Combinatorics of Schubert Polynomials

Zachary Hamaker

University of Florida

Email address: zhamaker@ufl.edu

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Contents

Preface

The goal of this text is to provide a concise introduction to the combinatorics of Schubert polynomials. While Schubert polynomials originate in algebraic geometry and are defined algebraically, we aim to apply a combinatorial approach whenever possible.

The reader is assumed to be familiar with the algebraic aspects of the undergraduate curriculum. In particular, we make extensive use of concepts from abstract algebra such as group actions and assume familiarity with linear algebra topics that may not appear in a first course. Our treatment of combinatorial content in the main text is self-contained, but exercises may assume familiarity with additional material.

Zachary Hamaker

Chapter 1

Permutations

Before discussing Schubert polynomials, we require some fundamental facts about permutations. We begin by presenting basic properties of the symmetric group as a Coxeter group, leading to the weak and strong Bruhat orders. Next, we give several different combinatorial models for permutations, each of which will see use in several ways.

1.1. The symmetric group

A *permutation* of the set $[n] := \{1, 2, \ldots, n\}$ is a bijection w from $[n]$ to itself. The set S_n of permutations of $[n]$ is a group under function composition. We typically write the permutation w either in *one-line notation* $w = w(1)\, w(2) \ldots w(n)$ or as product of other permutations. When $w(i) = k$, we say k *appears* in the ith *position* of w.

Definition 1.1. For $w \in S_n$, the *inversion set* of w is

$$\mathrm{Inv}(w) := \{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}.$$

Elements of $\mathrm{Inv}(w)$ are *inversions.* The *length* of w is $\ell(w) := \#\,\mathrm{Inv}(w)$.

Note (i, j) is an inversion for $w \in S_n$ if the value in position i is greater than the value in position j. For example, with $u = 2413$ and

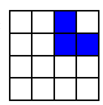
$v = 3142$ we have

$$\mathrm{Inv}(u) = \{(1,3),(2,3),(2,4)\},$$
$$\mathrm{Inv}(v) = \{(1,2),(1,4),(3,4)\}.$$

Note that the inversions in v correspond the values that appear out of order in u and visa versa. This is not a coincidence, but is instead a consequence of the fact that $v = u^{-1}$. More generally,

$$\mathrm{Inv}(w^{-1}) = \{(w(i), w(j)) : (i,j) \in \mathrm{Inv}(w)\}. \tag{1.1}$$

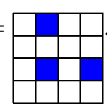
The inversion set $\mathrm{Inv}(w)$ is our first example of a *diagram*, which is a subset of $[n] \times [n]$. Diagrams are depicted using matrix coordinates. With $u = 2413$ as above we have

$\mathrm{Inv}(u) =$ 

where the blue shaded cells are elements of $\mathrm{Inv}(u)$. There is a natural $S_n \times S_n$ action on diagrams – for $u, v \in S_n$ and $D \subseteq [n] \times [n]$ we have

$$(u \times v) \cdot D = \{(u(i), v(j)) : (i,j) \in D\}.$$

Similarly, we let S_n act on diagrams diagonally: $u \cdot D = (u \times u) \cdot D$. For example with $u = 2413$ we have $1324 \cdot \mathrm{Inv}(u) = \{(1,2),(3,2),(3,4)\}$, which can be visualized as

$1324 \cdot \mathrm{Inv}(u) =$  .

The *simple transpositions* $s_1, s_2, \ldots, s_{n-1}$ in S_n are defined by

$$s_i := (i, i{+}1) = 1 \ldots i{-}1 \;\; i{+}1 \;\; i \;\; i{+}2 \;\; \ldots n.$$

Given $w \in S_n$, multiplying on the right by s_i changes the positions i and $i{+}1$ while multiplying on the left exchanges the values i and $i{+}1$ wherever they appear in w. For example, with $u = 2413$ we have $us_2 = 2143$ and $s_2 u = 1423$. The reader should verify that simple transpositions satisfy the following relations: for i, j positive integers

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad (|i-j| > 1), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{1.2}$$

Relations of the second and types are called *commutation* and *braid* relations, respectively.

Definition 1.2. For $w \in S_n$, the *right descent set* of w is

$$\mathrm{Des_R}(w) := \{k \in [n{-}1] : w(k) > w(k+1)\}.$$

The *left descent set* of w is $\mathrm{Des_L}(w) := \mathrm{Des_R}(w^{-1})$. The elements of $\mathrm{Des_R}(w)$ and $\mathrm{Des_L}(w)$ are *right* and *left descents*, respectively.

For $w \in S_n$, we see $k \in \mathrm{Des_R}(w)$ the ith position of w is greater than the $i{+}1$th, while $k \in \mathrm{Des_L}(w)$ if $k{+}1$ appears before k in w. For example, with $u = 2413$ we have $\mathrm{Des_R}(u) = \{2\}$ and $\mathrm{Des_L}(u) = \{1,3\}$.

For $w \in S_n$ and $k \in [n{-}1]$, we have

$$\mathrm{Inv}(ws_k) = \begin{cases} s_k \cdot \mathrm{Inv}(w) \cup \{(k,k{+}1)\} & k \notin \mathrm{Des_R}(w), \\ s_k \cdot \mathrm{Inv}(w) \setminus \{(k,k{+}1)\} & k \in \mathrm{Des_R}(w). \end{cases} \tag{1.3}$$

As a consequence, we have:

Corollary 1.3. *Let $w \in S_n$ and $k \in [n{-}1]$. Then*

(1) $\ell(ws_k) = \begin{cases} \ell(w)+1 & k \notin \mathrm{Des_R}(w), \\ \ell(w)-1 & k \in \mathrm{Des_R}(w). \end{cases}$

(2) $\ell(s_kw) = \begin{cases} \ell(w)+1 & k \notin \mathrm{Des_L}(w), \\ \ell(w)-1 & k \in \mathrm{Des_L}(w). \end{cases}$

Proof. Part (1) follows from Equation (1.3) while part (2) follows from part (1), Equation (1.1) and the fact that $s_kw = (w^{-1}s_k)^{-1}$. □

As a consequence, we show simple transpositions generate S_n.

Proposition 1.4. *We have $S_n = \langle s_1, \ldots, s_{n-1}\rangle$.*

Proof. Clearly S_n contains $G := \langle s_1, \ldots, s_{n-1}\rangle$, so we need only show the reverse containment. We proceed by induction on length. Note $\ell(w) = 0$ only for the identity permutation, which is in G. If $\ell(w) > 0$, we see $\mathrm{Inv}(w) \neq \varnothing$. Therefore, $\mathrm{Inv}(w)$ contains a lexicographically maximal pair (i,j) (that is i is maximal so that some $(i,k) \in \mathrm{Inv}(w)$ and j is maximal amongst entries such k). Either $j = i{+}1$ so $w(i) > w(i{+}1)$ or $(i{+}1,j) \notin \mathrm{Inv}(w)$ and $w(i) > w(j) > w(i{+}1)$. Therefore $i \in \mathrm{Des_R}(w)$, so by Corollary 1.3 (1) we have $\ell(ws_i) = \ell(w) - 1$, and the result follows by induction. □

This is the first of many proofs by induction on length. In later instances, we frequently outline the inductive step and leave the remaining details to the reader.

1.2. Reduced words

An *expression* or *decomposition* of $w \in S_n$ is a product

$$s_{a_1} s_{a_2} \dots s_{a_p} = w,$$

with associated *word* $(a_1, a_2, \dots, a_p)$. The expression is *reduced* if p is minimal amongst all expressions for w. Let $\mathrm{Red}(w)$ be the set of reduced words for w. When $w = 1$, $\mathrm{Red}(w)$ is the empty word. With this convention, Proposition 1.4 guarantees $\mathrm{Red}(w)$ is non-empty for every permutation w.

Proposition 1.5. *For $w \in S_n$ and $(a_1, \dots, a_p) \in \mathrm{Red}(w)$, $p = \ell(w)$.*

Proof. By Corollary (1.3) (1), for all $i \in [p-1]$ we see

$$\ell(s_{a_1} \dots s_{a_i} s_{a_{i+1}}) \leq \ell(s_{a_1} \dots s_{a_i}) + 1.$$

Repeated applications of this observation gives $\ell(w) \leq p$. Our proof of Proposition 1.4 outlines an inductive construction for an expression for w of length $\ell(w)$, so we see $\ell(w) \geq p$, hence we have equality. □

For example, with $u = 2413$ we see $u = s_1 s_3 s_2 = s_3 s_1 s_2$. One can check that these are the only expressions for u with three terms. Since $\ell(u) = 3$, Proposition 1.5 shows $\mathrm{Red}(u) = \{(1,3,2),(3,1,2)\}$. There are many non-reduced expressions for u, for instance $u = s_3 s_2 s_1 s_2 s_1$.

The permutation $w_0^{(n)} = n \dots 21 \in S_n$ is the *longest* or *reverse permutation*. When the context is clear, we write $w_0 = w_0^{(n)}$. The term 'reverse permutation' is apt since for $w = w(1) \dots w(n) \in S_n$

$$ww_0 = w(n) \dots w(1) \quad \text{and} \quad w_0 w = n{+}1 - w(1) \dots n{+}1 - w(n).$$

For example, with $w = 4132$ we have $ww_0 = 2314$ and $w_0 w = 1423$.

Corollary 1.6. *Let $w \in S_n$. Then*

(1) $\ell(w) = 0$ *if and only if* $w = 1$.
(2) $\ell(w) = 1$ *if and only if* $w = s_k$ *with* $k \in [n-1]$.
(3) $\ell(w) = \ell(w^{-1})$.

(4) $\ell(w_0 w) = \ell(w w_0) = \binom{n}{2} - \ell(w)$.

Proof. Parts (1) and (2) follow by examining $\mathrm{Red}(w)$. For (3), note $s_{a_1} \dots s_{a_p} = w$ if and only if $s_{a_p} \dots s_{a_1} = w^{-1}$, so the result follows by Proposition 1.5.

Now, note $\mathrm{Inv}(w_0) = \{(i, j) : 1 \le i < j \le n\}$, so $\ell(w_0) = \binom{n}{2}$. Since $ww_0 = w(n) \dots w(1)$, we see $\mathrm{Inv}(ww_0) = \mathrm{Inv}(w_0) \setminus \mathrm{Inv}(w)$, hence $\ell(ww_0) = \binom{n}{2} - \ell(w)$. Also, since $w_0 = w_0^{-1}$

$$\ell(w_0 w) = \ell(w^{-1} w_0) = \binom{n}{2} - \ell(w^{-1}) = \binom{n}{2} - \ell(w),$$

with the first and last equalities by Part (3). □

Note that Part (3) implies the stronger relationship

$$\mathrm{Red}(w^{-1}) = \{(a_p, \dots, a_1) : (a_1, \dots, a_p) \in \mathrm{Red}(w)\}.$$

In light of Proposition 1.5, letters in a reduced word for w and inversions for w are equinumerous. This relationship can be realized explicitly by a simple correspondence.

Proposition 1.7. *Let $w \in S_n$ and $(a_1, \dots, a_p) \in \mathrm{Red}(w)$. Then*

$$\mathrm{Inv}(w) = \{s_{a_p} \dots s_{a_{k+1}} \cdot (a_k, a_k{+}1) : k \in [p]\}.$$

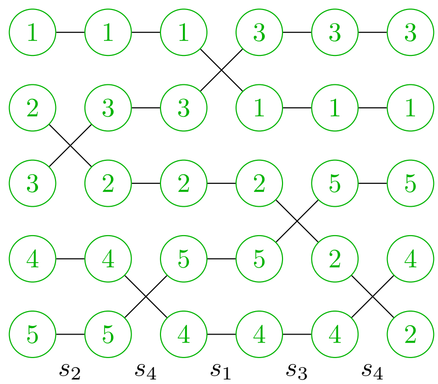
Proof. Let $v = ws_{a_p} = s_{a_1} \dots s_{a_{p-1}}$. Then by Equation (1.3)

$$\mathrm{Inv}(w) = s_{a_p} \mathrm{Inv}(v) \cup \{(a_p, a_p{+}1)\}.$$

Since $\ell(w) > \ell(v)$, the result follows by induction on $\ell(w)$. □

Proposition 1.7 has a natural visualization via certain diagrams. When first studying functions in a set-theoretic context, they are frequently visualized by drawing the domain and range side by side with arrows illustrating the image of each value. Function composition is performed by concatenating diagrams. Applying this perspective to a reduced expression, one constructs its *wiring diagram*.

Example 1.8. Let $w = 31542$. Note $\ell(w) = 6$ and $w = s_2 s_4 s_1 s_3 s_4 s$, so $\mathbf{a} = (2, 4, 1, 3, 4) \in \mathrm{Red}(w)$. The wiring diagram of $\mathbf{a}$ is



From the wiring diagram, it easy to see the values placed out of order by each letter in $\mathbf{a}$. For instance, s_3 puts the values 2 and 5 out of order. This means $(2, 5) \in \mathrm{Inv}(w^{-1})$, which in turn implies that

$$(w^{-1}(5), w^{-1}(3)) = (3, 5) \in \mathrm{Inv}(w).$$

This is preciesly the correspondence identified by Proposition 1.7.

We now prove several important properties of reduced words. For $\mathbf{a} = (a_1, \ldots, a_p)$ a word, let $(a_1, \ldots, \widehat{a_i}, \ldots, a_p)$ be the word obtained from $\mathbf{a}$ by omitting the ith letter.

