MAT 6932 — Seminar in Number Theory — Homework Problems

Problems (41a, 41b, 49, 52, 53g, and 55) for the final due 9:35 a.m. April 23 are this color.

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1. Prove that $E^p(q) \equiv E(q^p) \pmod{p}$ for any prime p, where $E(q) := (q;q)_\infty$ and $E^n(q) := (E(q))^n$.

   Is it possible to replace p with p^2 or with p^3 and still have a valid statement?

2. Use Euler's Pentagonal Number Theorem (EPNT) to show that

$$p(n) = \sum_{0<\omega_j\le n} (-1)^{j+1} p(n-\omega_j),$$

   where $\omega_j := \dfrac{j(3j-1)}{2}$ and $p(n)$ is the number of partitions of n.

3. Use the fact that

$$E^3(q) = \sum_{n\ge 0} (2n+1)(-1)^n q^{\binom{n+1}{2}}$$

   to prove that

$$E^3(q) \equiv J_0(q^5) + qJ_1(q^5) \pmod{5},$$

   where $J_0(q)$ and $J_1(q)$ are q-series with integer coefficients.

4. Use

$$\frac{1}{E(q)} = \frac{E^6(q)}{E^7(q)} \equiv \frac{(E^3(q))^2}{E(q^7)} \pmod{7}$$

   to prove that $7 \mid p(7n+5)$.

5. Use a combinatorial argument to show that

$$\frac{(zt)_\infty}{(z)_\infty} = \sum_{n\ge 0} \frac{(t)_n}{(q)_n} z^n.$$

6. Show that

$$E(q) \equiv K_0(q^5) + qK_1(q^5) + q^2K_2(q^5) \pmod{5},$$

   where $K_0(q)$, $K_1(q)$, and $K_2(q)$ are q-series with integer coefficients. Then, use this result to prove that

$$J_0^2(q) \equiv E(q)K_0(q) \pmod{5}.$$

7. Show that

$$\begin{bmatrix} L+1 \\ n \end{bmatrix}_q = \begin{bmatrix} L \\ n \end{bmatrix}_q q^n + \begin{bmatrix} L \\ n-1 \end{bmatrix}_q = \begin{bmatrix} L \\ n \end{bmatrix}_q + \begin{bmatrix} L \\ n-1 \end{bmatrix}_q q^{L+1-n},$$

and then use these recurrence relations to prove that, for $n, m \in \mathbb{Z}_{\geq 0}$, $\begin{bmatrix} n+m \\ n \end{bmatrix}_q$ is the generating function for partitions into parts such that the largest part is $\leq n$ and the number of parts is $\leq m$.

8. Problem 7 implies that

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \frac{(q^{m+1})_n}{(q)_n}$$

is a polynomial in q. Use this definition to show that this polynomial has degree nm.

9. Prove that

$$\begin{bmatrix} -L \\ n \end{bmatrix}_q = \frac{(q^L)_n}{(q)_n}(-1)^n q^{-\binom{n}{2}-Ln} = \begin{bmatrix} L-1+n \\ n \end{bmatrix}_q (-1)^n q^{-\binom{n}{2}-Ln}.$$

10. Use the q-Chu-Vandermonde summation identity (for $N \in \mathbb{Z}_{\geq 0}$)

$${}_2\phi_1\left[\begin{matrix} a, q^{-N} \\ c \end{matrix}; q, \frac{cq^N}{a}\right] = \frac{(c/a)_N}{(c)_N}$$

to show that, for $j \in \mathbb{Z}$ and $a \in \{0,1\}$,

$$\sum_{r=0}^{L} \frac{q^{r^2+ar}(q)_{2L+a}}{(q)_{L-r}(q)_{2r+a}} \begin{bmatrix} 2r+a \\ r-j \end{bmatrix}_q = q^{j^2+aj} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q.$$

11. Use the q-Chu-Vandermonde summation identity from problem 10 to show that

$${}_2\phi_1\left[\begin{matrix} a, q^{-N} \\ c \end{matrix}; q, q\right] = \frac{(c/a)_N}{(c)_N} a^N.$$

(Hint: the formula only involves a finite sum.)

