda, S \setminus \lambda + 1)$ holds in V if and only if it holds in $V^{\mathrm{Coll}(\omega,\lambda)}$, and then (1) implies that this holds if and only if $V \models \phi'(\omega_1, S)$. In particular, the truth of $\phi'(\lambda, S \setminus \lambda + 1)$ does not depend on the choice of $\lambda \in S \cap \mu$, contradicting the choice of μ. This contradiction completes the proof of Lemma 15. $\square$

We will now introduce some notation for the rest of this section. We write $\lim(S)$ for the set of limit points of S, and we write S^+ for the set of cardinal successors of members of S.

Definition 16. Suppose $\lambda \in \lim(S)$. Then we write $\mathcal{S}_\lambda \subseteq \mathcal{P}(\lambda)$ for the collection of sets of the form $\{\, \nu \in \lambda : \phi(\alpha, \nu, S \setminus \nu) \,\}$, where α is any ordinal and ϕ is a formula satisfying (2).

We write $\mathcal{F}_\lambda$ for the set of functions f with domain λ which are defined by $f(\nu) = y \iff \phi(\alpha, \nu, y, S \setminus \nu)$, where ϕ is a a formula, with ordinal parameter α, such that

$$\phi(\alpha, \nu, y, S \setminus \nu) \iff V^{\mathrm{Coll}(\omega,\nu)} \models \phi(\alpha, \nu, y, S \setminus \nu) \tag{4}$$

for all $\nu \in S^+$, all $\nu' < \nu$, and all sets y. For any limit cardinal λ we write U^*_λ for the filter on $\mathcal{P}(\lambda)$ generated by $S \cap \lambda$; that is, $x \in U^*_\lambda$ if and only if $S \setminus x$ is bounded in λ.

[[Well order on $\mathcal{S}_\lambda$ or $\mathcal{F}_\lambda$: Like the definable well ordering of OD; for every member of $\mathcal{S}_\lambda$ or $\mathcal{F}_\lambda$ we can take ϕ to have the form, with ordinal parameters γ and α, $V_\gamma \models \phi(\alpha, \nu, S \setminus \nu)$ or $V_\gamma \models \phi(\alpha, \nu, y, S \setminus \nu)$. Now well order these sets by looking at the triples $(\gamma, \alpha, \ulcorner\phi\urcorner)$ where $\ulcorner\phi\urcorner$ is a Gödel number for the formula ϕ.]]

[[Another complication: $\mathcal{S}_\lambda$ and $\mathcal{F}_\lambda$ may not be the same in $V^{\mathrm{Coll}(\omega,\delta)}$ as in V. They may be larger in $V^{\mathrm{Coll}(\omega,\delta)}$ since there are fewer λ to consider in (2).]]

Note that Lemma 15 implies that U^*_λ is an ultrafilter on $\mathcal{S}_\lambda$, and hence we can consider equivalence classes $[f]_{U^*_\lambda}$ for functions $f \in \mathcal{F}_\lambda$.

Proposition 17. *For every limit cardinal λ, the filter U^*_λ is a well founded ultrafilter on $\mathcal{F}_\lambda$.*

Proof. This means that there is no infinite U^*_λ-descending sequence $\langle\, f_n : n \in \omega \,\rangle$ of functions in $\mathcal{F}_\lambda$. Suppose the contrary, and let λ be the least limit cardinal such that U^*_λ is not well founded for functions in $\mathcal{F}_\lambda$. A canonical witness $\langle\, f_n : n \in \omega \,\rangle$ to this illfoundedness can be defined by recursion on $n \in \omega$: Supposing that $\langle\, f_0, \dots, f_{n-1} \,\rangle$ has been defined, define f_n to be the least function f, in the previously described well order of $\mathcal{F}_\lambda$, such that there is a U^*_λ-descending sequence starting with $\langle\, f_0, \dots, f_{n-1}, f \,\rangle$.