Lemma 1.9 (Exchange Lemma)**.** *Let $w \in S_n$ with $k \in \mathrm{Des}_{\mathrm{R}}(w)$, and let $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$. Then there exists $i \in [p]$ such that*

$$(a_1, \ldots, \widehat{a_i}, \ldots, a_p, k) \in \mathrm{Red}(w).$$

Proof. Since $k \in \mathrm{Des}_{\mathrm{R}}(w)$, we have $(k, k{+}1) \in I(w)$. By Proposition 1.7, there exists $i \in [p]$ such that

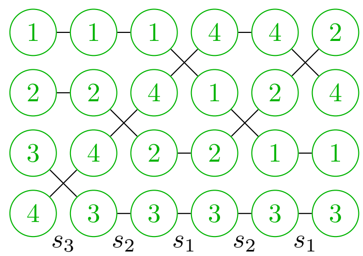
$$(k, k{+}1) = s_{a_p} \ldots s_{a_{i+1}} \cdot (a_i, a_i + 1).$$

Here, we are acting on (a_i, a_{i+1}) as an element of $[n] \times [n]$. Then

$$\begin{aligned} s_k &= (s_{a_p} \ldots s_{a_{i+1}}) s_{a_i} (s_{a_p} \ldots s_{a_{i+1}})^{-1} \\ &= (s_{a_p} \ldots s_{a_{i+1}}) s_{a_i} (s_{a_{i+1}} \ldots s_{a_p}), \end{aligned}$$

so $s_{a_{i+1}} \ldots s_{a_p} s_k = s_{a_i} \ldots s_{a_p}$, hence $s_{a_1} \ldots \widehat{s_{a_i}} \ldots s_{a_p} = w$ and the result follows. $\square$

The Exchange Lemma can be illustrated using wiring diagrams. Let $v = 4213$. Note $\mathbf{a} = (3, 2, 1, 2) \in \mathrm{Red}(v)$ and $1 \in \mathrm{Des}_{\mathrm{R}}(w)$. The word $(3, 2, 1, 2, 1)$ has the wiring diagram



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The final letter undoes the inversion $(1, 2)$, which corresponds by Proposition 1.7 to the first instance of 2 in $\mathbf{a}$. By omitting this letter from $(3, 2, 1, 2, 1)$ we undo the first crossing of the values 2 and 4, resulting in the new word $(3, 1, 2, 1) \in \mathrm{Red}(v)$.

The Exchange Lemma is a key tool when working with reduced words. We state some important consequences.

Corollary 1.10. *Let $w \in S_n$.*

(1) *For $k \in \mathrm{Des}_{\mathrm{R}}(w)$, there exists $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$ with $a_p = k$.*

(2) *For $\ell \in \mathrm{Des}_{\mathrm{L}}(w)$, there exists $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$ with $a_1 = \ell$.*

Proof. Part (1) is immediate from the Exchange Lemma, while (2) follows from (1) and the fact that $\mathrm{Des}_{\mathrm{L}}(w) = \mathrm{Des}_{\mathrm{R}}(w^{-1})$. $\square$

Corollary 1.11. *Let $w \in S_n$ so that $w = s_{a_1} \ldots s_{a_r}$ with $r > \ell(w)$. Then there exist $i \neq j$ such that*

$$w = s_{a_1} \ldots \widehat{s_{a_i}} \ldots \widehat{s_{a_j}} \ldots s_{a_p}.$$

Proof. Since $r > \ell(w)$, there exists some index j such that

$$\ell(s_{a_1} \ldots s_{a_j} s_{a_{j+1}}) < \ell(s_{a_1} \ldots s_{a_j}).$$

Assume that j is minimal, so for $v = s_{a_1} \ldots s_{a_j}$ we see $a_{j+1} \in \mathrm{Des}_{\mathrm{R}}(v)$ and $(a_1, \ldots, a_j) \in \mathrm{Red}(v)$. The Exchange Lemma implies there exists

$i \in [j-1]$ so that $s_{a_1} \ldots s_{a_j} = s_{a_1} \ldots \widehat{s_{a_i}} \ldots s_{a_j} s_{a_{j+1}}$. Multiplying both sides on the right by $s_{a_{j+1}}$ we see

$$s_{a_1} \ldots \widehat{s_{a_i}} \ldots s_{a_j} = s_{a_1} \ldots s_{a_j} s_{a_{j+1}},$$

so the result follows. □

Given two words $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_p)$, say $\mathbf{a} \sim \mathbf{b}$ if they differ by a commutation or braid relation. For instance,

$$(1, 3, 2, 1) \sim (3, 1, 2, 1) \sim (3, 2, 1, 2),$$

with the first relation arising from a commutation relation at the first two indices and the second from a braid relation at the last three. Let $G_{\mathrm{Red}}(w)$ be the graph where $(\mathbf{a}, \mathbf{b})$ is an edge if $\mathbf{a} \sim \mathbf{b}$.

Theorem 1.12 (Matsumoto-Tits)**.** *For $w \in S_n$, the graph $G_{\mathrm{Red}}(w)$ is connected.*

Equivalently, for $\mathbf{a}, \mathbf{b} \in \mathrm{Red}(w)$ there exists a chain of relations so that $\mathbf{a} \sim \cdots \sim \mathbf{b}$. We then say $\mathbf{a}$ is *connected* to $\mathbf{b}$.

Proof. We argue by induction on $\ell(w)$. Let $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_p)$ with $\mathbf{a}, \mathbf{b} \in \mathrm{Red}(w)$. If $a_p = b_p$,

$$(a_1, \ldots, a_{p-1}), (b_1, \ldots, b_{p-1}) \in \mathrm{Red}(ws_{a_p})$$

which are connected by the induction hypothesis. Otherwise, $a_p \neq b_p$ with $a_p, b_p \in \mathrm{Des}_{\mathrm{R}}(w)$. The Exchange Lemma implies for some $i \in [p]$

$$\mathbf{c} = (a_1, \ldots, \widehat{a_i}, \ldots, a_p, b_p) \in \mathrm{Red}(w).$$

Note $i \neq p$, since this would imply $a_p = b_p$.

If $|a_p - b_p| > 1$, applying a commutation move at end of $\mathbf{c}$ gives

$$\mathbf{c}' = (a_1, \ldots, \widehat{a_i}, \ldots, a_{p-1}, b_p, a_p).$$

Since $\mathbf{a}$ is connected to $\mathbf{c}'$, $\mathbf{b}$ is connected to $\mathbf{c}$ and $\mathbf{c} \sim \mathbf{c}'$, we see $\mathbf{a}$ is connected to $\mathbf{b}$.

If $|a_p - b_p| = 1$, applying the Exchange Lemma to $\mathbf{c}$ implies

$$\mathbf{d} = (a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{p-1}, a_p, b_p, a_p) \in \mathrm{Red}(w).$$

Since (a_p, b_p, a_p) is reduced, the inversions corresponding to these letters are distinct so $j \neq p$. Apply a braid relation to these values to get

$$\mathbf{d} = (a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_{p-1}, b_p, a_p, b_p) \in \mathrm{Red}(w).$$

Since $\mathbf{a}$ is connected to $\mathbf{d}$, $\mathbf{b}$ is connected to $\mathbf{d}'$ and $\mathbf{d} \sim \mathbf{d}$, we see $\mathbf{a}$ is connected to $\mathbf{b}$ and the result follows. □

1.3. Weak and Strong Bruhat orders

For $1 \leq i < j \leq n$, write t_{ij} for the transposition $(i, j) \in S_n$.

Let $u, v \in S_n$. We say $u \lessdot v$ if $\ell(v) = \ell(u) + 1$ and there exist i, j so that $ut_{ij} = v$. Equivalently, $u \lessdot ut_{ij}$ if $u(i) < u(j)$ and for $i < k < j$ either $u(k) < u(i)$ or $u(k) > u(j)$. The *strong Bruhat order* or *Bruhat order* on S_n is the transitive closure $\leq$ of $\lessdot$.

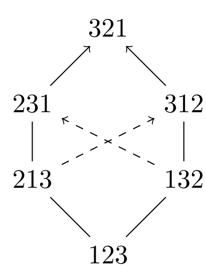
Proposition 1.13. *Let $u, v \in S_n$. The following are equivalent:*

(1) $u \leq v$, (2) $u^{-1} \leq v^{-1}$, (3) $vw_0 \leq uw_0$, (4) $w_0v \leq w_0u$.

Proof. See Exercise (2). □

Similarly, for $u, v \in S_n$ we say $u \lessdot_W v$ if $\ell(v) = \ell(u) + 1$ and there exists k so that $us_k = v$. The *right weak Bruhat order* or *weak order* on S_n is the transitive closure $\leq_W$ of $\lessdot_W$. Since $u \lessdot_W v$ implies $u \lessdot v$, we see $u \leq_W v$ implies $u \leq v$ so Bruhat order refines weak order.

Example 1.14. The Hasse diagram of $(S_3, \leq)$ is



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The dashed lines indicate cover relations for Bruhat order that are not relations for weak order.

Proposition 1.15. *For $u, v \in S_n$, $u \leq_W v$ if and only if any reduced word of u can be extended to a reduced word for v.*

Proof. Extending a reduced word for u by a single letter corresponds to a cover relation $\lessdot_W$, so the equivalence is immediate. □

In light of Proposition 1.15, many of the results in the Section 1.2 can be reinterpreted as statements about weak order. We would like to understand Bruhat order in terms of reduced words as well. To do so, we first understand $\lessdot$ in these terms.

Lemma 1.16. *Let $v, w \in S_n$ and $(a_1, \dots, a_p) \in \mathrm{Red}(w)$. Then $\ell(v) < \ell(w)$ and $v^{-1}w$ is a transposition if and only if there exists $r \in [p]$ so that $v = s_{a_1} \dots \widehat{s_{a_r}}, \dots, s_{a_p}$.*

Proof. Let $v \lessdot w$, so $vt_{ij} = w$ for some transposition t_{ij}. Since $\ell(v) < \ell(w)$, we have $(i, j) \in \mathrm{Inv}(w)$. Then by Proposition 1.7

$$(i, j) = (s_{a_p} \dots s_{a_{r+1}}) \cdot (a_r, a_r{+}1)$$

for some $r \in [p]$. Then

$$t_{ij} = (s_{a_p} \dots s_{a_{r+1}}) s_{a_r} (s_{a_{r+1}} \dots s_{a_p})$$

so $v = wt_{ij} = s_{a_1} \dots \widehat{s_{a_r}} \dots s_{a_p}$. The converse follows by reversing this computation. □

Note the expression for v appearing in Lemma 1.16 need not be reduced. For example, let $v = 123$, $w = 321$ and $(1, 2, 1) \in \mathrm{Red}(w)$. We have $v^{-1}w = w = t_{13}$ and $\ell(v) = 0 < 3 = \ell(w)$, while $s_1 \widehat{s_2} s_1 = v$.

Corollary 1.17. *Let $v, w \in S_n$ and $(a_1, \dots, a_p) \in \mathrm{Red}(w)$.*

(1) *$v \lessdot w$ if and only if there exists $r \in [p]$ so that*

$$(a_1, \dots, \widehat{a_r}, \dots, a_p) \in \mathrm{Red}(v).$$

(2) *If $v \leq w$, there exist $r_1, \dots, r_k \in [p]$ so that*

$$(a_1, \dots, \widehat{a_{r_1}}, \dots, \widehat{a_{r_k}}, \dots, a_p) \in \mathrm{Red}(v).$$

Proof. Part (1) follows immediately from Lemmat 1.16, and (2) follows from repeated application of the forwards direction in (1). □

The converse of (2) is not obvious since one could have $v, w \in S_n$ with $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$ and

$$(a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_p) \in \mathrm{Red}(v),$$

while $(a_1, \ldots, \widehat{a_i}, \ldots, a_p)$ and $(a_1, \ldots, \widehat{a_j}, \ldots, a_p)$ are **not** reduced. Ruling out this possibility will require several additional properties of Bruhat order.

Lemma 1.18. *For $w \in S_n$ and $k \in [n-1]$, either $w \lessdot s_k w$ or $s_k w \lessdot w$.*

Proof. The result follows from Corollary 1.3 (2) and the fact that $s_k w = w t_{ij}$ for $i = w^{-1}(k)$ and $j = w^{-1}(k+1)$. □

Lemma 1.19. *Let $v, w \in S_n$ so that $v \lessdot s_k v \neq w$. Then $v \lessdot w$ if nad only if $w \lessdot s_k w$ and $s_k v \lessdot s_k w$.*

Proof. For the forward direction, let $v \lessdot w$ and $\mathbf{a} = (a_1, \ldots, a_p)$ in $\mathrm{Red}(w)$. Then $w = v t_{ij}$ for some i, j. By Lemma 1.16,

$$\mathbf{a}' = (a_1, \ldots, \widehat{a_r}, \ldots, a_p) \in \mathrm{Red}(v)$$

for some $r \in [p]$. Note $a_1 \neq k$ since $v \neq s_k w$. Therefore $s_k \notin \mathrm{Des}_{\mathrm{L}}(w)$, so $w \lessdot s_k w$ by Lemma 1.18, while prepending k to $\mathbf{a}$ and $\mathbf{a}'$ shows $s_k v \lessdot s_k w$ by Lemma 1.16.

For the converse, assume $w \lessdot s_k w$ and $v \lessdot s_k v$. Then

$$(k, a_1, \ldots, a_p) \in \mathrm{Red}(s_k w) \text{ and } (k, a_1, \ldots, \ldots, \ldots, \widehat{a_i}, \ldots, a_p) \in \mathrm{Red}(s_k v)$$

so $v \lessdot w$ by Lemma 1.16. □

Proposition 1.20. *Let $v, w \in S_n$ such that $s_k v \lessdot v$ and $s_k w \lessdot w$. The following are equivalent:*

$$(1)\ v \leq w, \quad (2)\ s_k < w, \quad (3)\ s_k v \leq s_k w.$$

Proof. The equivalence (1) and (3) is Exercise 1.6.4. Also, (1) implies (2) since $s_k v \lessdot v \leq w$.

To show (2) implies (1), consider the chain

$$s_k v = v^0 \lessdot v^1 \lessdot v^2 \lessdot \cdots \lessdot v^m = w.$$

Note $v^0 \lessdot s_k v^0$ and $s_k v^m \lessdot v^m$. Therefore, there exists j so that $v^j \lessdot s_k v^j$ and $s_k v^{j+1} \lessdot v^{j+1}$. Also, $s_k v^j \neq s_k v^{j+1}$ since their lengths differ. If we assume $s_k v^j \neq s_k v^j$, then applying Lemma 1.19 with $v^j \lessdot s_k v^j, v^j \lessdot v^{j+1}$ implies $v^{j+1} \lessdot s_k v^{j+1}$, a contradiction. Therefore

$$v = s_k v^0 \lessdot s_k v^1 \lessdot \cdots \lessdot s_k v^j = v^{j+1} \lessdot \cdots \lessdot w,$$

hence $v \leq w$. □

We can now prepared to prove the converse of Corollary 1.17 (2), which says Bruhat order can be described by subword containment of reduced words.

Theorem 1.21. *For $v, w \in S_n$ we have $v \leq w$ if and only if for any $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$, $(a_{i_1}, \ldots, a_{i_k}) \in \mathrm{Red}(v)$ for some $i_1 < \cdots < i_k$.*

Proof. The forward direction is Corollary 1.17. For the converse, we proceed by induction on $\ell(v) + \ell(w)$. The base case is $v = w = 1$, which is vacuously true. Assume $\mathbf{a} = (a_1, \ldots, a_p) \in \mathrm{Red}(w)$ and there exist $i_1 < \cdots < i_k$ with $\mathbf{a}' = (a_{i_1}, \ldots, a_{i_k}) \in \mathrm{Red}(v)$.