12. Use the q-binomial theorem to evaluate ${}_1\Psi_1\left[\begin{smallmatrix} a \\ b \end{smallmatrix}; q, z\right]$ for $b = q^N$, $N \in \mathbb{Z}_{>0}$.

13. a. Use the q-Gauss summation identity,

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right] = \frac{\left(\frac{c}{a}, \frac{c}{b}\right)_\infty}{\left(c, \frac{c}{ab}\right)_\infty}, \qquad |q| < 1, \quad \left|\frac{c}{ab}\right| < 1,$$

to prove that

$$\sum_{n \geq 0} \frac{q^{2n^2-n}(-q;q^4)_n}{(q^2;q^2)_{2n}} = \frac{(-q;q^4)_\infty}{(q^2;q^4)_\infty}.$$

(Hint:

$$\frac{(\rho)_k}{\rho^k} \to q^{\binom{k}{2}}(-1)^k \quad \text{as} \quad \rho \to \infty.)$$

b. Find a partition theoretic interpretation of the summand

$$\frac{q^{2n^2-n}(-q;q^4)_n}{(q^2;q^2)_{2n}}.$$

14. Use the q-Pfaff-Saalschütz summation identity,

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-N} \\ c, \frac{ab}{c}q^{1-N} \end{matrix}; q, q\right] = \frac{\left(\frac{c}{a}, \frac{c}{b}\right)_N}{\left(c, \frac{c}{ab}\right)_N}$$

to prove that

$$\sum_{r\geq 0} \begin{bmatrix} M-m \\ r \end{bmatrix}_q \begin{bmatrix} N+m \\ m+r \end{bmatrix}_q \begin{bmatrix} m+n+r \\ M+N \end{bmatrix}_q q^{(N-r)(M-m-r)} = \begin{bmatrix} m+n \\ M \end{bmatrix}_q \begin{bmatrix} n \\ N \end{bmatrix}_q.$$

15. In class, we saw how we could use a "seed" of the form (for $a = 0, 1$)

$$B_a(r) = \sum_{j=-r-a}^{r} A(j) \begin{bmatrix} 2r+a \\ r-j \end{bmatrix}_q,$$

apply some "water" in the form of the identity

$$\sum_{r=0}^{L} \frac{q^{r^2+ar}(q)_{2L+a}}{(q)_{L-r}(q)_{2r+a}} \begin{bmatrix} 2r+a \\ r-j \end{bmatrix}_q = q^{j^2+aj} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q$$

and get "growth" in the form of

$$\tilde{B}_a(L) = \sum_{j=-L-a}^{L} \tilde{A}(j) \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q,$$

where

$$\tilde{A}(j) = A(j)q^{j^2+aj}$$

and

$$\tilde{B}_a(L) = \sum_{r=0}^{L} \frac{q^{r^2+ar}(q)_{2L+a}}{(q)_{L-r}(q)_{2r+a}} B_a(r).$$

When the "water" was applied to the "seed"

$$(q)_a \delta_{L,0} = \sum_{j=-L-a}^{L} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q,$$

we were able to conclude that

$$\frac{(q)_{2L+a}}{(q)_L} = \sum_{j=-L-a}^{L} (-1)^j q^{\frac{3j^2-j}{2}+aj} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q,$$

which we saw (by letting $L \to \infty$) is a finitized version of the EPNT. Now, apply the "water" two more times to obtain finitized versions of the Rogers-Ramanujan (modulo 5) and Andrews-Gordon (modulo 7) identities.

