MAS 4301, Abstract Algebra I

Notes of Krishna Alladi's Lecture on Mon, Apr 6, 2020

Chapter 9: Normal subgroups and factor groups

Some preliminaries

We will discuss some preliminary notions before we define normal subgroups and study their properties.

~~In~~ Let G be a group and H and K subgroups of G. Just as we defined cosets aH and Ha. when $a \in G$, we can define HK as follows:

$$HK = \{hk \mid h \in H, k \in K\}. \tag{1}$$

Thus HK is a subset of G and not a subgroup in general. But whether HK is a subgroup or not, we can determine the size of HK when H and K are finite. This is given by Thm 7.2, but we prove it here as a preliminary to our discussion of normal subgroups.

Theorem 7.2: Let G be a group and H and K two finite subgroups of G. Then

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Note: We can divide by $|H \cap K|$ because $H \cap K$ is a group and hence non-empty; it contains the identity.

Proof: Clearly we have $|H| \cdot |K|$ elements of the form hk with $h \in H$ and $k \in K$ but these elements are not necessarily all distinct. To evaluate $|HK|$, we need to understand the repetition in the values of hk.

Suppose $t \in H \cap K$ (including $t = e$, the identity). Observe that

$$hk = ht \cdot t^{-1}k \tag{2}$$

Set $h' = ht$ and $k' = t^{-1}k$ so that $h' \in H$ and $k' \in K$ since H and K are subgroups and we have $t \in H$ & $t \in K$. Thus $h'k' \in HK$

is another representation of hk, which means the values hk will have <u>at least</u> $|H \cap K|$ repetitions.

Next consider $h \in H$, $k \in K$, $h' \in H$ and $k' \in K$ such that

$$hk = h'k' \qquad (3)$$

Rewrite (3) as

$$h^{-1}h' = kk'^{-1} \qquad (4)$$

Note that $h^{-1}h' \in H$ and $kk'^{-1} \in K$ (because H and K are subgroups) and so the common (equal) value in (4), denoted by t will belong to $H \cap K$. So with $t = h^{-1}h' = kk'^{-1}$, we have

$$h' = ht \quad \& \quad k' = t^{-1}k, \qquad (5)$$

which is precisely how we formed the repetition above. Thus each value hk repeats exactly $|H \cap K|$ times. Hence the assertion of Thm 7.2 follows.

<u>WARNING</u>: Although H and K are groups, the set HK need not be a group, as the following example shows:

Let $G = S_3$, the group of permutations on 3 letters. We know that $|S_3| = 3! = 6$. Let $\alpha = (1,2)$ and $\beta = (2,3)$ be two transpositions in S_3. Since $\alpha^2 = \beta^2 = $ identity, $\alpha = \alpha^{-1}$ & $\beta^{-1} = \beta$, and so the subgroups H and K generated by α and β are:

$$H = \langle \alpha \rangle = \{i, \alpha\}, \quad K = \langle \beta \rangle = \{i, \beta\}, \quad \text{with } |H \cap K| = 1,$$

where i is the identity permutation. Now

$$HK = \{1, \alpha, \beta, \alpha\beta\}. \qquad (6)$$

Note $|HK| = |H| \cdot |K| / |H \cap K| = \frac{2.2}{1} = 4$, by Thm 7.2, and confirmed by (6).

Observe that

$$\alpha\beta = (1,2)(2,3) = (1,2,3) \text{ — a 3-cycle} \qquad (7)$$

Since $\alpha\beta \in HK$, but $(\alpha\beta)^{-1} = (3,2,1) \notin HK$, we see that HK is not a subgroup of S_3. Another way to realise that HK is <u>not</u> a subgroup of $G = S_3$ is to note that $4 \nmid 6$, $|HK| = 4$ & $|S_3| = 6$.

In view of the above example, we now ask

QUESTION: If H and K are subgroups of G, what condition can we impose on H and/or K to ensure that HK is a subgroup of G?

ANSWER: If at least one of H or K is a normal subgroup, then HK will be a subgroup.

We will establish this below after we define normal subgroups.

But before that, we consider another notion.

Theorem C: Let G be a group and H a subgroup of G. Then for each $a \in G$, the set

$$aHa^{-1} = \{aha^{-1} \mid h \in H\}$$

is a subgroup of G.

Proof: Let e be the identity in G. Then $aea^{-1} = e \in aHa^{-1}$ since $e \in H$. Thus aHa^{-1} contains the identity.

Next consider any two elements aha^{-1} and $ah'a^{-1}$ of aHa^{-1}. Observe that

$$aha^{-1} \cdot ah'a^{-1} = ahh'a^{-1} \in aHa^{-1} \text{ because } hh' \in H.$$

Thus aHa^{-1} is closed under multiplication.

Finally consider any $aha^{-1} \in H$. Then

$$(aha^{-1})^{-1} = (a^{-1})^{-1}h^{-1}a^{-1} = ah^{-1}a^{-1} \in aHa^{-1}, \text{ because } h^{-1} \in H.$$

Thus for each element of aHa^{-1}, its inverse belongs to aHa^{-1}.