Now define a function $g \colon \lambda \to \omega$ by letting $g(\nu)$ be the least n such that $f_{n+1}(\nu) \geq f_n(\nu)$. The function g is definable from $S \setminus \delta$ for any $\delta < \lambda$, and hence the same formula defines g in $V^{\mathrm{Coll}(\omega,\delta)}$ as in V. Let n_0 be such that

$V \models g(\omega_1) = n_0$. Then $V^{\mathrm{Coll}(\omega,\delta)} \models g(\delta^+) = n_0$ for all $\delta \in S \cap \lambda$, and since g is unchanged in $V^{\mathrm{Coll}(\omega,\delta)}$ it follows that $g(\delta^+) = n_0$ for all $\delta \in S \cap \lambda$, contradicting the fact that $[f_{n_0+1}]_{U^*_\lambda} < [f_{n_0}]_{U^*_\lambda}$. $\square$

Proposition 18. *Suppose that $\lambda \in \lim(S)$ and $f \colon \lambda \to \lambda$ is in $\mathcal{F}_\lambda$. Then $f{\restriction}(S^+ \cap \lambda)$ is eventually nondecreasing.*

Proof. Fix any λ and $f \in \mathcal{F}_\lambda \cap {}^\lambda\lambda$. Let $\phi(\nu)$ be the sentence asserting that $\forall \nu' \in (S^+ \cap \lambda \setminus \nu)\ f(\nu) \leq f(\nu')$. This formula, using the parameter $S \setminus \nu$ and the ordinal parameter used to define f, satisfies (2), so by Lemma 15 the formula $\phi(\nu)$ is either true for all sufficiently large $\nu \in S^+ \cap \lambda$ or false for all sufficiently large $\nu \in S^+ \cap \lambda$. However it is certainly true for arbitrarily large $\nu \in S^+ \cap \lambda$, so the first alternative must hold; and this implies that f is eventually nondecreasing on $S^+ \cap \lambda$ $\square$

Proposition 19. *For any cardinal δ there are only boundedly many $\lambda \in \lim(S)$ having a function $f \in \mathcal{F}_\lambda$ such that $f[S^+ \cap \lambda] \subseteq \delta$ but f is not eventually constant on $f{\restriction}(S \cap \lambda)$.*

Proof. Assume the contrary, let δ be the least counterexample, and let X be the unbounded class of $\lambda \in \lim(S)$ such that there is $f \in \mathcal{F}_\lambda$ with $\delta = \limsup_{\nu \in S^+ \cap \lambda} f(\nu)$ which is not eventually constant on $S^+ \cap \lambda$. For $\lambda \in X$ let f_λ be the least such function, in the well ordering of $\mathcal{F}_\lambda$, and let ϕ be the formula such that $\phi(\alpha, \delta, \nu, S \setminus \nu)$ asserts, for $\alpha < \delta$, that $f_{\lambda_\nu}(\nu) > \alpha$ where λ_ν is the least member of X above ν. If $\lambda \in X \setminus \lim(X)$ and $\alpha < \delta$ then the sentence $\phi(\alpha, \delta, \nu, S \setminus \nu)$ is true for all sufficiently large $\nu \in S \cap \lambda$, and since X is a proper class it follows by Lemma 15 that there is an ordinal γ_α such that $\phi(\alpha, \delta, \nu, S \setminus \nu)$ is true for all $\nu > \gamma_\alpha$. It follows that if $\nu > \sup\{\, \gamma_\alpha : \alpha < \delta \,\}$ then $f_{\lambda_\nu}(\nu) \geq \delta$, contradicting the choice of f_{λ_ν}. $\square$

Lemma 20. *There is a closed unbounded class $C \subseteq \lim(S)$ such that if $\lambda \in C$ and $f \in \lambda^\lambda \cap \mathcal{F}_\lambda$ then $f{\restriction}(S^+ \cap \lambda)$ either is eventually constant or is cofinal in λ.*

Proof. For each cardinal δ, let ζ_δ be the upper bound given by Proposition 19 on the set of $\lambda \in \lim(S)$ such that there is a function $f \in \mathcal{F}_\lambda$ with range contained in δ which is not cofinal. Now let C be the class of ordinals $\lambda \in \lim(S)$ such that $\zeta_\delta < \lambda$ for all $\delta < \lambda$. Then C is as required. $\square$