16. a. Show that for $j \geq 0$, $a \in \{0,1\}$, $n \geq 0$, and $(a,n) \neq (0,0)$,

$$\sum_{r=0}^{n} \frac{1-q^{2n+a}}{1-q^{n+r+a}} \begin{bmatrix} n+r+a \\ 2r+a \end{bmatrix}_q \begin{bmatrix} 2r+a \\ r-j \end{bmatrix}_q (-1)^{n+r} q^{\binom{n-r}{2}} = \delta_{n,j}.$$

b. Show that if

$$B_a(r) = \sum_{j \geq 0} A(j) \begin{bmatrix} 2r+a \\ r-j \end{bmatrix}_q$$

then

$$\sum_{r=0}^{n} \frac{1-q^{2n+a}}{1-q^{n+r+a}} \begin{bmatrix} n+r+a \\ 2r+a \end{bmatrix}_q (-1)^{n+r} q^{\binom{n-r}{2}} B_a(r) = A(n),$$

provided $(a,n) \neq (0,0)$.

c. Determine the $A(j)$'s if $B_0(r) = (-q;q^2)_r$. (Hint: there are two cases to consider.)

17. Use the Andrews-Gordon identities to prove that $P_{v,s+1} \succcurlyeq P_{v,s}$, for $v \in \mathbb{Z}_{>0}$ and $1 \leq s \leq v$, where

$$P_{v,s} := \prod_{\substack{j>0 \\ j \not\equiv 0, \pm s \\ (\mathrm{mod}\ 2v+3)}} \frac{1}{1-q^j}.$$

(We say that $\sum a_n q^n \succcurlyeq \sum b_n q^n$ iff $a_n - b_n \geq 0$ for all $n \geq 0$.)

18. We proved

$$\sum_{j=-L-a}^{L} (-1)^j q^{j^2} \begin{bmatrix} 2L+a \\ L-j \end{bmatrix}_q = (q;q^2)_{L+a}$$

when $a = 0$ in class; prove the formula when $a = 1$.

19. a. Prove that (for $v \in \mathbb{Z}_{>0}$)

$$(q)_{2L} q^L \sum_{n_1,\ldots,n_v \geq 0} \frac{q^{N_1^2+\cdots+N_v^2+N_1+\cdots+N_v}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_v}} = \sum_{j=-L}^{L} (-1)^j q^{(2v+3)\binom{j}{2}} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q,$$

where

$$N_i := \sum_{j=i}^{v} n_j.$$

b. Use the identity from 19a to prove all $v+1$ Andrews-Gordon identities (modulo $2v+3$).

20. Use the Bressoud identities to prove that $B_{v,s+1} \succcurlyeq B_{v,s}$, for $v \in \mathbb{Z}_{>0}$ and $1 \leq s < v$, where

$$B_{v,s} := \prod_{\substack{j>0 \\ j \not\equiv 0, \pm s \\ (\mathrm{mod}\ 2v+2)}} \frac{1}{1-q^j}.$$

What is the relationship between the identity in problem 18 and the Bressoud identities?

21. a. Prove that (for $v \in \mathbb{Z}_{>0}$)

$$q^L \sum_{n_1,\dots,n_v \geq 0} \frac{(q)_{2L} q^{N_2^2+\cdots+N_v^2+N_1+\cdots+N_v}}{(q)_{L-N_1}(q)_{n_1}\cdots(q)_{n_{v-1}}(q^2;q^2)_{n_v}} = \sum_{j=-L}^{L} (-1)^j q^{(2v+2)\binom{j}{2}} \begin{bmatrix} 2L \\ L-j \end{bmatrix}_q,$$

where

$$N_i := \sum_{j=i}^{v} n_j.$$

b. Use the identity from 21a to prove all $v+1$ Bressoud identities (modulo $2v+2$).

22. Apply the Berkovich-Garvan injection to the partition $\langle 2_1^2, 2_3^3, 7_3, 7_{10}^2 \rangle$ to get its image. How does this compare with the example on page 32 of the presentation notes from Keith Grizzell?