If $v \lessdot s_{a_1} v$, then $a_1 \in \mathrm{Des_L}(v)$ so $i_1 > 1$. Then $\mathbf{a}'$ is a subword of $(a_2, \ldots, a_p) \in \mathrm{Red}(s_{a_1} w)$. By the inductive hypothesis, we then have $v \leq s_{a_1} w < w$.

If $s_{a_1} v \lessdot v$, by the Exchange Lemma there exists $j \in [k]$ such that $(a_1, a_{i_1}, \ldots, \widehat{a_{i_j}}, \ldots, a_{i_k}) \in \mathrm{Red}(s_{a_1} v)$. Then by the inductive hypothesis $s_{a_1} v < w$, so by Proposition 1.20 $v \leq w$. □

We now give an important alternate description of Bruhat order.

Corollary 1.22. *Let $w \in S_n$ so that $\ell(w) < \ell(wt_{ij})$. Then $w < wt_{ij}$.*

Proof. By Proposition 1.18, any reduced expression for wt_{ij} contains an expression for w as a subword. By reprated application of Corollary 1.11, this expression contains a reduced subexpression so the result follows by Theorem 1.21. □

1.4. Combinatorial models for permutations

There are many equivalent ways to describe a permutation, each of which provides a valuable alternate perspective. For example, the permutation w is uniquely determined by $\mathrm{Inv}(w)$, and we have already seen many benefits to understanding w by its inversions. We now present several additional models for permutations.

Definition 1.23 (Triangular array)**.** For $w \in S_n$, we see

$$\{w(1)\} \subseteq \{w(1), w(2)\} \subseteq \dots \{w(1), \dots, w(n)\} = [n].$$

Let $T^w_{i\bullet} = \{w(1), \dots, w(i)\}$ and T^w_{ij} be the jth largest element in $T^w_{i\bullet}$. The *triangular array* for w is $T^w = \{T^w_{ij}\}_{1 \le j \le i \le n}$.

Since $w(i) = T^w_{i\bullet} \setminus T^w_{i-1\bullet}$, we see w can be reconstructed from T^w. For example, with $w = 351642$

$$T^w = \begin{array}{c} 3 \\ 3\ 5 \\ 1\ 3\ 5 \\ 1\ 3\ 5\ 6 \\ 1\ 3\ 4\ 5\ 6 \end{array} .$$

We omit the last line since it will always be $1\ 2\ \dots\ n$.

Theorem 1.24. *For $v, w \in S_n$, $v \le w$ if and only if $T^v_{ij} \le T^w_{ij}$ for all $1 \le j \le i \le n$.*

Proof. Assume $v \lessdot w$, so $w = vt_{k\ell}$ with $v(k) < v(\ell)$. Then for $i < k$ and $i > \ell$ we have $T^v_{ij} = T^w_{ij}$. For $k \le i \le \ell$, we have

$$T^w_{i\bullet} = T^v_{i\bullet} \cup \{v(\ell)\} - \{v(k)\},$$

so $T^w_{ij} \ge T^v_{ij}$ for all $j \in [i]$. This gives the forwards direction.

For the converse, let $T^v_{ij} \le T^w_{ij}$ for all $1 \le j \le i \le n$, and let k be maximal so that $T^v_{k\bullet} \ne T^w_{k\bullet}$. We argue by downward induction on k. Note $k < n$ if $v \ne w$. In this case, since $T^v_{k+1\bullet} = T^w_{k+1\bullet}$ we have $T^w_{k\bullet} \setminus T^v_{k\bullet} = \{q\}$ and $T^v_{k\bullet} \setminus T^w_{k\bullet} = \{p\}$ with $p < q$. Therefore $w(k{+}1) = p$ and $v(k{+}1) = q$ so $T^{t_{pq}w}_{\ell\bullet} = T^w_{\ell\bullet}$ for $\ell \ge k$ and $t_{pq}w \le w$ by Corollary 1.17, and the result follows by induction. □

Definition 1.25 (Rank matrices)**.** For $w \in S_n$, let $r^w = (r^w_{ij})^n_{i,j=1}$ where $r^w_{ij} = \#\{k \in [i] : w(k) \le j\}$.

Equivalently, for M^w the matrix that permutes a vector by w, r^w_{ij} counts the number of 1's in the submatrix $M^w_{[i][j]}$. For example, with

$w = 351642$ we have

$$M^w = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad r^w = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Theorem 1.26. *For $v, w \in S_n$, we have $v \leq w$ if and only if $r^w_{ij} \leq r^v_{ij}$ for all $i, j \in [n]$.*

Proof. This is Exercise 1.6.5. □

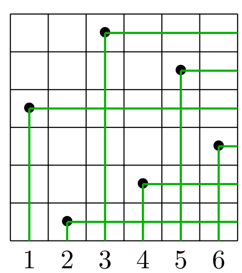
The next combinatorial models for permutations are closely related to inversion sets.

Definition 1.27. For $w \in S_n$, the *Rothe diagram* of w is

$$\begin{aligned} D(w) &:= (1 \times w) \cdot \mathrm{Inv}(w) = \{(i, w(j)) : (i,j) \in \mathrm{Inv}(w)\} \\ &= \{(k, \ell) : k < w^{-1}(\ell), w(k) > \ell\}. \end{aligned}$$

This last description says the complement of $D(w)$ in $[n] \times [n]$ is the set of cells weakly below or weakly to the write of a 1 in M^w. After crossing out these cells, the remaining cells will be $D(w)$. For example, with $w = 351642$ we have

$$D(w) = \{(1,1),(1,2),(2,1),(2,2),(2,4),(4,2),(4,4),(5,4)\} \tag{1.4}$$



Here, the $\bullet$'s are cells with 1 in M^w and the □'s are the cells in $D(w)$.

The following are consequences of the fact $D(w) = (1 \times w) \cdot \mathrm{Inv}(w)$.

Proposition 1.28. *Let $w \in S_n$.*

(1) $D(w^{-1})$ *is the transpose of* $D(w)$.

(2) $|D(w)| = \ell(w)$.

(3) *If* $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$, *then*

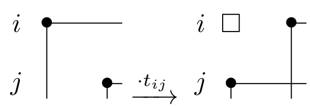
$$D(w) = \{s_{a_p} \ldots s_{a_{k+1}}(a_k), s_{a_1} \ldots s_{a_{k-1}}(a_{k+1}) : k \in [p]\}.$$

(4) *If* $k \in \mathrm{Des}_{\mathrm{R}}(w)$. *then*

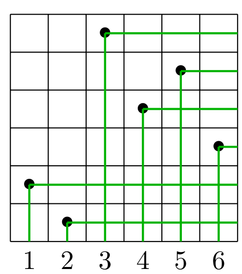
$$D(ws_k) = (s_k \times 1) \cdot D(w) \cup \{(k, w(k))\}.$$

Proof. This is Exercise 1.6.6. □

There is a natural interpretation of Bruhat order covers for permutation matrices that extends to Rothe diagrams. For $v \in S_n$, we have $v \lessdot vt_{ij}$ if $v(i) < v(j)$ and the submatrix of M^v in rows $\{i{+}1, \ldots, j{-}1\}$ and columns $\{v(i){+}1, \ldots, v(j){-}1\}$ is all zero. This can be easily observed for the depiction $D(v)$ as well. In this case, $(i, v(i)) \in D(vt_{ij}) \setminus D(v)$:



Note the other cells of $D(vt_{ij})$ may move. For example, with w as in Equation (1.4), we see $w \lessdot wt_{35}$ and $(4,4), (5,3) \in D(w) \setminus D(wt_{35})$:



In particular, $D(wt_{35}) = D(w) \cup \{(3,1), (3,2), (4,1)\} \setminus \{(4,4), (5,3)\}$.

We introduce one final model for permutations.

Definition 1.29 (Code)**.** For $w \in S_n$, the *Lehmer code* or *code* of w is $c(w) := (c_1(w), c_2(w), \ldots, c_{n-1}(w))$ where

$$c_i(w) = \#\{j : (i,j) \in \mathrm{Inv}(w)\} = \#\{j > i : w(i) > w(j)\}.$$

For example, with $w = 351642$ we have $c(w) = (2, 3, 0, 2, 1)$. At first glance, $c(w)$ contains significantly less information than $\mathrm{Inv}(w)$.

Proposition 1.30. *The code is a bijection*

$$c : S_n \to \{(c_1, \ldots, c_{n-1}) : 0 \le c_i \le n - i\}.$$

Proof. Note $0 \le c_i(w) \le n-i$ since there are only $n-i$ positions to the right of position i. Therefore c is well-defined. We invert c one entry at a time, beginning with $w(1) = c_1 + 1$. More generally, we see $w(i)$ is the (c_i+1)th largest element in the set $[n] \setminus T^w_{i-1\bullet}$. □

The code of a permutation is a *weak composition*, that is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where each α_i is a non-negative integer. The *size* of the weak composition α is $|\alpha| = \sum_{i=1}^k \alpha_i$. The symmetric group acts on weak compositions by permuting entries. Weak compositions can be added point-wise:

$$(\alpha_1, \ldots, \alpha_k) + (\beta_1, \ldots, \beta_k) = (\alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k).$$

Define ε^i by $\varepsilon^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$

Proposition 1.31. *Let $w \in S_n$.*

(1) $\ell(w) = \sum_{i \in [n-1]} c_i(w)$.
(2) $\max \mathrm{Des}_{\mathrm{R}}(w) = \max\{r : c_r(w) > 0\}$.
(3) *If $k \in \mathrm{Des}_{\mathrm{R}}(w)$, then $c(ws_k) = s_k \cdot c(w) + \varepsilon^k$.*
(4) *If $(a_1, \ldots, a_p) \in \mathrm{Red}(w)$, then $c(w) = \sum_{i=1}^p s_{a_p} \ldots s_{a_{i+1}} \varepsilon^{a_i}$.*
(5) *$c_i(w) > c_{i+1}(w)$ if and only if $i \in \mathrm{Des}_{\mathrm{R}}(w)$.*

Proof. By definition $\sum_{i \in [n-1]} c_i(w) = \#\,\mathrm{Inv}(w) = \ell(w)$, which implies (1). For Property (2), let $r = \max \mathrm{Des}_{\mathrm{R}}(w)$. Then $w(r) > w(r{+}1)$ so $c_r(w) > 0$, while $w(r{+}1) < w(r{+}2) < \cdots < w(n)$, so $c_s(w) = 0$ for $s > r$. For (3), we have $w(k) < w(k{+}1)$ so

$$\{j > k : w(j) < ws_k(k)\} = \{j > k : w(j) < w(k{+}1)\} \cup \{k{+}1\}.$$

Therefore $c_k(ws_k) = c_{k+1}(w) + 1$, while a similar argument shows $c_{k+1}(ws_k) = c_k(w)$. All other values remain unchanged, hence the result follows. Next, (4) follows by repeated application of (3). Finally, to prove (5) note $w(i{+}1)$ is the $(c_{i+1}(w){+}1)$th smallest entry in $\{w(i+1), \dots, w(n)\}$ and $w(i)$ is greater than at least that many entries in this set. $\square$

1.5. Important families of permutations

Many families of permutations have particularly nice descriptions with respect to the various combinatorial models. The ones we consider can all be described in terms of pattern avoidance.

For $v \in S_m, w \in S_n$, we say w *contains* v if for some $i_1 < \cdots < i_m$ both $w(i_1) \dots w(i_m)$ and $v(1) \dots v(m)$ have the same relative order. When w does not contain v, we say w *avoids* v. For example, the permutation $w = 351642$ contains 2143 since

$$w(1)w(3)w(4)w(5) = 3164$$

has the same relative order. However, w avoids 4321 since it does not contain a decreasing subsequence of length 4.

A permutation w is *vexillary* if it avoids 2143. There are many equivalent descriptions.

1.6. Exercises

1.6.1. For $w \in S_n$, demonstrate a bijection from $\mathrm{Inv}(w)$ to $\mathrm{Inv}(w^{-1})$.

1.6.2. For $w \in S_n$, demonstrate a bijection from $\mathrm{Inv}(w)$ to $\mathrm{Inv}(w^{-1})$.

1.6.3. Identify and prove the properties from Proposition 1.13 that are equivalent for $\leq_W$.

1.6.4. Prove the equivalence of (1) and (3) in Proposition 1.20.

1.6.5. Prove Theorem 1.26. You may want to use Theorem 1.24.

1.6.6. Prove Proposition 1.28.

Chapter 2

Schubert Polynomials

To define Schubert polynomials, we first introduce basic facts about divided difference operators. After proving some basic properties of Schubert polynomials, we introduce Monks formula. Using Monks formula, we introduce two combinatorial models for Schubert polynomials: pipe dreams and bumpless pipe dreams.

2.1. Divided difference operators

For a polynomial f in the variables x and y, define:

$$\partial_{xy}(f) = \frac{f(x,y) - f(y,x)}{x - y}.$$

When f has more variables, we may view these as coefficients.

Let $\mathcal{P}_n = \mathbb{C}[x_1, \ldots, x_n]$, the algebra of polynomials in the variables $x_1, \ldots, x_n$. Recall a *weak composition* is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \mathbb{N}$ for all $i \in [n]$. To each weak composition α we associate the monomial $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. For $f \in \mathcal{P}_n$, $[x^\alpha]f$ denotes the coefficient of x^α in f. The permutation $w \in S_n$ acts on $\mathcal{P}_n$ by

$$w \cdot f(x_1, \ldots, x_n) = f(x_{w^{-1}(1)}, \ldots, x_{w^{-1}(n)}).$$

The *ith divided difference operator* $\partial_i : \mathcal{P}_n \to \mathcal{P}_n$ is defined by

$$\partial_i(f) := \partial_{x_i x_{i+1}} = \frac{f - s_i f}{x_i - x_{i+1}}. \tag{2.1}$$

The reader should check that ∂_i is a linear operator. This means we can understand ∂_i by its action on monomials: for $a > b$

$$\partial_i x_i^a x_{i+1}^b = -\partial_i x_i^b x_{i+1}^a = x_i^a x_{i+1}^a \sum_{j=0}^{a-b-1} x_i^j x_{i+1}^{a-b-1-i}, \tag{2.2}$$

which can be verified by multiplying the righthand side by $x_i - x_{i+1}$.

We now establish some important basic properties of divided difference operators. All but the last follow from straightforward computations, and we encourage the reader to attempt their own proofs before reading on. Recall $f \in \mathcal{P}_n$ is *symmetric* in x_i and x_j if $f = t_{ij} \cdot f$.

