MAS 4301, Abstract Algebra I

Notes of Krishna Alladi's Lecture on Mon, Mar 23, 2020

Chapter 7: Cosets & Lagrange's Theorem

We begin with

Definition of a Coset: Let G be a group and H a subgroup of G. Let $g \in G$. Then the left (resp. right) coset of H generated by g, and denoted by gH (resp. Hg) is defined as

$$gH = \{gh \mid h \in H\} \quad (\text{resp: } Hg = \{h.g \mid h \in H\}) \tag{1}$$

We note that (1) immediately gives

Lemma (i) (in book p 139).

Let G be a group and $H < G$. Let $g \in G$. Then

$$g \in gH \quad (\text{resp. } g \in Hg)$$

Proof: $g = ge$ and $e \in H$. $\therefore$ $ge \in gH$. Hence $g \in gH$. Similarly $g \in Hg$.

Terminology: In view of Lemma (i), the cosets gH and Hg are ~~etc~~ called the left (resp. right) coset (s) containing g.

Next we discuss various properties of cosets as given in the Lemma, p. 139, but we prove the parts of the Lemma in a different order.

Lemma part (V): Let G be a group and $H < G$. Let $a, b \in G$. Then either $aH \cap bH = \phi$, or $aH = bH$.

Proof: What we need to show is that

$$aH \cap bH \neq \phi \implies aH = bH. \tag{2}$$

So let us start with $aH \cap bH \neq \phi$. This means there is an element common to aH & bH. Thus there exists $h_1, h_2 \in H$ (not necessarily the same) such that

$$ah_1 = bh_2, \text{ with } h_1, h_2 \in H \tag{3}$$

Now (3) yields

$$ah_1 h_2^{-1} = b \quad , \text{ with } h_1 h_2^{-1} \in H, \text{ since } H \text{ is a subgroup} \tag{4}$$

Now consider any element bh of bH. Then (4) yields

$$bh = ah_1h_2^{-1}h \in aH, \text{ because } h_1h_2^{-1}h \in H \qquad (5)$$

So (5) yields $bH \subseteq aH$. Similarly instead of writing down (4) as a consequence of (3), we write

$$a = bh_2h_1^{-1}, \quad \text{with } h_2h_1^{-1} \in H,$$

then the above argument will yield $aH \subseteq bH$. Thus $aH = bH$ which proves (2) and hence Lemma, part (V).

Remark, Analogous to Lemma, part (V), we have its companion for right cosets:

$$Ha \cap Hb = \phi \quad \text{or} \quad Ha = Hb. \qquad (6).$$

Lemma, part (Vi): Let G be a group and $H < G$. Let $a, b \in G$. Then

$$aH = bH \iff a^{-1}b \in H$$

Proof: By Lemma, part (V), we have

$$aH = bH \iff aH \cap bH \neq \phi \iff \exists\, h_1, h_2 \in H \text{ such that } ah_1 = bh_2$$

$$\Rightarrow a^{-1}b = h_1h_2^{-1} \in H. \text{ Thus } a^{-1}b \in H.$$

Now if $a^{-1}b \in H$, then $a^{-1}b = h \in H$. So $b = ah$. Now $b = be \in bH$ and $ah \in aH$. Hence $aH \cap bH \neq \phi \Rightarrow aH = bH$ by Lemma, part (V). Thus part (Vi) is proved.

Remark: In Lemma, part (Vi), instead of $a^{-1}b \in H$, we could also say

$$b^{-1}a \in H \qquad (7)$$

either by interchanging 'a' & b or by noting $b^{-1}a = (a^{-1}b)^{-1}$!

Lemma, part (ii): Let G be a group and $H < G$. Let $a \in H$. Then

$$aH = H \iff a \in H.$$

Proof: Write $aH = H$ in the form $aH = eH$, where e is the identity in G. Then by (7) we have

$$aH = eH \iff e^{-1}a \in H \iff a \in H.$$

Hence part (ii).