23. Use the anti-telescoping technique presented by Mitchell Harris, i.e.

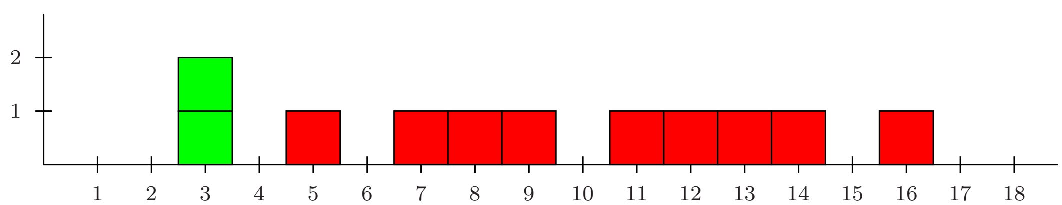
$$\frac{1}{P(L)} - \frac{1}{Q(L)} = \sum_{i=1}^{L} \frac{\frac{Q(i)}{Q(i-1)} - \frac{P(i)}{P(i-1)}}{P(i) \cdot \frac{Q(L)}{Q(i-1)}},$$

where $P(0) = Q(0) = 1$, to show that

$$\frac{1}{(q,q^2,q^5,q^6;q^7)_L} \succcurlyeq \frac{1}{(q,q^3,q^4,q^6;q^7)_L} \succcurlyeq \frac{1}{(q^2,q^3,q^4,q^5;q^7)_L}.$$

24. Determine n_1, n_2, $(\lambda_1,\dots)$, and $(\Lambda_1,\dots)$ for the path space corresponding to frequencies

$$f_i = \begin{cases} 2 & \text{if } i = 3, \\ 1 & \text{if } i \in \{5,7,8,9,11,12,13,14,16\}, \\ 0 & \text{otherwise.} \end{cases}$$



25. Verify the following theorem for $m = 6$, $s = 1$, and $1 \leq n \leq 10$. (If you're ambitious, feel free to verify it when $s = 2$ and $s = 3$ as well!)

> The number of partitions of n into parts incongruent with $-s$, 0, and s (modulo m), where $0 < s \leq m/2$, is equal to the number of partitions of n whose successive ranks lie in the interval $[2-s, m-s-2]$.

(The set of successive ranks of a partition with Fröbenius symbol $\begin{pmatrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{pmatrix}$ is given by $\{a_i - b_i \mid 1 \leq i \leq r\}$.)

26. Use

$$\sum_{N_1+N_2+N_3=3} q^{N_1^2+N_2^2+N_3^2} \begin{bmatrix} \hat{L}+n_1-2N_1 \\ n_1 \end{bmatrix}_q \begin{bmatrix} 2\hat{L}+n_2-2(N_1+N_2) \\ n_2 \end{bmatrix}_q \begin{bmatrix} 3\hat{L}+n_3-2(N_1+N_2+N_3) \\ n_3 \end{bmatrix}_q = q^3 \begin{bmatrix} 3+L-1 \\ 3 \end{bmatrix}_q$$

to show that, for $L \geq 3$, $\begin{bmatrix} L \\ 3 \end{bmatrix}_q$ is a unimodal, palindromic polynomial of degree $3(L-3)$. (Here, $\hat{L} := L+1$ and $N_i := \sum_{j=i}^{3} n_j$ for $1 \leq i \leq 3$.)

27. a. Use the binomial recurrences given in problem 7 to prove the following finitizations of EPNT and the Rogers-Ramanujan identities due to Issai Schur.

i.

$$1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j^2+j}{2}} \begin{bmatrix} L \\ \lfloor \frac{L-3j}{2} \rfloor \end{bmatrix}_q$$

ii.

$$\sum_{n=0}^{\infty} q^{n^2} \begin{bmatrix} L-n \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{5j^2+j}{2}} \begin{bmatrix} L \\ \lfloor \frac{L-5j}{2} \rfloor \end{bmatrix}_q$$

iii.

$$\sum_{n=0}^{\infty} q^{n^2+n} \begin{bmatrix} L-n-1 \\ n \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{5j^2+3j}{2}} \begin{bmatrix} L \\ \lfloor \frac{L-5j-1}{2} \rfloor \end{bmatrix}_q$$

b. Apply "water" (see problem 15) to each of the finitizations in part a and see what happens.