Since any function in K with domain λ is in $\mathcal{F}_\lambda$, this leads to an immediate corollary:

Corollary 21. *Every member of C is measurable in K; indeed, if $\lambda \in C$ and $\omega < \mathrm{cf}(C) < \lambda$ then $o^K(\lambda) \geq \mathrm{cf}(\lambda)$.*

Proof. For each $\lambda \in \lim(S)$, let σ_λ be the U^*_λ-least function in $\mathcal{F}_\lambda$ such that $f{\restriction}(S \cap \lambda)$ is not eventually constant. Now set $U_\lambda = \{\, x : \sigma_\lambda^{-1}[x] \in U^*_\lambda \,\}$. Then U_λ is a normal, iterable ultrafilter on λ and therefore is a member of K.

In particular, every member of C is regular in K. By [3, Theorem 0.1] this implies that $o^K(\lambda) \geq \mathrm{cf}(\lambda)$ for all singular $\lambda \in C$ of uncontable cofinality. $\square$

These results give almost all of the information we have on the lower bound for the strength of the property $V[H] \equiv V$ when H is any generic collapse of a successor cardinal. They imply that for every ordinal α the core model, the class of cardinals κ with $o(\kappa) \geq \alpha$ K is stationary in K. However if $\lambda \in C$ is regular in V (or if $\lambda = \Omega$, the class of all ordinals) then we do not know whether it follows that $o(\lambda) > 1$.

The next proposition perhaps gives a bit more information in this case; more importantly it will be a critical tool in the proof of Theorem 4.

Proposition 22. *If $\lambda \in C$ then $[\sigma_{\lambda'}]_{U^*_{\lambda'}} = [\sigma_\lambda \restriction \lambda']_{U^*_{\lambda'}}$ for all sufficiently large $\lambda' \in C \cap \lambda$ such that $\sup(\sigma_\lambda[S \cap \lambda']) = \lambda'$.*

Thus, if $\operatorname{cf}(\lambda) > \omega$ then there is a closed unbounded set $C_\lambda \subseteq \lambda \cap C$ such that if x is any member of U_λ then $x \cap \lambda' \in U_{\lambda'}$ for all sufficiently large $\lambda' \in C_\lambda$.

Proof. Suppose that $\lambda' \in C \cap \lambda$ and $\sigma_\lambda \restriction \lambda'$ maps $\lambda' \cap S$ cofinally into λ'. Since $\sigma_\lambda \restriction \lambda' \in {}^{\lambda'}\lambda' \cap \mathcal{F}_{\lambda'}$ and $\sigma_{\lambda'} \restriction \lambda$ is not eventually constant, it must be that $[\sigma_{\lambda'}]_{U^*_{\lambda'}} \leq [\sigma_\lambda \restriction \lambda']_{U^*_{\lambda'}}$. Suppose that the set

$$Y = \{\, \lambda' \in C \cap \lambda : \lambda' = \sup \sigma_\lambda[\lambda' \cap S] \;\;\&\;\; [\sigma_{\lambda'}]_{U^*_{\lambda'}} < [\sigma_\lambda \restriction \lambda']_{U^*_{\lambda'}} \,\}$$

is unbounded in λ and define $\sigma\colon \lambda \to \lambda$ by setting $\sigma(\nu) = \sigma_{\lambda_\nu}(\nu)$ where $\lambda_\nu = \min(Y \setminus \nu + 1)$. Then $\sigma(\nu) < \sigma_\lambda(\nu)$ for unboundedly many $\nu \in S \cap \lambda$, since if λ' is a successor member of Y then this is true for all sufficiently large $\nu \in S \cap \lambda'$. But the function σ is in $\mathcal{F}_\lambda$, so it follows that $[\sigma]_{U^*_\lambda} < [\sigma_\lambda]_{U^*_\lambda}$, contradicting the choice of σ_λ. $\square$

2.2 Collapsing all cardinals

For the remainder of this section we allow any cardinal to be collapsed, so that the following results hold only when S is the class of all cardinals.