Proposition 2.1. *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then*

(1) $\partial_i(f) = 0$ *if and only if f is symmetric in x_i and x_{i+1}.*

(2) $\partial_i(f)$ *is symmetric in the variables x_i and x_{i+1}.*

Proof. Property (2) follows from Equation (2.2).

For (1), the converse is immediate from Equation 2.1. Now view f as a polynomial in the variables x_i and x_{i+1} with coefficients in $\mathbb{C}[x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n]$ and assume it is not symmetric. Equivalently, for some a, b we have $[x_i^a x_{i+1}^b]f = g_{ab} \neq g_{ba} = [x_i^b x_{i+1}^a]f$. Assume $a > b$ maximizes first $a + b$ and then $|a - b|$. By Equation (2.2) $[x_i^{a-1} x_{i+1}^b]f = g_{ab} - g_{ba} \neq 0$ so $\partial_i(f) \neq 0$. □

Divided difference operators satisfy a version of Leibnitz's rule.

Proposition 2.2. *Let $f, g \in \mathcal{P}_n$ and $i \in [n-1]$. Then*

$$\partial_i(fg) = \partial_i(f)g + (s_i \cdot f)\partial_i(g).$$

Proof. We compute

$$\begin{aligned}\partial_i(fg) &= \frac{fg - s_i \cdot (fg)}{x_i - x_{i+1}} = \frac{fg - (s_i \cdot f)g + (s_i \cdot f)g - s_i \cdot (fg)}{x_i - x_{i+1}} \\ &= g\frac{f - s_i \cdot f}{x_i - x_{i+1}} + (s_i \cdot f)\frac{g - s_i \cdot g}{x_i - x_{i+1}} = \partial_i(f)g + (s_i \cdot f)\partial_i(g).\end{aligned}$$

□

Corollary 2.3. *Let $f, g \in \mathcal{P}_n$ and $i \in [n-1]$ so that f is symmetric in x_i and x_{i+1}. Then $\partial_i(fg) = f\partial_i(g)$.*

Proof. Applying Proposition 2.2, we have

$$\partial_i(fg) = \partial_i(f)g + (s_i \cdot f)\partial_i(g) = f\partial_i(g)$$

since $\partial_i(f) = 0$ by Proposition 2.1 (1) and $s_i \cdot f = f$ by symmetry. □

The divided difference operators satisfy relations similar to those for simple transpositions.

Proposition 2.4. *For i, j positive integers,*

(1) $\partial_i^2 = 0$, (2) $\partial_i\partial_j = \partial_j\partial_i$ $|i-j| > 1$, (3) $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$.

Proof. The first relation follows by Proposition 2.1 (1) and (2), while the commutation relation follows from the computation

$$\partial_i\partial_j(f) = \frac{f - s_i \cdot f - s_j \cdot f + +s_i s_j f}{(x_i - x_{i+1})(x_j - x_{j+1})}$$

and the fact $s_i s_j = s_j s_i$. The third relation can also be proved by direct computation and an application of the braid relation:

$$\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1} = \frac{1 - s_i - s_{i+1} + s_i s_{i+1} + s_{i+1} s_i - s_i s_{i+1} s_i}{(x_i - x_{i+1})(x_i - x_{i+2})(x_{i+1} - x_{i+2})}.$$

□

For $\mathbf{a} = (a_1, \ldots, a_p)$ a tuple of positive integers, define

$$\partial_{\mathbf{a}} := \partial_{a_1} \ldots \partial_{a_p}.$$

Corollary 2.5. *For $w \in S_n$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Red}(w)$, we have $\partial_{\boldsymbol{a}} = \partial_{\boldsymbol{b}}$.*

Proof. By Proposition 2.4 and Theorem 1.12, the result follows. □

Definition 2.6. For $w \in S_n$, define $\partial_w := \partial_{\mathbf{a}}$ for some $\mathbf{a} \in \mathrm{Red}(w)$.

By Corollary 2.5 ∂_w is well defined. Note $\partial_{s_i} = \partial_i$ and $\partial_1 = 1$.

Proposition 2.7. *Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ be non-reduced. Then $\partial_{\boldsymbol{a}} = 0$.*

Proof. There exists some minimal $j \in [p]$ so that $(a_1, \ldots, a_j)$ is reduced and $(a_1, \ldots, a_j, a_{j+1})$ is non-reduced. Let $v = s_{a_1} \ldots s_{a_j}$. Then $a_{j+1} \in \mathrm{Des}_{\mathrm{R}}(v)$ so by Corollary 1.10 some $(b_1, \ldots, b_j) \in \mathrm{Red}(v)$ has $b_j = a_{j+1}$. For $\mathbf{b} = (b_1, \ldots, b_j, a_{j+1}, \ldots, a_p)$ we see $\partial_{\mathbf{a}} = \partial_{\mathbf{b}} = 0$ since $\partial_{b_j}\partial_{a_{j+1}} = 0$ by Proposition 2.4 (2). □

Corollary 2.8. *For $u, v \in S_n$, $\partial_u \partial_v = \begin{cases} \partial_{uv} & \ell(uv) = \ell(u) + \ell(v), \\ 0 & \text{else.} \end{cases}$*

Proof. The result follows by Definition 2.6, Proposition 2.7 and properties or reduced words. □

By Definition 2.6 and Proposition 2.7, compositions of divided difference operators are either 0 or ∂_w for some $w \in S_n$. In general, computing ∂_w is non-trivial. Here is an important special case.

Theorem 2.9. *Let $w \in S_n$, $g \in \mathcal{P}_n$ and $f = \sum_{i=1}^n \alpha_i x_i$. Then*

$$\partial_w(fg) = (w \cdot f)\partial_w(g) + \sum_{u = wt_{ij} \lessdot w} (\alpha_i - \alpha_j)\partial_u(g).$$

Proof. Let $(a_1, \dots, a_p) \in \mathrm{Red}(w)$. Since f is linear, $\partial_i \partial_j(f) = 0$ for any $i, j \in [n-1]$ so by repeated application of Proposition 2.2

$$\begin{aligned} \partial_w(fg) &= \partial_{a_1} \dots \partial_{a_p}(fg) \\ &= (s_{a_1} \dots s_{a_p}) \cdot f \ \partial_{a_1} \dots \partial_{a_p}(g) + \\ &\sum_{i=1}^{p} (s_{a_1} \dots s_{a_{i-1}}) \cdot \partial_{a_i}((s_{a_{i+1}} \dots s_{a_p}) \cdot f) \ \partial_{a_1} \dots \widehat{\partial_{a_i}} \dots \partial_{a_p}(g). \end{aligned}$$

When $(a_1, \dots, \widehat{a_i}, \dots, a_p)$ is non-reduced, $\partial_{a_1} \dots \widehat{\partial_{a_i}} \dots \partial_{a_p} = 0$ by Proposition 2.7. When $(a_1, \dots, \widehat{a_i}, \dots, a_p) \in \mathrm{Red}(u)$ for some $u \in S_n$, we see $u = wt_{ij} \lessdot w$ for some $i < j$ by Lemma 1.16. Moreover, by Proposition 1.7 we see $(i, j) = s_{a_p} \dots s_{a_{i+1}}(a_i, a_i + 1)$ so

$$\partial_{a_i}((s_{a_{i+1}} \dots s_{a_p}) \cdot f) = \alpha_i - \alpha_j,$$

which completes our proof. □

2.2. Schubert polynomials

Recall $\delta_n = (n-1, n-2, \dots, 2, 1, 0)$.

Definition 2.10. For $w \in S_n$, the *Schubert polynomial* $\mathfrak{S}_w$ is

$$\mathfrak{S}_w := \delta_{w^{-1}w_0}(x^{\delta_n}).$$

In particular, $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$.

From the definition, there's no guarantee that $\mathfrak{S}_w \neq 0$ for $w \in S_n$. We will prove this fact later as Corollary 2.14, but first use it to prove several several important properties of Schubert polynomials.

Proposition 2.11. *Let $w \in S_n$ and $i \in [n-1]$. Then*

(1) $\partial_i \mathfrak{S}_w = \mathfrak{S}_{ws_i}$ *if* $i \in \mathrm{Des}_{\mathrm{R}}(w)$ *and* 0 *otherwise.*

(2) $\mathfrak{S}_w \in \mathcal{P}_n$ *is homogeneous of degree* $\ell(w)$.

(3) *If* $[x^\alpha]\mathfrak{S}_w \neq 0$, *then* x^α *divides* x^{δ_n}.

(4) $\mathfrak{S}_w$ *is symmetric in* x_i, x_{i+1} *if and only if* $i \notin \mathrm{Des}_{\mathrm{R}}(w)$.

(5) *If* $r = \max \mathrm{Des}_{\mathrm{R}}(w)$, *then* $\mathfrak{S}_w \in \mathcal{P}_r$.

Proof. Property (1) holds by Corollary 2.8 since

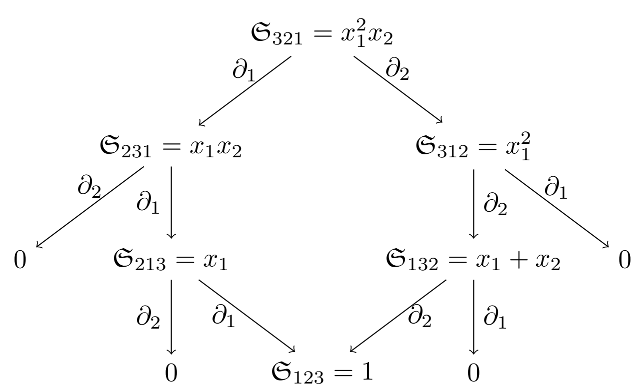
$$\partial_i \partial_{w^{-1}w_0} = \begin{cases} \partial_{s_i w^{-1} w_0} = \partial_{(ws_i)^{-1}w_0} & i \in \mathrm{Des}_{\mathrm{L}}(w^{-1}) = \mathrm{Des}_{\mathrm{R}}(w) \\ 0 & \text{else.} \end{cases}$$

Assuming $\mathfrak{S}_w \neq 0$, since its degree is

$$\deg(x^{\delta_n}) - \ell(w^{-1}w_0) = \binom{n}{2} - \binom{n}{2} + \ell(w^{-1}) = \ell(w).$$

For (3), note for $a \leq n-i$ and $b \leq n-i-1$ that if $[x_i^c x_{i+1}^d]\partial_i(x_i^a x_{i+1}^n) \neq 0$ then by Equation (2.2) $c \leq n-i$ and $d \leq n-i-1$ as well. Property (4) follows by (1) and Proposition 2.1, while (5) follows from the fact that $\mathfrak{S}_w$ is symmetric in $x_{r+1}, \dots, x_n$ and x_n never appears. $\square$

Using Proposition 2.11 (1), for $n = 3$ we compute



Recall $w \in S_n$ is *dominant* if $c(w)$ is a partition.

Theorem 2.12. *For $w \in S_n$ dominant, $\mathfrak{S}_w = x^{c(w)}$.*

Proof. Our proof is by descending induction on length. For $w = w_0$, we have $\mathfrak{S}_{w_0} = x^{c(w_0)}$ since $c(w_0) = \delta_n$.

Assume w is dominant with $\ell(w) < \binom{n}{2}$. Since

$$n-1 \geq c_1(w) \geq \cdots \geq c_n(0) = 0,$$

the pigeonhole principle there exists $i \in [n-1]$ so that $c_i(w) = c_{i+1}(w)$. Note $i \notin \mathrm{Des}_{\mathrm{R}}(w)$ by Proposition 1.31 (5). Taking i minimal, we see $c(ws_i) = s_i \cdot c(w) + \varepsilon_i = c(w) + \varepsilon_i$, which is also an integer partition. By Proposition 2.11 (1) and the inductive hypothesis

$$\mathfrak{S}_w = \partial_i(\mathfrak{S}_{ws_i}) = x^{c(w)+\varepsilon_i} = x^{c(w)},$$

and the result follows by induction. □

Define $\iota : S_n \to S_{n+1}$ by $\iota(w) = w(1) \ldots w(n)\ n{+}1$.

Corollary 2.13. *For $w \in S_n$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. In particular $\mathfrak{S}_{1\ldots n} = 1$.*

Proof. For $w_0^{(n)} = n \ldots 1\ n{+}1 \in S_{n+1}$, we see $w_0^{(n)}$ is dominant. Then by Theorem 2.12 we have $\mathfrak{S}_{w_0^{(n)}} = x^{\delta_n}$. Then for $w \in S_n$, since $\iota(w)^{-1} w_0^{(n+1)} = \iota(w^{-1}) w_0^{(n)} w_0^{(n)} w_0^{(n+1)}$ with

$$\ell(\iota(w)^{-1} w_0^{(n+1)}) = \ell(\iota(w^{-1}) w_0^{(n)}) + \ell(w_0^{(n)} w_0^{(n+1)})$$

we have

$$\mathfrak{S}_{\iota(w)} = \partial_{w^{-1} w_0^{(n)}} \partial_{w_0^{(n)} w_0^{(n+1)}} x^{\delta_{n+1}} = \partial_{w^{-1} w_0^{(n)}} x^{\delta_n} = \mathfrak{S}_w.$$

□

Note that Corollary 2.13 has a simple direct proof from the fact that $\partial_1 \ldots \partial_n x^{\delta_{n+1}} = x^{\delta_n}$. The key takeaway of the corollary is that Schubert polynomials are a well-defined family of objects indexed by permutations in S_∞. In Theorem 2.31 we will show how this property uniquely characterizes Schubert polynomials. We now show Schubert polynomials are non-zero.

Corollary 2.14. *For $w \in S_n$, $\mathfrak{S}_w \neq 0$.*

Proof. Using Corollaries 2.8 and 2.13 we compute

$$\partial_w \mathfrak{S}_w = \partial_w \partial_{w^{-1}w_0} x^{\delta_n} = \partial_{w_0} x^{\delta_n} = \mathfrak{S}_1 = 1.$$

Since $\partial_w(0) = 0$. we see $\mathfrak{S}_w \neq 0$. □

We now prove a critical lemma about the relationship between Schubert polynomials and divided difference operators.