c. For each of the right-hand sides of i–iii in part a, find values for K, i, n, m, α, and β to interpret the right-hand side as $G_{K,i}(n, m, \alpha, \beta)$, where

$$G_{K,i}(n,m,\alpha,\beta) = \sum_{j=-\infty}^{\infty} \left\{ q^{j(Kj+i)(\alpha+\beta)-K\beta j} \begin{bmatrix} n+m \\ n-Kj \end{bmatrix}_q - q^{j(Kj-i)(\alpha+\beta)-K\beta j+\beta i} \begin{bmatrix} n+m \\ n-Kj+i \end{bmatrix}_q \right\}.$$

28. We make the following definitions.

- An RR partition is a partition in which the difference between parts is at least 2.
- A chain of an RR partition is a $\subseteq$-maximal subpartition with parts differing by exactly 2.
- A chain of an RR partition is even (resp. odd) iff it has all even (resp. odd) parts.
- $\mathbb{R}_i(n) :=$ the set of all RR partitions of n with smallest part $\geq i$.
- $\mathbb{R}_i := \bigcup_{n \in \mathbb{N}} \mathbb{R}_i(n)$.
- $A_{k,i}(n) :=$ the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{k}$.
- $\omega_1(\pi) := 2^{k(\pi)}$, where $\pi \in \mathbb{R}_2$ and $k(\pi)$ is the number of even chains (of π) in $\mathbb{R}_4$.

- $\omega_2(\pi) := 2^{k(\pi)}$, where $\pi \in \mathbb{R}_1$ and $k(\pi)$ is the number of odd chains (of π) in $\mathbb{R}_3$.
- $\omega_3(\pi) := 2^{k(\pi)}$, where $\pi \in \mathbb{R}_1$ and $k(\pi)$ is the number of even chains (of π) in $\mathbb{R}_2$.

Prove the following:

a.
$$\sum_{\pi \in \mathbb{R}_2(n)} \omega_1(\pi) = A_{6,1}(n).$$

b.
$$\sum_{\pi \in \mathbb{R}_1(n)} \omega_2(\pi) = A_{6,2}(n).$$

c.
$$\sum_{\pi \in \mathbb{R}_1(n)} \omega_3(\pi) = A_{6,3}(n).$$

29. Taking the definitions from problem 28, we add the following new definitions.

- A string of an RR partition is a $\subseteq$-maximal subpartition with parts differing by ≤ 3.
- $\eta(\psi) :=$ the number of times the difference is 3 between successive parts in the string ψ.
- $\omega_5(\psi) := \begin{cases} F_{\eta(\psi)+3} & \text{if } \psi \in \mathbb{R}_2, \\ F_{\eta(\psi)+2} & \text{otherwise,} \end{cases}$ where ψ is a string and F_k represents the kth Fibonacci number ($F_1 = F_2 = 1$ and $F_j = F_{j-1} + F_{j-2}$ for all $j > 2$).
- $\omega_5(\pi) := \prod_{i=1}^{r} \omega_5(\psi_i)$, where $\pi \in \mathbb{R}_1$; $\psi_1, \ldots, \psi_r$ are distinct strings (of π); and $\pi = \bigcup_{i=1}^{r} \psi_i$.

Prove that
$$\sum_{\pi \in \mathbb{R}_1(n)} \omega_5(\pi) = A_{7,3}(n).$$

30. Let Q_m^L be the generating function for partitions with rank at least m and largest part no more than L. Prove the following.

a. If $L > m$, then
$$Q_m^L + Q_{1-m}^{L-m} + 1 = \begin{bmatrix} 2L - m \\ L \end{bmatrix}_q.$$

b. If $L > m \geq 0$, then
$$Q_m^L = q^{m+1} \left(Q_{-2-m}^{L-1-m} + 1 \right).$$

c. 30a and 30b together imply 27(a)i.

