MAS 4301, Abstract Algebra I

Notes of Krishna Alladi's Lecture on Wed, Apr 8, 2020

Chapter 9: Normal subgroups and factor groups cont'd.

Our definition of a normal subgroup H of a group G is that for every $a \in G$, the left coset aH must equal the right coset Ha. The equality

$$aH = Ha, \quad \forall a \in G \tag{1}$$

is the same as saying the following:

$$\forall a \in G, \text{ and } \forall h \in H, \ \exists h' \in H, \text{ such that } ah = h'a \tag{2}$$

& $$\forall a \in G \text{ and } \forall h' \in H, \ \exists h \in H, \text{ such that } ah = h'a. \tag{3}$$

Note that (2) implies $aH \subseteq Ha$, and (3) implies $Ha \subseteq aH$. Thus (2) & (3) imply $aH = Ha$ which is (1).

We now establish

Theorem 9.1 (Normal subgroup test).

Let G be a group and H a subgroup of G. Then H is a normal subgroup if and only if $xHx^{-1} \subseteq H$, $\forall x \in G$.

Proof:

($\Rightarrow$) Suppose H is normal in G. Pick any $x \in G$. Since $H \triangleleft G$, we know that $xH = Hx$. So by (2), given $h \in H$, we know $\exists h' \in H$ such that

$$xh = h'x \iff xhx^{-1} = h' \tag{4}$$

In (4), interpret $xhx^{-1} \in xHx^{-1}$ and $h' \in H$. Since (4) holds $\forall h \in H$, we see that $xHx^{-1} \subseteq H$. This proves ($\Rightarrow$).

($\Leftarrow$) Conversely, let $xHx^{-1} \subseteq H$, $\forall x \in G$. So given any $h \in H$, we have $xhx^{-1} \in H \Rightarrow xhx^{-1} = h' \in H$. This means $\forall h \in H, \exists h' \in H$, such that

$$xh = h'x \in Hx.$$

Thus $xH \subseteq Hx$, $\forall x \in G$.

Next since $xHx^{-1} \subseteq H$ $\forall x \in G$, we may write this as

$$x^{-1}H(x^{-1})^{-1} = x^{-1}Hx \subseteq H, \quad \forall x \in G \tag{5}$$

since x^{-1} ranges over all elements of G as x ranges over all elements of G. Now interpret (5) as follows:

$\forall x \in G$ and $\forall h \in H$, $\exists h' \in H$ such that $x^{-1}hx = h' \in H$. (6)

(with $x^{-1}hx \in x^{-1}Hx$). Finally rewrite the equality in (6) as:

$\forall x \in G$, and $\forall x \in G$, $\exists h' \in H$ such that $hx = xh'$ (7)

and interpret (7) as

$$Hx \subseteq xH, \quad \forall x \in G.$$

Hence $xH = Hx$, $\forall x \in G$, which means $H \triangleleft G$.

Remark: Theorem 9.1 can be recast as follows:

Theorem 9.1′: Let G be a group and H a subgroup of G. Then H is normal in G if and only if $xHx^{-1} = H$, $\forall x \in G$.

Proof: The condition $xHx^{-1} = H$, $\forall x \in G$ is stronger than the condition $xHx^{-1} \subseteq H$, $\forall x \in G$. We know from the proof of Theorem 9.1, that the weaker condition $xHx^{-1} \subseteq H$ implies $H \triangleleft G$. Hence $xHx^{-1} = H$, $\forall x \in G$ ~~at~~ implies $H \triangleleft G$.

In the other direction if we start with $H \triangleleft G$, we interpreted (4) as saying $xHx^{-1} \subseteq H$, $\forall x \in G$. By we can also interpret (4) as follows;

For every $x \in G$ and $h \in H$, $\exists h'$ such that $h = x^{-1}h'x = x^{-1}h'(x^{-1})^{-1}$ (8)

In (8), we can replace x by x^{-1}, since x^{-1} ranges over all elements of G as x does. Thus

$\forall x \in G$ & $\forall h \in H$, $\exists h'$ such that $h = xh'x^{-1} \in xHx^{-1}$.

Interpret this as $H \subseteq xHx^{-1}$. This yields $xHx^{-1} = H$, $\forall x \in G$ and proves Theorem 9.1′!

Remark cont'd: The book gives Thm 9.1 as the normal subgroup test. Other books give Thm 9.1′ as the normal subgroup test. One reason to prefer Thm 9.1 is because $xHx^{-1} \subseteq H$ is easier to verify than $xHx^{-1} = H$!

Some properties of normal subgroups

Problem 56, p.191: If $H \lhd G$ and $K \lhd G$, prove that $H \cap K \lhd G$.