Lemma 2.15. *Let $u, v \in S_n$ with $\ell(u) = \ell(v)$. Then $\partial_u \mathfrak{S}_v = \delta_{uv}$.*

Proof. Note $\partial_u \mathfrak{S}_v = \partial_u \partial_{v^{-1}w_0} x^{\delta_n}$. If $u = v$, then $\partial_u \partial_{v^{-1}w_0} = \partial_{w_0}$ so $\partial_u \mathfrak{S}_v = 1$ by Corollary 2.13. Otherwise, $uv^{-1}w_0 \neq w_0$ so

$$\ell(uv^{-1}w_0) < \ell(w_0) = \ell(v) + \ell(v^{-1}w_0) = \ell(u) + \ell(v^{-1}w_0).$$

Then $\partial_u \partial_{v^{-1}w_0} = 0$ by Proposition 2.8, and the result follows. □

For $w \in S_\infty$ with $\ell(w) = p$ and $\max \mathrm{Des}_{\mathrm{R}}(w) = r$, the divided difference operator ∂_w is a functional on the vector space

$$\mathrm{span}(\{x^\alpha : \alpha \in \mathbb{N}^r, |\alpha| = p\}) \subseteq \mathcal{P}_r.$$

Lemma 2.15 shows the Schubert polynomial $\mathfrak{S}_w$ is dual to ∂_w. Since our vector space is finite dimensional, we see Schubert polynomials are the dual basis to divided difference operators. As a consequence:

Corollary 2.16.

(1) $\{\mathfrak{S}_w\}_{w \in S_n}$ *is a basis for* $\mathrm{span}(x^\alpha : \alpha \subseteq x^{\delta_n})$.

(2) $\{\mathfrak{S}_w\}_{w \in S_\infty, \max \mathrm{Des}_{\mathrm{R}}(w) = r}$ *is a basis for* $\mathcal{P}_n$.

(3) $\{\mathfrak{S}_w\}_{w \in S_\infty}$ *is a basis for* $\mathcal{P}$.

Proof. Since $\mathrm{span}(x^\alpha : \alpha \subseteq x^{\delta_n})$ has dimension $n!$, (1) follows by the previous discussion. Properties (2) and (3) follow since for any $f \in \mathcal{P}$, there exists n sufficiently large so that $f \in \mathrm{span}(x^\alpha : \alpha \subseteq x^{\delta_n})$. □

2.3. Monk's formula

Since $\mathfrak{S}_{s_k}$ is homogeneous of degree 1 for $k > 0$, in principle we can compute the product $\mathfrak{S}_{s_k}\mathfrak{S}_w$ for any $w \in S_\infty$ using Theorem 2.9. Monk's formula, originally proved in a geometric context, describes this product. Before stating the formula, we give a formula for $\mathfrak{S}_{s_k}$.

Proposition 2.17. *For k a positive integer* $\mathfrak{S}_k = x_1 + x_2 + \cdots + x_k$.

Proof. By Proposition 2.11 (5) the variables $x_{k+1}, x_{k+2}, \ldots$ do not occur in $\mathfrak{S}_{s_k}$. Since $\partial_k \mathfrak{S}_{s_k} = 1$, we then see $[x_k]\mathfrak{S}_{s_k} = 1$. By Proposition 2.11 (4), $\mathfrak{S}_{s_k}$ is symmetric in $x_1, \ldots, x_k$, giving the result. □

Theorem 2.18 (Monk's formula)**.** *For $r > 0$ and $v \in S_\infty$,*

$$\mathfrak{S}_{s_r}\mathfrak{S}_v = \sum_{\substack{v \lessdot vt_{is} = w \\ i \le r < s}} \mathfrak{S}_w.$$

Proof. Note $\mathfrak{S}_{s_k}\mathfrak{S}_v$ is a polynomial of degree $\ell(v) + 1$. Then

$$\mathfrak{S}_{s_k}\mathfrak{S}_v = \sum_{\ell(w) = \ell(v)+1} c_w \mathfrak{S}_w$$

since Schubert polynomials are a basis of $\mathcal{P}$. For $w \in S_\infty$ with $\ell(w) = \ell(v) + 1$, by Lemma 2.15 and Theorem 2.9 we compute

$$c_w = \partial_w(\mathfrak{S}_{s_r}\mathfrak{S}_v) = w \cdot (\mathfrak{S}_{s_r})\partial_w(\mathfrak{S}_v) + \sum_{u = wt_{ij} \lessdot w} (\alpha_i - \alpha_j)\partial_u(\mathfrak{S}_v)$$

where $\alpha_i = 1$ if $i \le r$ and 0 otherwise. Since $\ell(w) > \ell(v)$, we see $\partial_w(\mathfrak{S}_v) = 0$. By Lemma 2.15 we have $\partial_u(\mathfrak{S}_v) = \delta_{uv}$. Note

$$\alpha_i - \alpha_j = \begin{cases} 1 & i \le r < j \\ 0 & \text{else} \end{cases}, \quad \text{so} \quad c_w = \begin{cases} 1 & v \lessdot vt_{ij} = w,\ i \le r < j \\ 0 & \text{else} \end{cases}.$$

The result now follows. □

Monk's formula in turn implies another important relation. For $v \in S_n$ and $r > 0$, define

$$I(v, r) := \{i < r : v \lessdot vt_{ir}\}, \quad J(v, r) := \{s > r : v \lessdot vt_{rs}\},$$
$$\Phi(v, r) := \{vt_{ir} : i \in I(v, r), \quad \Psi(v, r) := \{vt_{rs} : s \in J(v, r).$$

See Example 2.20 for worked examples of these terms.

Corollary 2.19 (Monk's recurrence)**.** *For $r > 0$ and $v \in S_\infty$,*

$$x_r\mathfrak{S}_v = \sum_{w \in \Psi(v,r)} \mathfrak{S}_w - \sum_{u \in \Phi(v,r)} \mathfrak{S}_u.$$

Proof. By Proposition 2.17 and Theorem 2.18

$$x_r\mathfrak{S}_v = (\mathfrak{S}_{s_r} - \mathfrak{S}_{s_{r-1}})\mathfrak{S}_v = \sum_{\substack{v \lessdot vt_{is}=w \\ i \le r < s}} \mathfrak{S}_w - \sum_{\substack{v \lessdot vt_{is}=w \\ i \le r-1 < s}} \mathfrak{S}_u.$$

The terms on the righthand side that don't cancel are those in the first summand with $r = i$ and those in the second where $r = s$. These terms correspond precisely to $J(v, r)$ and $I(v, r)$, respectively, so the result follows from the definitions of $\Phi(v, r)$ and $\Psi(v, r)$. □

Example 2.20. Let $v = 4317526$ and $r = 5$. Then

$$I(v, r) = \{1, 2, 3\}, \quad J(v, r) = \{7\}$$

then $\Phi(v, r) = \{u_1 = 5317426, u_2 = 4517326, u_3 = 4357126\}$ and $\Psi(v, r) = \{w = 4317625\}$. Therefore, we have

$$x_5\mathfrak{S}_v = \mathfrak{S}_w - \mathfrak{S}_{u_1} - \mathfrak{S}_{u_2} - \mathfrak{S}_{u_3},$$

or equivalently $\mathfrak{S}_w = x_5\mathfrak{S}_v + \mathfrak{S}_{u_1} + \mathfrak{S}_{u_2} + \mathfrak{S}_{u_3}$.

As highlighted in Example 2.20, when $\Psi(v, r) = \{w\}$, we obtain an expansion for $\mathfrak{S}_w$ in terms of other Schubert polynomials. As we will see, this observation can be applied to any permutation, leading to a recursive formula for Schubert polynomials.

Recall *lexicographic order* is the total order $\le_{\text{lex}}$ where

$$(a, b) \le_{\text{lex}} (c, d)$$

if $a < c$ or $a \le c$ and $b \le d$.

Lemma 2.21. *Let $w \in S_\infty$ and (r, s) be $\le_{\text{lex}}$–maximal in* $\text{Inv}(w)$. *Then $v := wt_{rs} \lessdot w$ and $\Psi(v, r) = \{w\}$.*

Proof. The result is more easily understood working with $D(w)$. To see $v \lessdot w$, we must check that the submatrix of M^w with corners $(r, w(r))$ and $(s, w(s))$ has all other entries 0. If such an entry (k, ℓ) is non-zero, then $i < k < s$ with $w(i) > \ell = w(k) > w(s)$. This means $(\ell, s) \in \text{Inv}(w)$, but $(r, s) \le_{\text{lex}} (\ell, j)$, a contradiction.

To see $\Psi(v, r) = \{w\}$, let $vt_{r\ell} \in \Psi(v, r)$. If $\ell < s$, we see $w(\ell) = v(\ell) > v(r) = w(s)$ so $(\ell, s) \in \text{Inv}(w)$, a contradiction. If $\ell > s$, we see $w(r) = v(s) > v(\ell) = w(\ell)$ so $(r, \ell) \in \text{Inv}(w)$, a contradiction. Therefore $\ell = s$ and the result holds. □

Corollary 2.22 (Transition equations)**.** *Let $w \in S_\infty$ and (r,s) be $\leq_{\text{lex}}$–maximal in* $\text{Inv}(w)$*. Then for $v = wt_{rs}$*

$$\mathfrak{S}_w = x_r \mathfrak{S}_v + \sum_{u \in \Phi(v,r)} \mathfrak{S}_u.$$

Lemma 2.23. *Let $w \in S_\infty$, (r,s) be $\leq_{\text{lex}}$–maximal in* $\text{Inv}(w)$ *and $v = wt_{rs}$. Then if (k,ℓ) is the $\leq_{\text{lex}}$–maximal element of* $\text{Inv}(v)$ *or* $\text{Inv}(u)$ *with $u \in \Phi(v,r)$, we have $(k,\ell) \leq_{\text{lex}} (r,s)$.*

Proof. Homework for now. □

Theorem 2.24. *For $w \in S_\infty$, $\mathfrak{S}_w \in \mathbb{N}[x_1, x_2, \ldots]$.*

Proof. We argue by induction on the value (r,s) of the $\leq_{\text{lex}}$–maximal inversion of w. Our base case is $w = 1$, in which case we can say $(r,s) := (0,0)$ and $\mathfrak{S}_w = 1$. Otherwise, by Corollary 2.22 and Lemma 2.23 we see

$$w = x_r \mathfrak{S}_v + \sum_{u \in \Phi(v,r)} \mathfrak{S}_u$$

where if (k,ℓ) is the $\leq_{\text{lex}}$–maximal inversion of v or $u \in \Phi(v,r)$ then $(k,\ell) \leq_{\text{lex}} (r,s)$. The result follows from the inductive hypothesis. □

Corollary 2.25. *The Schubert polynomials $\{\mathfrak{S}_w\}_{w \in S_\infty}$ are the unique family of polynomials satisfying $\mathfrak{S}_1 = 1$ and Monk's formula (or the transition equations, or Corollary 2.22).*

In practice, using Corollary 2.22 is a horribly inefficient way to compute Schubert polynomials. However, we will see it can be extended to a more useful combinatorial approach in Section **??**.

Example 2.26. By repeated application of Corollary 2.22 we have

$$\begin{aligned}\mathfrak{S}_{1432} &= x_3 \mathfrak{S}_{1423} + \mathfrak{S}_{2413} \\ &= x_3(x_2 \mathfrak{S}_{1324} + \mathfrak{S}_{3124}) + x_2 \mathfrak{S}_{2314} + \mathfrak{S}_{3214}.\end{aligned}$$

Since $1324 = s_2$ and $3124, 2314, 3214$ are dominant, we have

$$\begin{aligned}\mathfrak{S}_{1432} &= x_2 x_3(x_1 + x_2) + x_3(x_1^2) + x_2(x_1 x_2) + x_1^2 x_2 \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3\end{aligned}$$

by Proposition 2.17 and Theorem 2.12. Since

$$\begin{aligned}\mathfrak{S}_{1432} &= \partial_1\partial_2\partial_3(x_1^3x_2^2x_3) = \partial_1\partial_2(x_1^3x_2^2\\ &= \partial_1(x_1^3(x_2+x_3)) = x_1x_2(x_1+x_2) + x_3(x_1^2+x_1x_2+x_2^2)\\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3,\end{aligned}$$

we see the transition approach agrees with Definition 2.10 in this case.

We now give an alternate approach to computing Schubert polynomials based on the transition equations. Starting with Monk's recurrence

$$x_r\mathfrak{S}_v = \sum_{w\in\Psi(v,r)} \mathfrak{S}_w - \sum_{u\in\Phi(v,r)} \mathfrak{S}_u,$$

the transition equations say given $w \in S_\infty$ one can choose v so that $\Psi(v,r) = \{w\}$. An alternate approach known as *co-transition* says given $v \in S_n$, we can choose r so that $\Phi(v,r) = \varnothing$. So long as $\Psi(v,r) \subseteq S_n$, we can then compute $\mathfrak{S}_v$ using upward induction and division. We will need some background to identify the correct r.

For $w \in S_\infty$, the *dominant component* of $D(w)$, denoted $\mathrm{Dom}(w)$, is the $\subset$–maximal partition λ so that $D_\lambda \subseteq D(w)$. Note $\mathrm{Dom}(w) = \varnothing$ when $(1,1) \notin D(w)$. For example, with $w = 351642$, we see from Equation (1.4) that $\mathrm{Dom}(w) = (2,2)$.

Lemma 2.27. *Let $v \in S_\infty$ and $\lambda = \mathrm{Dom}(v)$. If $r = 1$ or $\lambda_r < \lambda_{r-1}$, then $\Phi(v,r) = \varnothing$.*

Proof. The $r = 1$ case follows from the definition of $\Phi(v,r)$. Otherwise $(r,\lambda_r), (r-1,\lambda_r+1) \in D(v)$ but (r,λ_r+1) does not. Therefore $v(r) = \lambda_r+1$. Since $(r,j) \in D(w)$ for $j \le \lambda_r$, we see $v(i) > v(r)$ for $i < r$. Then $vt_{ir} < v$ for $i < r$ so $\Phi(v,r) = \varnothing$. □

Proposition 2.28 (Co-transition). *For $v \in S_n$ with $\lambda = \mathrm{Dom}(v)$. Then for $r = 1$ or r so that $\lambda_r < \lambda_{r-1}$, we have*

$$x_r\mathfrak{S}_v = \sum_{w\in\Psi(v,r)} \mathfrak{S}_w. \tag{2.3}$$

Moreover, if $\lambda_r < n - r$ then $\Psi(v,r) \subseteq S_n$.

Proof. Let $\lambda = \mathrm{Dom}(v)$ and $r = 1$ or r so that $\lambda_r < \lambda_{r-1}$. By Lemma 2.27 we see $\Phi(v,r) = \varnothing$ so Monk's recurrence becomes Equation (2.3). Now assume $\lambda_r < n - r$. Then $c_i(v) \geq \lambda_r + 1$ for $i < r$, so $v(i) \geq \lambda_r + 2$ for such values. Additionally, we have $v(r) = \lambda_r + 1$. Since $\lambda_r < n - r$, by the pigeon hole principle there must exist some minimal j so that $r < j \leq n$ and $v(j) > v(r)$. Then $v \lessdot vt_{rj}$ so $vt_{rj} \in \Psi(v,r)$. For $k > n$, $v(k) > v(j)$ so $\ell(vt_{rk}) > \ell(v) + 1$ and the result follows. □

We now give a co-transition proof of Theorem 2.24.