31. Use Heine's transformation to prove that, for $m > 0$,

$$(1-q)\sum_{k\geq 0}\frac{q^{(k+1)(k+m)+k}}{(q)_k(q)_{k+m}} = \sum_{j\geq 1}(-1)^{j-1}q^{T_{j-1}+mj}\frac{1-q^j}{(q)_\infty}.$$

32. We define a pair of sequences $(\alpha_L(a,q), \beta_L(a,q))$ to be a Bailey pair relative to a iff

$$\beta_L(a,q) = \sum_{r=0}^{L}\frac{\alpha_r(a,q)}{(q)_{L-r}(aq)_{L+r}}.$$

Let

$$\tilde{\beta}_L(a,q) := \frac{\left(\frac{aq}{\rho_1\rho_2}\right)_L}{\left(q, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}\right)_L}\sum_{r=0}^{L}\frac{(\rho_1, \rho_2, q^{-L})_r q^r}{\left(\frac{\rho_1\rho_2}{aq^L}\right)_r}\beta_r(a,q)$$

and

$$\tilde{\alpha}_L(a,q) := \frac{(\rho_1,\rho_2)_L}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}\right)_L}\left(\frac{aq}{\rho_1\rho_2}\right)^L\alpha_L(a,q).$$

Then Bailey's Lemma says that if $(\alpha_L(a,q), \beta_L(a,q))$ is a Bailey pair, then so must $(\tilde{\alpha}_L(a,q), \tilde{\beta}_L(a,q))$ be.

a. Take $\rho_1 \to \infty$ and $\rho_2 \to \infty$ in Bailey's Lemma and compare with problem 15.

b. Take $\rho_1 \to \infty$ and $\rho_2 = -\sqrt{aq}$ in Bailey's Lemma and determine the result.

c. Starting with a Bailey pair relative to 1 (which is associated with the penultimate identity in problem 15 with $a = 0$), use Bailey's Lemma $v \geq 1$ times with $\rho_1 \to \infty$ and $\rho_2 \to \infty$, and then use Bailey's Lemma one more time with $\rho_1 \to \infty$ and $\rho_2 = -\sqrt{q}$; determine the result.

33. Ramanujan showed that

$${}_1\Psi_1\left[\begin{matrix}a\\b\end{matrix}; q, z\right] = \frac{\left(q, \frac{b}{a}, az, \frac{q}{az}\right)_\infty}{\left(z, \frac{b}{az}, b, \frac{q}{a}\right)_\infty}$$

for $\left|\frac{b}{a}\right| < |z| < 1$, $|q| < 1$. Use this identity to prove that

$$\sum_{n=-\infty}^{\infty}\frac{x^n}{1-yq^n} = \frac{\left(xy, \frac{q}{xy}, q, q\right)_\infty}{\left(y, \frac{q}{y}, x, \frac{q}{x}\right)_\infty}$$

for $|q| < |x| < 1$.

34. Use the prime factorization of n to find a formula, in terms of those primes and their multiplicities, for the number of ways to represent $n \geq 1$ as each of the following:

a. the sum of a square plus two times a square,

$$r_{\square+2\square}(n) = 2 \sum_{0<d|n} \left(\frac{-2}{d}\right),$$

b. the sum of a square plus three times a square,

$$r_{\square+3\square}(n) = 4\left(d_{4,12}(n) - d_{8,12}(n)\right) + 2\left(d_{1,3}(n) - d_{2,3}(n)\right),$$

where $d_{i,j}(n)$ is the number of divisors of n congruent to $i \pmod{j}$.

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35. Prove that

$$\sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n} = \psi(q^2)E^2(q),$$

where $\psi(q) := \sum_{n\geq 0} q^{n(n+1)/2} = \dfrac{E^2(q^2)}{E(q)}$.

36. Prove that the discriminants $\Delta = -3$, $\Delta = -4$, $\Delta = -8$, and $\Delta = -12$ each admit exactly one class.