Proposition 23 (Philip Welch)**.** *Every member λ of S^+ is the successor of a regular cardinal in K.*

Proof. Since K satisfies the weak covering lemma, $\lambda^{+K} = \lambda^+$ for every singular cardinal λ. Furthermore, by Lemma 21 each λ is measurable in K. Since K is unchanged in $V^{\mathrm{Coll}(\omega,\lambda)}$, it follows from (1) that every member of S^+ is, in K, the successor of a measurable cardinal. $\square$

Note that it follows that $C = \lim(S)$, which is the class of all limit cardinals. Also, it follows that we can assume σ_λ is nondecreasing for all λ; otherwise we could use $\sigma'_\lambda(\nu) = \sup_{\nu' \leq \nu} \sigma_\lambda(\nu')$ instead of σ_λ.

If δ is any cardinal, then we write ν^* for the cardinal predecessor of ν in the core model K. Thus $\nu \leq \nu^* < \nu^+$.

Lemma 24. *If λ is any limit cardinal then $\sigma_\lambda(\nu^+) = \nu^*$ for every sufficiently large $\nu \in S \cap \lambda$.*

Proof. First we show that if λ is a limit cardinal of uncountable cofinality then $\sigma_\lambda(\nu^+) = \nu^*$ for all sufficiently large $\nu \in S \cap \lambda$. Otherwise Lemma 15 implies that $\sigma_\lambda(\nu^+) < \nu^*$ for all sufficient large $\nu \in S \cap \lambda$. In particular $\sigma_\lambda(\nu^+) < \nu^* = \nu$ for all sufficiently large singular cardinals $\nu < \lambda$, and it follows that there is $\gamma < \lambda$ such that $\{\, \nu : \sigma_\lambda(\nu^+) = \gamma \,\}$ is stationary in λ; however this is impossible since σ_λ is nondecreasing and cofinal in λ.

Now we show that the conclusion of the lemma holds for all sufficiently large cardinals $\lambda \in \lim(S)$. Otherwise let X be the class of cardinals $\lambda \in \lim(S)$ for which it fails, and let ξ be the supremum of the first ω_1 members of X. Then $\sigma_\xi(\nu^+) = \nu^* \geq \nu$ for all $\nu \in S \cap \xi$, and it follows that $\sigma_\xi[S^+ \cap \lambda]$ is cofinal in λ for all $\lambda \in \lim(S) \cap \xi$. It follows by Lemma 22 that for all sufficiently large $\lambda < \xi$ and all sufficiently large $\nu \in S \cap \lambda$, $\sigma_{\lambda'}(\nu^+) = \sigma_\lambda(\nu) = \nu^*$. Hence the conclusion of the lemma holds for all sufficiently large limit cardinals $\lambda < \xi$.

Finally, we show that the lemma holds for all $\lambda \in \lim(S)$. Otherwise let $\phi(\xi)$ be the formula asserting that there is some $\lambda \in \lim(S \setminus \xi)$ for which the lemma fails. Then we would have $V \models \phi(\omega_1)$, but $V^{\mathrm{Coll}(\xi)} \models \neg\phi(\xi^+)$ for sufficiently large $\xi \in S$; however this is impossible by (1). □

We are now ready to prove Theorem 4. In the proof, we assume that the elementary equivalence $V[H] \equiv V$ holds for formulas in the language of Morse-Kelly set theory in order to be able to quantify over all closed unbounded subclasses of the class Ω of all ordinals. Note that the filter U_Ω is definable in V using the class of cardinals. We can quantify over ultrafilters $U \lhd U_\Omega$ by quantifying over functions $f\colon \Omega \to \Omega$ and indentifying such a function with the ultrafilter $U = \{\, x \subseteq \Omega : f^{-1}[x] \,\} \in U_\Omega$.

We will regard a model of Morse-Kelly set theory as having the form K_{Ω^+}, and as containing the extended core model $L_{\Omega^{+K}}(K)$, where by Ω^{+K} we mean the least class-sized ordinal η such that $L_\eta(K)$ is a model of Morse-Kelley set theory.