Theorem. *For $v \in S_n$, $\mathfrak{S}_v \in \mathbb{N}[x_1, x_2, \ldots]$.*

Proof. We argue by downward induction on $\ell(w)$. For $v = w_0$ the result follows by definition. Otherwise, by Proposition 2.28 there exists r so that

$$x_r \mathfrak{S}_v = \sum_{w \in \Psi(v,r)} \mathfrak{S}_w$$

with $\Psi(v,r) \subseteq S_n$. By the inductive hypothesis, the righthand side is in $\mathbb{N}[x_1, x_2, \ldots]$. Dividing by x_r, we have

$$\mathfrak{S}_v = \left(\sum_{w \in \Psi(v,r)} \mathfrak{S}_w \right) / x_r.$$

Since $\mathfrak{S}_v \in \mathcal{P}_n$, we see x_r divides $\mathfrak{S}_w$ so the result follows by the inductive hypothesis. □

Example 2.29. By repeated application of Proposition 2.28 we have

$$\begin{aligned} x_1 \mathfrak{S}_{1432} &= \mathfrak{S}_{4132} + \mathfrak{S}_{\underline{3412}} + \mathfrak{S}_{2431} \\ x_2 \mathfrak{S}_{4132} &= \mathfrak{S}_{\underline{4312}} + \mathfrak{S}_{\underline{4231}} \\ x_1 \mathfrak{S}_{2431} &= \mathfrak{S}_{\underline{4231}} + \mathfrak{S}_{\underline{3421}}. \end{aligned}$$

The underlined permutations are dominant, so by Theorem 2.12 we have

$$\mathfrak{S}_{4213} = \frac{x_1^3 x_2^2 + x_1^3 x_2 x_3}{x_2} \quad \text{and} \quad \mathfrak{S}_{2431} = \frac{x_1^3 x_2 x_3 + x_1^2 x_2^2 x_3}{x_1},$$

so

$$\begin{aligned}\mathfrak{S}_{1432} &= \frac{x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3}{x_1} \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2^2x_3.\end{aligned}$$

Note our computation agrees with Example 2.20.

As a consequence of our proof of co-transition, we see certain variables must divide Schubert polynomials.

Proposition 2.30. *For $w \in S_\infty$, we have $x^{\mathrm{Dom}(w)} \mid \mathfrak{S}_w$.*

Proof. Let $\lambda = \mathrm{Dom}(w)$. We argue by induction on $|\lambda|$. The base case where $|\lambda| = 0$ is true since $x^{(0)} = 1$. Now assume $|\lambda| > 0$, and let r be maximal so that $\lambda_r > 0$. Note for $j = w^{-1}(\lambda_r)$ the position of λ_r in v we have $j > r$ since otherwise $(r, \lambda_r) \notin D(w)$. Therefore $v := wt_{rj} \lessdot w$. Either $r = 1$ or for $\mu = \mathrm{Dom}(v)$ we have $\mu_r = \lambda_r - 1 < \lambda_{r-1}$. Then by Lemma 2.27 we see $w \in \Psi(v, r)$ and $\Phi(v, r) = \varnothing$ so $x_r \mid \mathfrak{S}_w$. By the inductive hypothesis, we then have $x_r x^\mu = x^\lambda \mid \mathfrak{S}_w$. □

2.4. Schubert polynomials as a basis

One motivation for studying Schubert polynomials is that in many combinatorial contexts they are the most natural basis for $\mathcal{P} := \mathbb{C}[x_1, x_2, \dots]$. A direct proof based of this fact on properties of divided difference operators appears as Exercise 2.5.1. We give a stronger proof showing in a precise sense that the leading monomial of the Schubert polynomial $\mathfrak{S}_w$ is $x^{c(w)}$.

For α and β weak compositions, let

Theorem 2.31. *The Schubert polynomials $\{\mathfrak{S}_w\}_{w \in S_\infty}$ are the unique family of polynomials satisfying:*

(1) $\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & i \in \mathrm{Des}_{\mathrm{R}}(w) \\ 0 & \textit{else.} \end{cases}$ *for all $i \geq 1$.*

(2) $\mathfrak{S}_1 = 1$ *and* $[1]\mathfrak{S}_w = 0$ *for all other $w \in S_\infty$.*

Proof. By combining Proposition 2.11 (1) and (4) with Corollary 2.13, we see $\{\mathfrak{S}_w\}_{w\in S_\infty}$ has the desired properties. For the converse, assume there exists w dominant so that $\mathfrak{S}_w \neq x^{c(w)}$.

$$w_\Box = n{+}1 \ldots 2n \; 1 \ldots n$$

□

2.5. Exercises

2.5.1. Using divided difference operators, show that $\{\mathfrak{S}_w\}_{w\in S_n}$ is linearly independent. Explain why this implies $\{\mathfrak{S}_w\}_{w\in S_n}$ is a basis for $\{x^\alpha : \alpha \subseteq \delta_n\}$ and $\{\mathfrak{S}_w\}_{w\in S_\infty}$ is a basis for $\mathcal{P}$.

2.5.2. Let $w \in S_n$, (r,s) be the $\leq_{\mathrm{lex}}$–maximal inversion in w and $v = wt_{rs}$. Show $\Phi(v,r)$ is empty if and only if w is dominant.

Chapter 3

Combinatorial formulas

Recall from Theorem 2.24) that for $w \in S_\infty$ we have

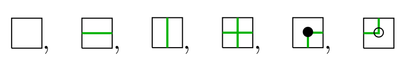
$$\mathfrak{S}_w = \sum_{\alpha \vDash \ell(w)} c_\alpha x^\alpha$$

with $c_\alpha \in \mathbb{N}$. In combinatorics, the natural desire when encountering non-negative integers is for them to count some combinatorial object. In this section we give two such descriptions of Schubert polynomials: the pipe dream formula and the bumpless pipe dream formula. Each formula offers distinct benefits for deducing properties of Schubert polynomials, and we will present many such applications.

3.1. Bumpless pipe dreams

We begin with a definition:

Definition 3.1. A *bumpless pipe dream* B is a filling of $[n] \times [n]$ with

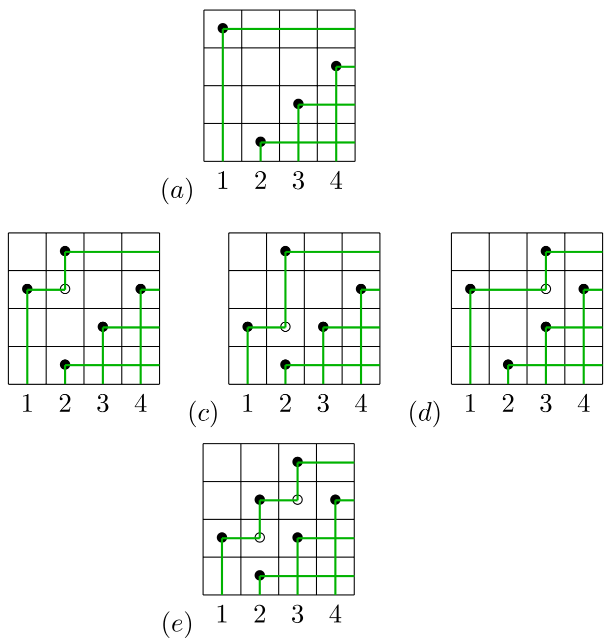


so that wires are connected, entering from the bottom of the diagram and exiting to the right. For $(i, j) \in [n] \times [n]$, let B_{ij} denote the filling of position (i, j). Let

$$B(\square) = \{(i, j) \in [n] \times [n] : B_{ij} = \square\}$$

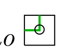
and likewise for the other possible fillings. Labeling the wires from right to left as $1, 2, \ldots n$, the *permutation* of B is the permutation obtained by reading of the wires along the right of B from top to bottom. A bumpless pipe dream is *reduced* if no two wires cross more than once. Let $\mathrm{BPD}(w)$ be the set of reduced bumpless pipe dreams for w.

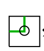
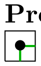
Example 3.2. There are five bumpless pipe dreams in BPD(1432):



(3.1)

Note that bumpless pipe dream (a) in Example 3.2 is the Rothe diagram $D(1432)$. This is not a coincidence.

Proposition 3.3. *A bumpless pipe dream B has no* *'s if and only if $B = D(w)$ for some $w \in S_n$. In particular $D(w) \in \mathrm{BPD}(w)$.*

Proof. If B has no 's, the jth wire enters the bottom, meets some  in position (i, j) and exits row i. Therefore, if w is the permutation of B we see $B = D(w)$. The converse is immediate. □

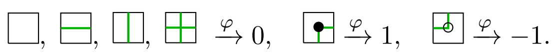
Bumpless pipe dreams have many alternate descriptions.

Definition 3.4. A vector $\vec{a} = [a_1 \ \ldots \ a_p] \in \{-1, 0, 1\}^p$ is *alternating* if the non-zero entries alternate in sign, beginning and ending with 1. This condition implies $\vec{a}$ is not the zero vector. An *alternating sign matrix* A is a square matrix with every row and column alternating. Let $\mathrm{ASM}(n)$ be the set of $n \times n$ alternating sign matrices.

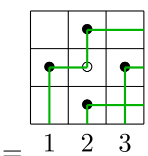
Note for $w \in S_n$ that the permutation matrix $M^w \in \mathrm{ASM}(n)$. There are seven matrices in ASM(3) – six permutation matrices and

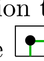
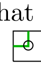
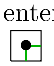
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Define a map φ on bumpless pipe dreams by



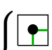
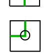
For example, with

$B =$ 

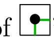
we have $\varphi(B) = A$. The condition that wires enter from the bottom ensures columns of B alternate , , ...,  and the condition that wires exit to the right guarantees rows alternate in the same way. This shows φ is well-defined.

Proposition 3.5. *The map ϕ is a bijection from bumpless pipe dreams of size n to* $\mathrm{ASM}(n)$.

Proof. For $A \in \mathrm{ASM}(n)$, we define $B = \varphi^{-1}(A)$ as follows. Clearly

$B_{ij} =$  $A_{ij} = 1$;  $A_{ij} = -1$.

When A_{ij} is 0, place a vertical wire in B_{ij} if the first non-zero entry above A_{ij} is 1 and a horizontal wire in B_{ij} if the first non-zero entry to the left of A_{ij} is 1. By construction, φ^{-1} is the inverse of φ. □

One takeaway from Proposition 3.5 is that a bumpless pipe dream B can be reconstructed from the locations of 's and 's. In fact, since the signing is determined by their locations, we only need the set of elbow locations to reconstruct B. We will show B can also be reconstructed using two other collections of data.

For $A \in \mathrm{ASM}(n)$, define the *rank matrix* r^A by

$$r^A_{k\ell} = \sum_{i\in[k],\ j\in[l]} A_{ij}.$$

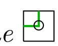
Note for $w \in S_n$ that $r^{M^w} = r^w$ from Definition 1.25. Let $B = \varphi^{-1}(A)$. With the convection that $r^A_{0j} = r^A_{i0} = 0$, if $r^A_{ij} = k$ the entry B_{ij} determines the following values of r^A:

(3.2)

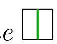
B_{ij}						
$\begin{matrix} r^A_{i-1\ j-1} & r^A_{i-1\ j} \\ r^A_{i\ j-1} & r^A_{ij} \end{matrix}$	$\begin{matrix} k & k \\ k & k \end{matrix}$	$\begin{matrix} k-1 & k-1 \\ k & k \end{matrix}$	$\begin{matrix} k-1 & k \\ k-1 & k \end{matrix}$	$\begin{matrix} k-2 & k-1 \\ k-1 & k \end{matrix}$	$\begin{matrix} k-1 & k-1 \\ k-1 & k \end{matrix}$	$\begin{matrix} k-1 & k \\ k & k \end{matrix}$

Verifying this table is Exercise 3.5.1.

Lemma 3.6. *The bumpless pipe dream B is determined by:*

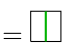
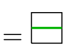
(1) *the locations of the* *'s and the* *'s;*

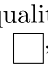
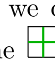
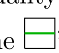
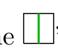
(2) *the locations of the* *'s and the* *'s;*

(3) *the locations of the* *'s and the* *'s.*

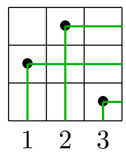
Proof. Part (1) is a corollary of Proposition 3.5. Let $A = \varphi(B)$, observe using Table (3.2) that

$r^A_{ij} - r^A_{i-1\ j-1} =$
- 2 if $B_{ij} =$ 
- 0 if $B_{ij} =$ ,
- 1 else

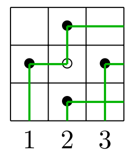
$r^A_{i-1\ j} - r^A_{i\ j-1} =$
- 1 if $B_{ij} =$ 
- -1 if $B_{ij} =$ .
- 0 else

For (2), the first equality shows we can fill in each diagonal using the locations of the 's and the 's using the initial conditions $r^A_{0j} = r^A_{i0} = 0$. Likewise, for (3) the second equality shows we can fill in each anti-diagonal using the locations of the 's and the 's using the same initial conditions. □

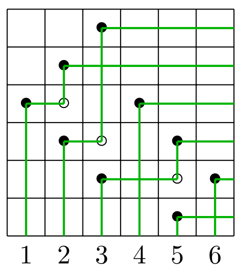
One might ask whether the conditions in Lemma 3.6 can be relaxed. Towards this end, note



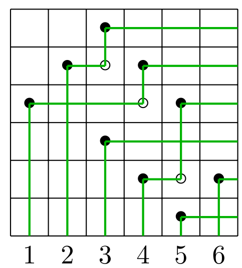
and



both have a single □ in position $(1,1)$. These bumpless pipe dreams have different permutations. However,



and



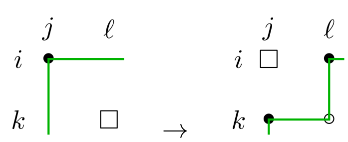
are distinct elements of $\mathrm{BPD}(214365)$ with the same set of □'s.