37. Determine the class group for the discriminant $\Delta = -23$.

38. Let $\mathrm{CL}(\Delta)$ denote the class group for the discriminant Δ, let $\mathrm{cl}(\Delta) := |\,\mathrm{CL}(\Delta)|$, let $\mathrm{SQ}(\Delta) := \{f^2 \mid f \in \mathrm{CL}(\Delta)\}$, and let $\mathrm{sq}(\Delta) := |\,\mathrm{SQ}(\Delta)|$.

a. Show that $\mathrm{sq}(\Delta)$ divides $\mathrm{cl}(\Delta)$.

b. Is there an odd prime p such that p divides $\dfrac{\mathrm{cl}(\Delta)}{\mathrm{sq}(\Delta)}$?

39. Show that

$$\frac{1}{2}\phi(q)\phi(q^5) = \sum_{n=-\infty}^{\infty} \left(\frac{q^{3n}}{1+q^{10n}} + \frac{q^{5n+1}}{1+q^{10n+2}}\right),$$

where $\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$.

40. Use the fact that

$$\phi(-q)\phi(-q^3) = 1 + 2\sum_{n=1}^{\infty} \frac{(-q)^n + (-q^2)^n}{1+q^{3n}}$$

to prove that

$$\phi(q)\phi(q^3) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n} + 4\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{4n}}{1-q^{4n}}.$$

Note that $\left(\frac{\bullet}{\bullet}\right)$ denotes the Jacobi symbol.

41. Define the r modulo m projection operator, $\mathrm{P}_{m,r}$, by

$$\mathrm{P}_{m,r}\left(\sum_{k=-\infty}^{\infty} a_k q^k\right) := \sum_{i=-\infty}^{\infty} a_{mi+r} q^{mi+r},$$

and note that

$$\mathrm{P}_{m,r}\left(\sum_{n\geq 0} a_n q^n\right) = \sum_{k\geq 0} a_{km+r} q^{km+r}$$

for $0 \leq r < m$.

a. Show that

$$3\sum_{x,y} q^{x^2+xy+7y^2} = 3\sum_{x,y} q^{9(x^2+xy+y^2)} + \mathrm{P}_{3,1}\left(\sum_{x,y} q^{x^2+xy+y^2}\right).$$

b. Determine $(1,1,7;n)$.

c. Prove that

$$3\sum_{x,y} q^{x^2+xy+7y^2} = \sum_{x,y} q^{x^2+xy+y^2} - \sum_{x,y} q^{3(x^2+xy+y^2)} + 3\sum_{x,y} q^{9(x^2+xy+y^2)}.$$

42. For each genus of $\Delta = -119$, find a set of rational transformations that relate the elements in the genus to each other.

43. Determine the associated characters for each form with discriminant $\Delta = -96$: $(1,0,24)$, $(3,0,8)$, $(4,4,7)$, $(5,2,5)$.

44. Suppose that $p \in \{7, 17\}$ (the prime factors from $\Delta = -119$); then the following statements are true.

I. $(1,1,30;pn) = (6,5,6;n)$ and $(6,5,6;pn) = (1,1,30;n)$.

II. $(2,1,15;pn) = (3,1,10;n)$ and $(3,1,10;pn) = (2,1,15;n)$.

III. $(4,3,8;pn) = (5,1,6;n)$ and $(5,1,6;pn) = (4,3,8;n)$.

Choose one of I, II, and III, and prove it.

45. Determine the number of representations by the principle genus of discriminant $\Delta = -231 = -3 \cdot 7 \cdot 11$ for any positive integer n. Your answer should be in terms of the prime factors of n.

46. Prove that every positive ternary form F with $H(F) = 16$ and $\min(F) = 3$ is equivalent to

$$3x^2 + 3y^2 + 3z^2 + 2yz + 2xz - 2xy.$$

(Recall that if $F = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy$, then $H(F) := \det \begin{pmatrix} a & t & s \\ t & b & r \\ s & r & c \end{pmatrix}$.)

47. Prove that every positive integer is representable either as the sum of three squares ($\square + \square + \square$) or as the sum of four squares with two of the four squares being identical ($\square + \square + 2 \cdot \square$).