Therefore aHa^{-1} is a subgroup, and so Theorem C is proved.

Terminology: The group $K = aHa^{-1}$ is called a conjugate of H.

Note that

$$K = aHa^{-1} \Rightarrow H = a^{-1}Ka = a^{-1}K(a^{-1})^{-1} \qquad (8).$$

which means H is a conjugate of K using a^{-1} instead of 'a'.

Thus H and K are conjugates (of each other), and so the terminology "conjugate" is appropriate. In view of this conjugacy, we called the above result as Theorem C.

Inner automorphisms and conjugate groups

Let G be a group and $a \in G$. We know that each $a \in G$ induces an automorphism of G called the inner automorphism induced by 'a', denoted by i_a and defined by

$$\left.\begin{array}{l} i_a : G \to G \\ \quad g \to aga^{-1} \end{array}\right\} \quad i_a(g) = aga^{-1} \qquad (9)$$

Let H be a subgroup of G. If we consider the restriction of i_a to H, then we get

$$\left.\begin{array}{l} i_a|_H : H \to aHa^{-1} \\ \qquad h \to aha^{-1} \end{array}\right\} \quad i_a(h) = aha^{-1}, \ \forall h \in H \qquad (10)$$

which is a map (isomorphism) from H to aHa^{-1}. The map $i_a|_H$ is a bijection and operation preserving, hence an isomorphism.

The groups H and aHa^{-1} are isomorphic but not the same in general. Let us look at the earlier example we had to illustrate this:

$G = S_3$ = the symmetric group on 3 letters.

Let $\alpha = (1,2)$ – a transposition and let $H = \langle \alpha \rangle = \{1, \alpha\}$.

Let $\beta = (2,3)$. Then since $\beta^{-1} = \beta$, we have

$$\beta H \beta^{-1} = \beta H \beta = \{ i, \beta\alpha\beta \}. \qquad i = \text{identity}$$

Note that

$$\beta\alpha\beta = (2,3)\cdot(1,2)\cdot(2,3) = (2,3)(1,2,3) = (1,3) \text{ – a transposition}$$

Thus

$$\beta H \beta^{-1} = \{ i, (1,3) \} \neq H = \{ i, (1,2) \}.$$

So this leads to

QUESTION: What condition do we need to impose on H such that $aHa^{-1} = H$, $\forall a \in G$?

ANSWER: If H is a normal subgroup, we will have $aHa^{-1} = H$, $\forall a \in G$.

We will prove this after defining a normal subgroups which is what we do next.

Normal subgroups

Previously for a subgroup H of a group G, we considered the left cosets aH and right cosets Ha, for $a \in H$. The left coset aH is not the same as the right coset Ha in general. So this leads to

<u>Definition of a Normal subgroup</u>: Let G be a group and H a subgroup. We say H is a <u>normal</u> subgroup if

$$aH = Ha, \quad \forall\, a \in G. \tag{11}$$

We write $H \lhd G$ to denote H is a <u>normal</u> subgroup of G.

<u>Examples</u>:

1) If G is Abelian, then all subgroups H of G are normal.

2) If $H = Z(G)$, the Center of G, then $Z(G)$ is a normal subgroup because all elements of $Z(G)$ commute with <u>all</u> elements <u>of G</u>.
That is $aH = \{ah \mid h \in Z(G)\} = \{ha \mid h \in Z(G)\} = Ha$, $\forall a \in G$.

3). Let $G = S_n$, the symmetric group on n letters and A_n the alternating group, namely the subgroup of even permutations. Then A_n is a normal subgroup of $G = S_n$.

To realize this consider any $\pi \in S_n$. We have two cases:

<u>Case 1</u>: π is even. That is $\pi \in A_n$. Then

$$\pi A_n = A_n = A_n \pi$$

<u>Case 2</u>: π is odd. That is $\pi \in O_n$ = set of odd permutations. Then

$$\pi A_n = O_n = A_n \pi.$$

Thus $\pi A_n = A_n \pi$, $\forall\, \pi \in S_n$. Hence $A_n \lhd S_n$.

4) Let D_{2n} be the Dihedral group of order $2n$ and $R^{(n)}$ the group of rotations in D_{2n}. Then $R^{(n)}$ is a normal subgroup of D_{2n}.
The proof of this is similar to the proof of $A_n \lhd S_n$. Any member of D_{2n} is either a rotation r or a reflection ρ. We know that

$$\left.\begin{aligned} r R^{(n)} &= R^{(n)} = R^{(n)} r, \quad \forall\, r \in R^{(n)} \\ \text{and } \rho R^{(n)} &= D_{2n} - R^{(n)} = R^{(n)} \rho, \quad \forall\, \rho \in D_{2n} - R^{(n)}. \end{aligned}\right\} \text{ Hence } R^{(n)} \lhd D_{2n}$$