Definition 3.7 (Droop move)**.** For B a bumpless pipe dream, say $[(i,j),(k,\ell)]$ is an *available interval* if

$B_{ij} =$ , $\quad B_{k\ell} =$ □, $\quad B_{pq} \neq$ ,  for $i \leq p \leq k,\ j \leq q \leq \ell.$

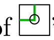
Viewing B as its set of elbows, we perform a *droop move* on B at the available interval $[(i,j),(k,\ell)]$ by removing the elbow (i,j) and adding elbows $(i,\ell),(k,j),(k,\ell)$:

(3.3)



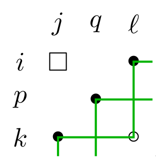
Note the underlying permutation is unchanged since the depicted wire enters and exits the same locations. Additionally, observe that a droop move is invertible.

Theorem 3.8. *For $w \in S_n$,* $\mathrm{BPD}(w)$ *is connected by droop moves.*

Proof. We argue by induction on the number of 's. If B has none, by Proposition 3.3 $B = D(w)$. Otherwise, let (k, ℓ) be $\leq_{\text{lex}}$–minimal so that $B_{k\ell} =$ . Necessarily, there exist $i < k$ and $j < \ell$ so that $B_{i\ell} = B_{kj} =$ . Let p, q so that $i \leq p \leq k$ and $j \leq q \leq \ell$. By the $\leq_{\text{lex}}$–minimality of (k, ℓ), we see:

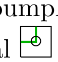
- i and j are unique;
- if $(p, q) \neq (k, \ell)$, then $B_{pq} \neq$ .

If $(p, q) \neq (i, \ell), (k, j)$ and $B_{pq} =$ , we see the wire entering B_{pq} from below passes through (k, q) and (p, ℓ). The wire entering $B_{k\ell}$ from the left passes through these positions as well, so B would not be reduced:



.

Therefore, $B_{pq} \neq$ , and we can perform an inverse droop move on the interval $[(i, j), (k, \ell)]$. The resulting bumpless pipe dream has one fewer , so the result follows by induction. $\square$

For $w \in S_n$, the proof of Theorem 3.8 gives a canonical construction for a path from $D(w)$ to any $B \in \text{BPD}(w)$. For example, in Example 3.2 we see there are droops from (a) to (b), (c) and (d) and from (c), (d) to (e). The canonical path from $D(1432) =$ (a) to (e) is (a) – (c) – (e). Here, the middle bumpless pipe dream is (c) and not (d) since $(2, 3)$ is the $\leq_{\text{lex}}$–minimal  in (e).

For $D \subseteq [n] \times [n]$, define $\rho(D)$ to be the $\subseteq$–minimal partition ρ so that $D \subseteq D_\rho$. Abusing notation, let $\rho(w) = \rho(D(w))$. Since available droop moves only move $\square$'s northwest, by Theorem 3.8 we see for $B \in \text{BPD}(w)$ that $B(\square) \subseteq D_{\rho(w)}$. Therefore, every available droop move for every bumpless pipe dream occurs in $D(\rho(w))$ as well. Then for $(i, j) \notin D_{\rho(w)}$, we see B_{ij} must be the same for every $B \in \text{BPD}(w)$. We call such cells *frozen*.

Corollary 3.9. *Let $w \in S_n$ be dominant. Then* $\text{BPD}(w) = \{D(w)\}$.

Proof. For w dominant, $\rho(w) = c(w)$ so $D_{\rho(w)} = D(w)$. In particular, no 's occur in $D_{\rho(w)}$ so there are no available intervals. Therefore by Theorem 3.8 we have $\mathrm{BPD}(w) = \{D(w)\}$. $\square$

For B a bumpless pipe dream, let

$$x^B = \prod_{(i,j)\in B(\square)} x_i.$$

Theorem 3.10. *For $w \in S_n$,*

$$\mathfrak{S}_w = \sum_{B\in \mathrm{BPD}(w)} x^B.$$

Proof. For the purposes of this proof, let

$$\mathfrak{B}_w = \sum_{B\in \mathrm{BPD}(w)} x^B.$$

We will argue by induction on the $\leq_{\mathrm{lex}}$ maximal inversion in w. For our base case, let w be dominant. By Corollary 3.9, we see $\mathrm{BPD}(w) = \{D(w)\}$ so $\mathfrak{B}_w = x^{c(w)} = \mathfrak{S}_w$ by Theorem 2.12.

Now let (r, s) be the $\leq_{\mathrm{lex}}$–maximal inversion in w. The transition equation Theorem 2.19 says

$$\mathfrak{S}_w = x_r\mathfrak{S}_v + \sum_{u\in\Phi(v,r)} \mathfrak{S}_u$$

where $v = wt_{rs}$. By the inductive hypothesis, we then have

$$\mathfrak{S}_w = x_r\mathfrak{B}_v + \sum_{u\in\Phi(v,r)} \mathfrak{B}_u.$$

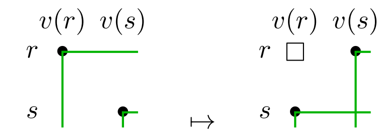
We will show the righthand side equals $\mathfrak{B}_w$ bijectively.

To do so, let $\Phi(v, r) = \{u^{(1)}, \ldots, u^{(k)}\}$ where $u^{(j)} = vt_{i_j r}$. Recall that $D(w) = D(v) \cup \{(r, w(s)\}$ and for all $j \in [k]$ that $(i_j, w(i_j))$ is a maximally south east $\bullet$ in $D(w)$ northwest of $(r, w(s))$. Therefore $[(i, w(i), (r, w(s)]$ is an available droop move in $D(w)$. Let $I(v, r) = \{i_1, \ldots, i_k\}$. Then

$$\mathrm{BPD}(w) = R_0 \sqcup R_1 \sqcup \cdots \sqcup R_k$$

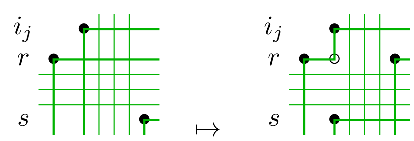
where $R_0 = \{B \in \mathrm{BPD}(w) : B_{rw(s)} = \square\}$ and R_j is the the set of bumpless pipe dreams whose canonical path from $D(w)$ has the first droop at the available interval $[(i_j, w(i_j)), (r, w(s))]$. We will identify R_0 with $\mathrm{BPD}(v)$ and R_j with $\mathrm{BPD}(u^{(j)})$.

Note that $(r, w(s)) = (r, v(r))$ is frozen for v. Therefore, for any $B \in \mathrm{BPD}(v)$ we can perform the transformation



to obtain $B \in R_0$. Note this transformation is invertible. Therefore $\sum_{B \in R_0} x^B = x_r \mathfrak{S}_v$.

Similarly, for $j \in [k]$ and $B \in \mathrm{BPD}(u^{(j)})$ the transformation



maps B to $B' \in \mathrm{BPD}(w)$ since all the illustrated positions are frozen in $u^{(j)}$. Note this transformation is invertible. Moreover, the first droop in a canonical path from B' to $D(w)$ will always include $(r, w(s))$ if possible, so $B' \in R_j$. Since $B(\square) = B'(\square)$, we have $\mathfrak{S}_{u^{(j)}} = \sum_{B \in R_j} x^B$ and the result follows. □

Example 3.11. For $w = 1432$ and labels as in Example 3.2, we have $(r, s) = (3, 4)$, $(r, w(s)) = (3, 2)$, $v = 1423$ and $\Phi(v, 3) = \{u^{(1)} = 2413\}$ Then $R_0 = \{\text{(a), (b), (d)}\}$, $R_1 = \{\text{(c), (e)}\}$. Figures will be added later comparing Rothe diagrams to bumpless pipe dreams.

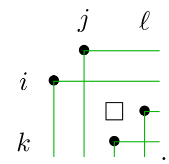
3.2. Vexillary bumpless pipe dreams

Recall a permutation v is *vexillary* if it avoids the pattern 2143. We will show that bumpless pipe dreams for vexillary permutations have a natural interpretation as non-intersecting lattice paths, or equivalently as tableaux.

First, we prove some properties of bumpless pipe dreams for vexillary permutations.

Lemma 3.12. *Let $w \in S_n$ and $B \in \mathrm{BPD}(w)$. Then $B(\boxplus) \cap D_{\rho(w)} = \varnothing$ if and only if w is vexillary.*

Proof. We argue by induction on $|B(\boxdot)|$ with $D(w)$ as our base case. Assume we have a $\boxplus$ in position $(i, j) \in D_{\rho(w)}$. Then there exists $(k, \ell) \in D(w)$ with $i < k, j < \ell$. Diagrammatically, we have

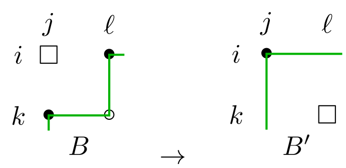


Equivalently, w contains 2143 as a pattern at positions

$$\{w^{-1}(j), i, w^{-1}(\ell), k\}$$

so w is not vexillary.

Now let $B \in \mathrm{BPD}(w)$ with $B \neq D(w)$. We can perform an inverse droop move on B at the interval $[(i, j), (k, \ell)]$:



By the induction hypothesis, the resulting bumpless pipe dream B' has no crosses in $D_{\rho(w)}$ if and only w is vexillary. Since $[(i, j), (k, \ell)]$ is an available interval, no elbows exist inside it. Therefore, any wire entering column j between rows i and k must continue through column ℓ. Likewise, any wire entering row i between columns j and ℓ must continue through row k. This guarantees any $\boxplus$ in B' occuring in the available interval corresponds to a $\boxplus$ in the available interval in B. All entries outside the available interval are unchanged. Since available intervals are contained in $D_{\rho(w)}$, the result follows. $\square$

When v is vexillary, our proof shows available intervals only contain wires along the boundary. Equivalently:

Corollary 3.13. *Let $v \in S_n$ be vexillary, $B \in \mathrm{BPD}(v)$ and $[(i,j),(k,\ell)]$ be an available interval for B. Then $(i,k] \times (j,\ell] \subseteq B(\square)$.*

Corollary 3.14. *Let $v \in S_n$ be vexillary. Then each $B \in \mathrm{BPD}(v)$ is determined by $B(\square)$.*

Proof. By Lemma 3.12 we see $B(\boxplus)$ is frozen. Therefore, $B(\boxplus)$ is determined by v, so by Lemma 3.6 (2) we can reconstruct B from $B(\square)$ and v. $\square$

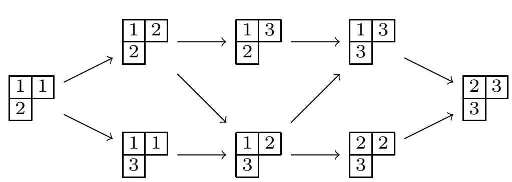
The kth *diagonal* of the diagram $D \subseteq [n] \times [n]$ is the subset $\{(i,j) \in D : i - j = k\}$. Let $d_k(D)$ be the size of the kth diagonal of D. Recall for α a weak composition that $\lambda(\alpha)$ is the integer partition obtained by sorting α.

Proposition 3.15. *Let $v \in S_n$ vexillary. Then for $B \in \mathrm{BPD}(w)$, $d_k(B(\square)) = d_k(D_{\lambda(c(w))})$ for all k.*

Proof. By Corollary 3.9, the result holds for v dominant. Otherwise, we perform a droop. By Corollary 3.13, we see the droop move shifts cells along diagonals. Therefore, the counts $d_i(B\square)$ are invariant under droop moves. To see the counts coincide with those for $D_{\lambda(c(v))}$, note by Exercise 2.5.2 that the Lascoux-Sch utzenberger tree of v has at least one dominant leaf. The shape of that permutation $\square$

Definition 3.16 (Semistandard tableaux)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be an integer partition. A *tableau* of shape λ is a function $T : D_\lambda \to \mathbb{Z}$. For $(i,j) \in D_\lambda$, we have $T_{ij} := T((i,j))$. A tableau in *semistandard* if $T_{ij} \leq T_{i\ j+1}$ and $T_{ij} < T_{i+1\ j}$ when T is well-defined and has *type* $\phi = (\phi_1, \ldots, \phi_k)$ if $T_{ij} \in [\phi_i]$ for all $(i,j) \in D_\lambda$. Let $\mathrm{SSYT}(\lambda, \phi)$ be the set of semistandard tableaux of shape λ and type ϕ. Additionally, define $\mathrm{SSYT}_n(\lambda) := \mathrm{SSYT}(\lambda, (n^k))$.

Example 3.17. We depict the tableaux in $\mathrm{SSYT}_3((2,1))$:



.

Each edge corresponds to incrementing a single entry by one.

There is alternate description of semistandard tableaux that closely resembles bumpless pipe dreams. The rth *diagonal* of $[n] \times [n]$ is the set $\{(i,j) \in [n] \times [n] : i - j = r\}$. For $D \subseteq [n] \times [n]$, let D^r be the rth diagonal of D. We say D^r interlaces $D^{r\pm 1}$ *interlaces* if for each $(i,j),(k,\ell) \in D^{r+1}$ with $i < k, j < \ell$ there exists $(i',j') \in D^r$ with

Definition 3.18. An *excited Young diagram* is a diagram $D \subseteq [n] \times [n]$ that is for $(i,j),(k,\ell)$ in the rth diagonal

3.3. Pipe dreams

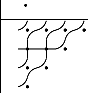
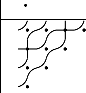
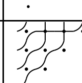
Definition 3.19. A *compatible sequence* is a biword

$$\binom{\mathbf{i}}{\mathbf{a}} = \binom{i_1, i_2, \ldots, i_p}{a_1, a_2, \ldots, a_p} \tag{3.4}$$

where $1 \leq i_1 \leq i_2 \leq \ldots i_p$ with $i_k = i_{k+1}$ implying $a_k > a_{k+1}$ and $i_k \leq a_k$ for all $k \in [p]$. Let $K(\mathbf{a})$ denote the set of $\mathbf{i}$ so that $\binom{\mathbf{i}}{\mathbf{a}}$ is a compatible sequence. The *permutation* of a compatible sequence is $s_{a_1} \ldots s_{a_p}$. A compatible sequence is *reduced* if $\mathbf{a}$ is.

To the compatible sequence in Equation 3.4 we associate the diagram $\{(i_k, a_k{+}1{-}i_k) : k \in [p]\}$, which we call its *pipe dream*. This procedure is inverted by reading the entries of P from right to left and top to bottom, mapping each $(i,j) \in P$ to $\binom{i}{i+j-1}$. For $w \in S_n$, let $\mathrm{PD}(w)$ be the set of pipe dreams whose associated compatible sequence is reduced with permutation w. Note a reduced pipe dream P has $|P| = \ell(w)$.