48. Prove that $x^2 + 2y^2 + 4z^2$ represents all positive integers not of the form $4^k(16m + 14)$.

49. Prove that $3x^2 + 3y^2 + 3z^2 + 2yz + 2xz - 2xy$ is regular. (Recall that a ternary quadratic form is regular iff it represents all positive integers except for those in some arithmetic progressions.)

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50. Prove that the following diagonal quaternary forms are universal. (Recall that a form is universal iff it represents all positive integers.)

    a. $x^2 + y^2 + 2z^2 + du^2$, where $2 \leq d \leq 14$

    b. $x^2 + 2y^2 + 5z^2 + du^2$, where $6 \leq d \leq 10$

51.
    a. Prove that $x^2 + y^2 + 10z^2$ represents all even positive integers except those of the form $2 \cdot 4^k \cdot (8m + 3)$.

    b. Is it proven that the set of odd positive integers not represented by $x^2 + y^2 + 10z^2$ is finite?

    c. Prove that $x^2 + y^2 + 10z^2$ represents all positive integers congruent to $5 \pmod{10}$.

52. Show that if F is a ternary quadratic form with $H(F) = 5$, then $\min F = 1$. (See problem 46 for the definition of H.)

53. Assuming that the following forms are regular, determine for each form the sequences of integers it does not represent.

a. $(3, 6, 14, 4, 2, 2)$
b. $(1, 5, 13, 2, 1, 1)$
c. $(1, 6, 13, 3, 1, 0)$
d. $(2, 5, 11, 2, 2, 1)$
e. $(3, 5, 15, 3, 3, 3)$
f. $(1, 10, 29, 5, 1, 0)$
g. $(5, 8, 11, -4, 1, 2)$
h. $(5, 9, 15, 9, 3, 3)$
i. $(5, 9, 17, 6, 5, 3)$
j. $(2, 15, 32, 15, 1, 0)$
k. $(5, 9, 27, 0, 3, 3)$
l. $(5, 13, 33, -6, 3, 1)$
m. $(9, 11, 29, -4, 3, 6)$
n. $(11, 15, 39, -3, 6, 3)$

54. Prove that $x^2 + 2y^2 + 7z^2 + 13w^2$ represents all positive integers except 5.

55. Prove that $x^2 + 2y^2 + cz^2$ is irregular if c is a positive odd integer greater than 5. (Hint: use the following theorem.)

Theorem: Let a, b, and c be positive integers, not all even, such that no odd prime divides any pair of them. Then $ax^2 + by^2 + cz^2$ is irregular if there is a positive odd integer k such that

(i) $\gcd(k, abc) = 1$,
(ii) k is not represented by $ax^2 + by^2 + cz^2$, and
(iii) $ax^2 + by^2 + cz^2 \equiv k \pmod 8$ is solvable.

BONUS PROBLEM

B1. Prove bijectively that, for every positive integer N,

> the number of partitions of N into parts with the difference between parts ≥ 2 and no consecutive odd parts

equals

> the number of partitions of N into distinct parts such that all all even-indexed parts are even.

(We assume the parts are indexed by consecutive positive integers, beginning with 1, starting from the largest part on down to the smallest part.)

Some Useful Formulae

$$(a;q)_{-n} = \frac{1}{\left(\frac{a}{q^n};q\right)_n} \tag{U1}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{\left(\frac{q^{1-n}}{a};q\right)_k} \left(-\frac{q}{a}\right)^k q^{\frac{(k-1)k}{2}-nk} \tag{U2}$$

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n}{(b;q)_n} \cdot \frac{\left(\frac{q^{1-n}}{b};q\right)_k}{\left(\frac{q^{1-n}}{a};q\right)_k} \cdot \left(\frac{b}{a}\right)^k \tag{U3}$$

$$\left(\frac{a}{q^n};q\right)_k = \frac{(a;q)_k \left(\frac{q}{a};q\right)_n}{\left(\frac{q^{1-k}}{a};q\right)_n q^{nk}} \tag{U4}$$