We say that a function $f\colon \kappa \to \kappa$ is *canonical* if there is a well ordering $\prec$ of κ such that $f(\nu) = \mathrm{otp}(\prec \cap (\nu \times \nu))$ for all cardinals ν. For such a function we write $f(\kappa) = \mathrm{otp}(\prec)$. Then we will have $f(\lambda) = [f\restriction\lambda]_U$ for any measurable $\lambda \leq \kappa$ and measure U on λ.

Proposition 25. *Suppose that κ is a regular limit cardinal, or that V is a model of Morse-Kelley set theory and κ is equal to the class Ω of all ordinals. Then for each canonical function $f\colon \kappa \to \kappa$ in K there is a closed and unbounded set of cardinals λ such that λ is an $f(\lambda)$-repeat point.*

Hence κ is a κ^+-repeat point.

Proof. We prove this by induction on $f(\kappa)$. For the case $f(\kappa) = 0$, we have $f(\lambda) = 0$ so being a $f(\lambda)$-repeat point only means that λ is measurable, which follows from Proposition 21 for all limit cardinals λ.

Now suppose that $\alpha = f(\lambda)$ is a limit, and let $\langle\, \alpha_\nu : \nu < \mathrm{cf}(\alpha) \,\rangle$ be increasing, continuous, and cofinal, and for $\nu < \mathrm{cf}(\alpha)$ let f_ν be a canonical function such that $f_\nu(\kappa) = \alpha_\nu$. Then we can assume that f is the diagonal supremum of the function f_ν: $f(\lambda) = \sup\{\, f_\nu(\lambda) : \nu < \lambda \,\}$. In this case, if C_ν is a closed unbounded set such that $\lambda \in C_\nu$ implies that λ is a $f_\nu(\nu)$-repeat point, then let C be the diagonal intersection $\{\, \lambda < \kappa : \lambda \in \bigcap_{\nu<\lambda} C_\nu \,\}$ of these sets. Then every member λ of C is a $f(\lambda)$-repeat point.

Finally, suppose that $f(\kappa) = \alpha + 1$. Then we can assume $f(\lambda) = g(\lambda) + 1$ for all $\lambda < \kappa$, where g is a canonical function such that $g(\kappa) = \alpha$. Now let C be a closed unbounded subset of cardinals κ such that $\lambda \in C$ implies that λ is an $g(\lambda)$-repeat point. I claim that every limit point of C is a $f(\lambda)$-repeat point. To see this, let $A \in U_\lambda$ be arbitrary. Then by Proposition 22, $A \cap \lambda' \in U_{\lambda'}$ for every sufficiently large limit cardinal $\lambda' < \lambda$. Since λ' is a $g(\lambda')$-repeat point for cofinally many $\lambda' < \lambda$, the sentence asserting that $A \cap \lambda' \in U$ for some $g(\lambda')$-repeat point is true for all sufficiently large limit cardinals $\lambda' < \lambda$, and hence is true in $\operatorname{ult}(V, U_\lambda)$. Thus there is $U \triangleleft U_\lambda$ such that U is a $i^{U_\lambda}(g)(\lambda) = g(\lambda)$-repeat point. Since A was arbitrary, U_λ is a $f(\lambda) = g(\lambda) + 1$-repeat point.

The final sentence of the proposition is now immediate, since for any limit ordinal α, a cardinal κ is a α-repeat point if and only it is a α'-repeat point for all $\alpha' < \alpha$. $\square$

References

[1] J. D. Hamkins and B. Löwe. The modal logic of forcing. *Transactions of the American Mathematical Society*, 2008? to appear. 1

[2] W. J. Mitchell. How weak is a closed unbounded ultrafilter? In M. Boffa, D. V. Dalen, and K. McAloon, editors, *Logic Colloquium '80*, pages 209–230. North-Holland, Amsterdam, 1982. 3

[3] W. J. Mitchell. Applications of the covering lemma for sequences of measures. *Transactions of the American Mathematical Society*, 299(1):41–58, 1987. 11

[4] W. J. Mitchell. A measurable cardinal with a closed unbounded set of inaccessibles from $o(\kappa) = \kappa$. *Trans. Amer. Math. Soc.*, 353:4863–4897, 2001. 2