Lemma, part (iv): Let G be a group and $H < G$. Let $a, b \in G$. Then

$$aH = bH \iff a \in bH. \qquad (8)$$

Proof: We know by part (vi) ~~that~~, equation (7) that

$$aH = bH \iff b^{-1}a \in H \iff b^{-1}a = h, \text{ for some } h \in H$$
$$\iff a = bh, \text{ for some } h \in H \iff a \in bH.$$

Hence part (iv).

Instead of (8), we may write

$$aH = bH \iff b \in aH \qquad (9)$$

by interchanging ~~the roles of~~ 'a' & 'b'.

Lemma, part (iii): Let G be a group and $H < G$. Let $a, b \in G$. Then

$$a(bH) = (ab)(H \quad \& \; (Ha)b = H(ab),$$

where for a subset S of G, by aS we mean $\{as \mid s \in S\}$ and $Sa = \{sa \mid s \in S\}$.

Proof: Part (iii) is a trivial consequence of the definition of cosets.

Lemma, Part (vii): Let G be a group and $H < G$. Let $a, b \in G$. Then

$$|aH| = |bH|,$$

where $|S|$ of a set means its cardinality (size).

Proof: We need to establish a bijection between aH and bH. This is easily given by

$$\begin{aligned} f: aH &\to bH \\ ah &\to bh \end{aligned} \qquad (10).$$

Every element of aH is of the form ah, $h \in H$. Now map ~~ah~~ ah to bh. Since every element of bH is of the form bh, with $h \in H$, this map is onto. The map is clearly one-to-one by the Latin square property or ~~by~~ as can be seen by cancellation:

$$ah_1 = ah_2 \implies h_1 = h_2.$$

Thus f is a bijection and that proves Part (vii).

We will do Lemma part (viii) later.

Examples of Cosets.

I) Let $G = (\mathbb{Z}, +)$ and $H = n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$, where n is a positive integer. Then the cosets of H are

$0 + H = n\mathbb{Z} = [0]_n$ = residue class 0 mod n

$1 + H = [1]_n$ = residue class 1 mod n

$\cdots\; j + H = [j]_n$ = residue class j mod n, $j = 0, 1, 2, \ldots, n-1$.

Thus the cosets of H are the elements of $\mathbb{Z}_n$.

II) Let $G = S_n$ the group of permutations on n letters. Let $H = A_n$ the subgroup of even permutations. Consider an odd permutation π and the identity permutation i. Then the cosets are

$i \circ A_n = A_n$ = set of even permutations

$\pi \circ A_n = O_n$ = set of odd permutations.

III) Let D_{2n} denote the Dihedral group of order $2n$, for $n \geq 3$. Let R denote the subgroup of rotations in D_n. Let R_0 denote the identity (rotation through angle 0) and ρ any reflection. Then the two cosets are

$R_0 \circ R = R$ = set of rotations in D_{2n}

$\rho \circ R$ = set of reflections in D_{2n}.

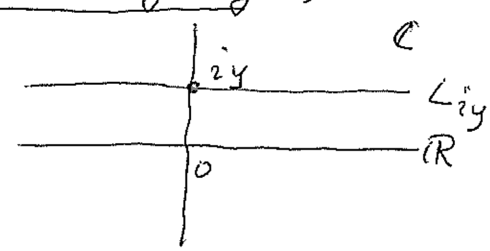
(IV) Let $G = (\mathbb{C}, +)$ = set of complex numbers under addition.

Let $H = (\mathbb{R}, +)$ = set of real numbers under addition.

Consider any number purely imaginary, say iy for $y \in \mathbb{R}$. – y arbitrary but fixed. Then the coset generated by iy is

$iy + \mathbb{R} = \{x + iy \mid x \in \mathbb{R}\} = L_{iy}$

= horizontal line!



Thus the set of all cosets of $\mathbb{R}$ is the set of all horizontal lines in $\mathbb{C}$.