MAS 4301, Abstract Algebra I

Notes of Krishna Alladi's Lecture on Mar 20, 2020, Friday

Chapter 6: Isomorphisms Cont'd

Problems:

We will discuss problem 44, p134 in a more general form as follows:

Lemma 1: Let G be an Abelian group. Then the mapping

$$\left.\begin{array}{l}\phi: G \to G \\ \quad x \to x^2\end{array}\right\} \phi(x) = x^2$$

is a homomorphism.

Proof: We need to show that ϕ is operation preserving. Since G is Abelian, we have for any pair $x, y \in G$,

$$\phi(xy) = (xy)^2 = xy \cdot xy = x \cdot x\, y \cdot y = x^2 y^2 = \phi(x)\phi(y).$$

Hence ϕ is a homomorphism.

Lemma 2: Let $\phi: G \to G'$ be a homomorphism of groups with e & e' the identities of G & G' respectively. Then ϕ is one-to-one if and only if $\phi(x) = e'$ has a unique solution, namely e ie $\phi(e) = e'$.

Proof: ($\Rightarrow$) If ϕ is one-to-one, then in particular only one value of x, namely e, can satisfy $\phi(x) = e$ $(x = e)$. So ϕ one-to-one implies $\phi(x) = e$ has a unique solution.

($\Leftarrow$) Conversely, let $\phi(x) = e$ have $x = e$ as the unique solution. To prove ϕ is one-to-one, consider any pair x, y for which $\phi(x) = \phi(y)$. Since ϕ is a homomorphism, we have

$$\phi(x) = \phi(y) \iff \phi(x)\phi(y)^{-1} = e' \Rightarrow \phi(x)\phi(y^{-1}) = e'$$

$$\Rightarrow \phi(xy^{-1}) = e' \Rightarrow xy^{-1} = e \text{ (by uniqueness)} \Rightarrow x = y.$$

Thus ϕ is one-to-one. This proves Lemma 2.

Remark: The importance/usefulness of Lemma 2 is that, if ϕ is a homomorphism, then in order to ~~verify~~ prove that ϕ is one-to-one, it is enough to verify that ϕ is one-to-one on the identity, that is $\phi(x) = e'$ has a unique solution.

Lemma 3: If G is Abelian and has no element of order 2, then the map $\phi: G \to G$ given by $\phi(x) = x^2$ is one-to-one.

Proof: We know by Lemma 1, that ϕ is a homomorphism. Next, we know by Lemma 2, that ϕ will be one-to-one, if $\phi(x) = e$ has a unique solution. Now in this case, $\phi(x) = e$ means $x^2 = e$. Clearly e is a solution of $x^2 = e$. If f is another solution satisfying $f^2 = e$, $f \neq e$, then the order of f will be 2. This is a contradiction to the hypothesis that G has no elements of order 2. Hence the only solution to $\phi(x) = e$ is $x = e$. This uniqueness proves that ϕ is one-to-one.

Theorem 1: (This is part 1 of Problem 44).

If G is a finite Abelian group and has no element of order 2, then $\phi: G \to G$ given by $\phi(x) = x^2$ is an automorphism.

Proof: We know that ϕ is a homomorphism (by Lemma 1). By Lemma 3, we know that ϕ is one-to-one. Since G is finite, any one-to-one map $\phi: G \to G$ is also onto. Hence ϕ is a bijection (one-to-one & onto). Hence ϕ is an isomorphism of G to itself. Thus ϕ is an automorphism.

Problem 44, part 2: Give an example of an infinite Abelian group G which has no elements of order 2, and which $\phi(x) = x^2$ is one-to-one but is not onto, hence not an automorphism.

Solution: Note that x^2 is multiplicative notation. In additive notation this is $x + x = 2x$. With this in mind, consider the group $G = \mathbb{Z}$ and the map

$$\left.\begin{array}{l} \phi: \mathbb{Z} \to \mathbb{Z}, \\ \quad x \to 2x \end{array}\right\} \quad \phi(x) = 2x, \ \forall\, x \in \mathbb{Z}.$$

Then ϕ is a homomorphism & is one-to-one. Except for $e = 0$, all elements of $\mathbb{Z}$ have infinite order. Note that ϕ is not onto. Hence ϕ is not an automorphism.

Problem 65, p. 136: Prove that every automorphism ϕ of $\mathbb{R}^{\times}$, the non-zero reals under multiplication, maps the positive reals to the positive reals and the negative reals to the negative reals.

Solution (Proof): Let us denote the positive reals and negative reals by $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively.

Consider any $x \in \mathbb{R}_+$. Since $x > 0$, we can write $x = y^2$, for some $y \in \mathbb{R}$. Thus

$$\phi(x) = \phi(y^2) = \phi(y \cdot y) = \phi(y)\phi(y) = \phi(y)^2 > 0$$

since $\phi(y) \in \mathbb{R}^{\times} = \mathbb{R} - \{0\}$. Hence $\phi: \mathbb{R}_+ \to \mathbb{R}_+$.

Next consider $-1 \in \mathbb{R}_-$. Suppose $\phi(-1) > 0$. Let $\phi(-1) = y > 0$. Then write $y = x^2$. Since ϕ^{-1} is also an automorphism, we have

$$-1 = \phi^{-1}(y) = \phi^{-1}(x^2) = \{\phi^{-1}(x)\}^2 > 0 \quad \text{— a contradiction.}$$

Thus $\phi(-1) < 0$. Finally, consider any $x \in \mathbb{R}_-$. Write $x = -1 \cdot y$, with $y \in \mathbb{R}_+$. Then

$$\phi(x) = \phi(-1 \cdot y) = \phi(-1)\phi(y) < 0, \text{ because } \phi(y) > 0 \text{ \& } \phi(-1) < 0.$$

Thus $\phi: \mathbb{R}_- \to \mathbb{R}_-$ and this completes the proof.

Problem 55 (p 135): Let $\phi: \mathbb{C}^{\times} \to \mathbb{C}^{\times}$ be an automorphism of the non-zero complex numbers $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ under multiplication. What are the possible values of $\phi(-1)$ and $\phi(i)$.

Solution: We know that -1 has order 2, and is the unique element of order 2. Thus $\phi(-1) = -1$.

Next note that ~~to~~ the order of i is 4. The set of all elements of order 4 is $\{i, -i\}$. Thus $\phi(i) = i$ or $-i$, since isomorphisms preserve orders.

Examples: $\phi(z) = z$, the identity map, is an automorphism. This maps i to i.

Next $\Phi(z) = \bar{z}$ is also an automorphism, mapping $a+ib$ to $a-ib$. This maps i to $-i$. Note that under both maps, -1 is mapped to -1.