For $w \in S_n$, the maximum value in a reduced word is $n{-}1$. With $P \in \mathrm{PD}(w)$ and $(i,j) \in P$, we then see $i + j \leq n{-}1$. Then $P \subseteq D_{\delta_n}$. For example, $\mathrm{Red}(1423) = \{(3,2)\}$. The reduced compatible sequences for 1423 and associated pipe dreams are

compatible sequence	$\binom{22}{32}$,	$\binom{12}{32}$,	$\binom{11}{32}$
pipe dream	$\begin{matrix} \cdot & \cdot & \cdot \\ + & + & \\ \cdot & & \end{matrix}$,	$\begin{matrix} \cdot & \cdot & + \\ + & \cdot & \\ \cdot & & \end{matrix}$,	$\begin{matrix} \cdot & + & + \\ \cdot & \cdot & \\ \cdot & & \end{matrix}$
wiring diagram			

Here, we depict the pipe dream P by placing $+$ in position (i, j) of D_{δ_n} if $(i, j) \in P$. Replacing each $\cdot$ with ╯╭ gives the wiring diagram. If we label the wires in each diagram of P from right to left across the top, their order on the left hand side is the permutation of P. This is true in general since the associated biword of P is read off from right to left, top to bottom.

From the definition, it is not obvious that every permutation has a pipe dream.

Lemma 3.20. *For $w \in S_\infty$, we have $D_{c(w)} \in \mathrm{PD}(w)$.*

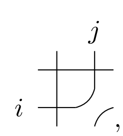
Proof. This is Exercise 3.5.3 . □

The main objective for this section is to prove a combinatorial formula for Schubert polynomials in terms of pipe dreams. Our proof is similar to that for the bumpless pipe dream formula (Theorem 3.10) with a key difference. While bumpless pipe dreams give a combinatorial realization of transition, pipe dreams give a combinatorial realization of cotransition. In order to see this, we must analyze the dominant component of pipe dreams.

For $D \subseteq [n] \times [n]$, the *dominant component* of D, denoted $\mathrm{Dom}(D)$, is the $\subseteq$–maximal partition λ so that $D_\lambda \subseteq D$. Note $\mathrm{Dom}(w) = \mathrm{Dom}(D(w))$ as defined in Section 2.3. Recall a permutation v is dominant if $D(v) = D_\lambda$ for some partition λ, or equivalently if $D(v) = D_{\mathrm{Dom}(v)}$. Moreover, the map $v \mapsto c(v)$ is a bijection from dominant permutations in S_n to partitions $\lambda \subseteq \delta_n$.

Lemma 3.21. *Let v be a dominant. Then $\mathrm{PD}(v) = \{D(v)\}$.*

Proof. By Lemma 3.20, we see $D(v) \in \mathrm{PD}(v)$. Now assume $P \in \mathrm{PD}(v)$ is a diagram with $D(v) \not\subseteq P$. Let (i, j) be the $\leq_{\mathrm{lex}}$–minimal element of $P \setminus D(v)$. Then the permutation of P maps j to i:

(3.5) 

but $v(j) > i$, a contradiction. Therefore $D(v) \subseteq \mathrm{Dom}(P)$. Since $|P| = |D(v)|$, we then have $P = D(v)$. □

Corollary 3.22. *Let $v, w \in S_\infty$ with v dominant. Then $v \le w$ if and only if $D(v) \subseteq P$ for all $P \in \mathrm{PD}(w)$.*

Proof. By subword containment, $v \le w$ if and only if $P = \binom{\mathbf{i}}{\mathbf{a}}$ has a subword $\binom{\mathbf{i}'}{\mathbf{a}'}$ corresponding to $P' \in \mathrm{PD}(v)$. In this case, Lemma 3.21 says $\mathrm{PD}(v) = \{D(v)\}$, so $P' = D(v) \subseteq P$. □

This says every pipe dream for $w \in S_n$ has the same dominant component. We now show this dominant component is $\mathrm{Dom}(w)$.

An *outer corner* of the partition λ is (i, j) where $\lambda + \varepsilon_i$ is a partition and $j = \lambda_i + 1$. Therefore $D_\lambda \cup \{(i, j)\} = D_{\lambda+\varepsilon_i}$. Note if $\lambda = \mathrm{Dom}(w)$, then $c_i(w) = \lambda_i$.

Proposition 3.23. *For $w \in S_n$ and $P \in \mathrm{PD}(w)$, $\mathrm{Dom}(w) = \mathrm{Dom}(P)$.*

Proof. By Corollary 3.22, the result is equivalent to showing $\mathrm{Dom}(w) = \mathrm{Dom}(P_{bot}(w))$. Note $\mathrm{Dom}(w) \subseteq c(w)$ by construction. By the same reasoning as in the proof of Lemma 3.21, if (i, j) is an outer corner for $\mathrm{Dom}(D(w))$, then it is also an outer corner of $P_{bot}(w)$. Thus $\mathrm{Dom}(P_{bot}(w)) = \mathrm{Dom}(w)$. □

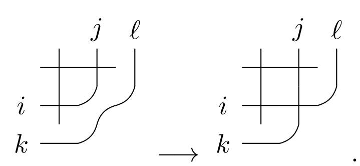
We are now ready to prove that pipe dreams satisfy a cotransition–type recurrence. For $v \in S_\infty$, recall from Section 2.3 that

$$\Psi(v, r) = \{v t_{rs} : v \lessdot t_{rs}, r < s\}.$$

Theorem 3.24. *Let $v \in S_\infty$ and (i, j) be an outer corner of $\mathrm{Dom}(v)$. The following map is a bijection:*

$$\begin{aligned} \mathrm{PD}(v) &\to \cup_{w \in \Psi(v,i)} \mathrm{PD}(w) \\ P &\mapsto P \cup \{(i, j)\} \end{aligned}$$

Proof. At the level of wiring diagrams, our map sends



Here, the locations of k and ℓ are not necessarily to scale, but $i < k$ and $j < \ell$. Therefore, $P \cup \{(i,j)\}$ has the permutation vt_{ik}. Then $v < vt_{j\ell}$ by Corollary 1.22, so $\ell(vt_{ik}) \geq \ell(v) + 1$. Moreover,

$$|P \cup \{(i,j)\}| = \ell(w) + 1,$$

so $\ell(vt_{ik}) = \ell(v) + 1$ and $v \lessdot vt_{ik}$. This shows our map is well-defined.

To the see the map is invertible, let $w = vt_{ik} \in \Psi(v,i)$ and $\lambda = \mathrm{Dom}(v)$. Note $c_i(w) = c_i(v) + 1$. Let u be the dominant permutation with $c(u) = \lambda$. Then $u \leq v \leq w$ so by Corollary 3.22 $\lambda \subseteq \mathrm{Dom}(w)$. Since $c_i(w) = c_i(v) + 1$, we see $\lambda + \varepsilon_i \subseteq \mathrm{Dom}(P_{bot}(w))$, which equals $\mathrm{Dom}(w)$ by Proposition 3.23. For $P \in \mathrm{PD}(w)$, we then have $(i,j) \in P$. By examining wiring diagrams, the permutation v' of $P \setminus \{(i,j)\}$ must map j to i. Since $w(j) = k$, we have $v' = v$ and the result follows. □

For P a diagram and $\mathbf{i} = (i_1, \ldots, i_p)$ a word, define

$$x^P = \prod_{(i,j) \in P} x_i \quad \text{and} \quad x^{\mathbf{i}} = x_{i_1} \ldots x_{i_p}.$$

Theorem 3.25. *For $v \in S_\infty$,*

$$\mathfrak{S}_v = \sum_{P \in \mathrm{PD}(v)} x^P = \sum_{\boldsymbol{a} \in \mathrm{Red}(v)} \sum_{\boldsymbol{i} \in K(\boldsymbol{a})} x^{\boldsymbol{i}}.$$

Proof. Assume $v \in S_n$. We argue by induction on $\binom{n}{2} - \ell(v)$. Our base case is $v = w_0$, where the result holds by Lemma 3.21 and Theorem 2.12. Otherwise, let $\lambda = \mathrm{Dom}(v)$ and note $\lambda \subset \delta_n$ without equality. There then exists an outer corner (i,j) of λ with $(i,j) \in D_{\delta_n}$. By Proposition 2.28 and the induction hypothersis, we compute

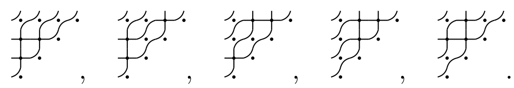
$$\begin{aligned} x_i \mathfrak{S}_v &= \sum_{w \in \Psi(v,i)} \mathfrak{S}_w \\ &= \sum_{w \in \Psi(v,i)} \sum_{P \in \mathrm{PD}(w)} x^P. \end{aligned}$$

Applying the inverse map from Theorem 3.24, we see

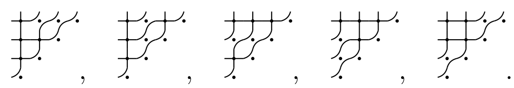
$$\frac{\sum_{w \in \Psi(v,i)} \sum_{P \in \mathrm{PD}(w)} x^P}{x_i} = \sum_{P \in \mathrm{PD}(v)} x^P$$

and the result follows. □

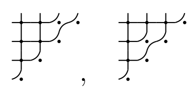
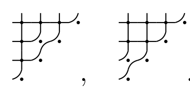
Example 3.26. For $w = 1432$, note $\mathrm{Dom}(w) = ()$ so its sole outer corner is $(1,1)$. The five pipe dreams of $\mathrm{PD}(w)$ are:



Adding the cell $(1,1)$ to each of these, we obtain



The first and second of these comprise $\mathrm{PD}(4132)$, the third and fourth comprise $\mathrm{PD}(2431)$ and the last is the sole element in $\mathrm{PD}(3412)$. Since $\Psi(w,1) = \{2431, 3412, 4132\}$, this is consistent with Theorem 3.24. Note 3412 is dominant, so $\mathfrak{S}_{3412} = x_1^2x_2^2$. Meanwhile, $\mathrm{Dom}(4132)$ has the outer corner $(1,2)$ and $\mathrm{Dom}(2431)$ has the outer corner $(2,1)$. Adding these to the appropriate pipe dreams, we get

 and 

These are pipe dreams for the dominant permutations 4312, 4231 (twice!) and 3421. The proof outline of Theorem 3.25 in this case gives the computation

$$\begin{aligned}\mathfrak{S}_w &= \frac{\frac{x_1^2x_2^2x_3 + x_1^3x_2x_3}{x_1} + \frac{x_1^3x_2x_3 + x_1^3x_2^2}{x_2} + x_1^2x_2^2}{x_1} \\ &= x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2\end{aligned}$$

as expected.

3.4. Pipe dreams and ladder moves

Our proof of Theorem 3.10 included an efficient method of generating the set of bumpless pipe dreams for a fixed permutation. In contrast, our proof of Theorem 3.25 does not offer an easy path to generating the set of pipe dreams for a permutation. In this section, we introduce local moves that allow one to generate the set of all pipe dreams.

Definition 3.27. A *chute move* is the transformation on diagrams

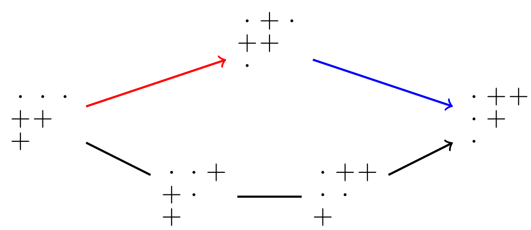
$$\begin{array}{cccccc}\cdot & + & \cdot & \cdot & + & \cdot \\ + & + & \cdot & \cdot & + & \cdot\end{array} \longrightarrow \begin{array}{cccccc}\cdot & + & \cdot & \cdot & + & + \\ \cdot & + & \cdot & \cdot & + & \cdot\end{array}.$$

Similarly, a *ladder move* is the transpose of a chute move. A move that is both a chute move and a ladder move is called *simple*. These moves can all be inverted.

The following follows by examining of wiring diagrams.

Lemma 3.28. *Let P, P' be two pipe dreams differing by a chute move (equivalently, a ladder move). Then the permutations of P and P' are the same.*

We depict the chute and ladder moves on PD(1432) below:



Here, the red edge is a ladder move, the blue edge is a chute move and the other edges are simple moves.

Theorem 3.29. *For $w \in S_\infty$,*

(1) *the directed graph on* $\mathrm{PD}(w)$ *given by ladder moves is connected with unique source $D_{c(w)}$.*
(2) *the directed graph on* $\mathrm{PD}(w)$ *given by chute moves is connected with unique sink $D^T_{c(w^{-1})}$.*
(3) *the directed graph on* $\mathrm{PD}(w)$ *given by both chute and ladder moves is connected with unique source $D_{c(w)}$ and sink $D^T_{c(w^{-1})}$.*

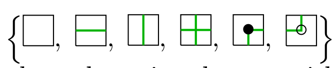
Proof. We first prove (a). It is easy to check that $D_{c(w)}$ has no available inverse ladder moves, so it is a sink in the graph on $\mathrm{PD}(w)$. Now let $P \in \mathrm{PD}(w)$ so that $P \neq D_{c(w)}$. Note $P \neq D_{c(v)}$ for any $v \in S_\infty$, so there must exist a row in P with a $+$ after a $\cdot$. Let i be the lowest row in P with a $+$ after a $\cdot$, and let j be a column so that $P_{ij} = \cdot$ and $P_{i\ j+1} = +$. Since P is reduced, we cannot have $P_{i+1\ j} = P_{i+1\ j+1} = \cdot$, and the maximality of i guarantees we do not have

$P_{i+1\ j} = \cdot, P_{i+1\ j+1} = +$. Therefore, either $P_{i+1\ j} = \cdot, P_{i+1\ j+1} = \cdot$ and we have an inverse ladder move or $P_{i+1\ j} = +, P_{i+1\ j+1} = +$, in which case the same applies to row $i+2$ and so on. Since P is finite, there exists minimal $k \geq 1$ so that $P_{i+k\ j} = \cdot, P_{i+k\ j+1} = \cdot$, giving an available inverse ladder move. Since our directed graph has a unique source, it must be connected.

For (b), use (a) for $\mathrm{PD}(w^{-1})$ and take the transpose. Then (c) follows from (a) and (b). □

3.5. Exercise

3.5.1. Justify the values in Table (3.2).

3.5.2. For each pair in  not covered by Lemma 3.6, find distinct bumpless pipe dreams with the same locations for these values or prove that one can reconstruct a bumpless pipe dream using this information.

3.5.3. Prove for all $w \in S_\infty$ that $D_{c(w)} \in \mathrm{PD}(w)$.

Bibliography

Index