Proof: Pick any $x \in G$ and any $g \in H \cap K$. Then $g \in H$ and $g \in K$.

$$\left.\begin{array}{l} \text{Since } H \lhd G, \text{ we have } xHx^{-1} \subseteq H \Rightarrow xgx^{-1} \in H. \\ \text{Since } K \lhd G, \text{ we have } xKx^{-1} \subseteq K \Rightarrow xgx^{-1} \in K. \end{array}\right\} xgx^{-1} \in H \cap K$$

Thus

$\forall x \in G$, $\forall g \in H \cap K$, we have $xgx^{-1} \in H \cap K$

which means

$$x(H \cap K)x^{-1} \subseteq H \cap K, \quad \forall x \in G.$$

Thus by Thm 9.1, we see that $H \cap K \lhd G$.

Remark: The above proof yields the following more general result:

If $H_1, H_2, \ldots, H_n$ is a finite collection of normal subgroups of G, then

$$\bigcap_{i=1}^{n} H_i \lhd G. \tag{9}$$

Example 5, p.175: If $H \lhd G$ and K any subgroup of G, then HK is a subgroup of G. Similarly, if $K \lhd G$ and H any subgroup of G, then HK is a subgroup of G.

Proof: We prove only the first statement in Example 5. The proof of the second statement is similar (follows by symmetry).

For any pair of subgroups H, K, of the group G, since $e \in H$ & $e \in K$ we have $e \cdot e = e \in H \cdot K$.

Now assume $H \lhd G$. We will prove that HK is a subgroup by the one-step subgroup test. To this end, consider any pair of elements $a, b \in HK$. Write $a = h_1 k_1$ and $b = h_2 k_2$, with $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then

$$ab^{-1} = h_1 k_1 (h_2 k_2)^{-1} = h_1 k_1 k_2^{-1} h_2^{-1} = h_1 (k_1 k_2^{-1}) h_2^{-1}. \tag{10}$$

In (10), note that $h_2^{-1} \in H$ and so $k_1 k_2^{-1} h_2^{-1} \in k_1 k_2^{-1} \cdot H$. Since $H \lhd G$, this means $\exists\, h'$ such that

$$k_1 k_2^{-1} h_2^{-1} = h' k_1 k_2^{-1}, \quad h' \in H \tag{11}$$

Thus by (10) and (11) we have

$$ab^{-1} = h_1 h' k_1 k_2^{-1} = (h_1 h')(k_1 k_2^{-1}) \in HK$$

Thus HK is a subgroup of G.

Remark: We saw an example of subgroups H and K of G for which HK was not a subgroup. Example 5 (which really should have been stated as a Theorem in the book!) asserts that if at least one of H or K is normal, then HK is a subgroup. So this raises the question as to what happens if both H and K are normal? In this case we write N and M instead of H and K and prove:

Problem 58, p. 191 (could be stated as a theorem!)

If N and M are both normal subgroups of G, then NM is a normal subgroup.

Proof: We know from Example 5 above that NM is a subgroup. We will therefore use the normal subgroup test to show that $NM \trianglelefteq G$. Consider any $x \in G$ and any $y \in NM$. Write $y = nm$, with $n \in N, m \in M$. Observe that

$$xyx^{-1} = xnmx^{-1} = xnx^{-1} \cdot xmx^{-1} \qquad (12)$$

Since $N \trianglelefteq G$, we have $xnx^{-1} \in N$. Since $M \trianglelefteq G$, we have $xmx^{-1} \in M$. So in (12)

$$xnx^{-1} \cdot xmx^{-1} \in NM \qquad (13).$$

Interpret (12) and (13) as yielding $xNMx^{-1} \subseteq NM$. So $NM \trianglelefteq G$ by the normal subgroup test.

Given subgroups H and K of G, we will now provide a necessary and sufficient condition for HK to be a subgroup, and connect this with normal subgroups.

Theorem 1: Let H and K be subgroups of a group G. Then HK is a subgroup of G if and only if $HK = KH$.

Proof: ($\Leftarrow$) We are given that $HK = KH$. We need to show that HK is a subgroup.

First observe that $e = e \cdot e \in HK$. Next pick any two elements of HK, say hk and $h'k'$. Then

$$hk \cdot h'k' = h(kh')k' \qquad (14).$$

In (14), the element $kh' \in KH$. Since $KH = HK$, $\exists\ h''k''$ such that $kh' = h''k''$. Hence (14) could be rewritten as

$$hk \cdot h'k' = h(h''k'')k' = (hh'')(k''k') \in HK.$$

Thus HK is closed under multiplication. Finally, given any $hk \in HK$, consider

$$(hk)^{-1} = k^{-1}h^{-1} \in KH = HK \text{ (given)} \Rightarrow (hk)^{-1} \in HK.$$

Thus for every element of HK, its inverse is in HK. Hence HK is a subgroup.

($\Rightarrow$) Now assume HK is a subgroup. Pick <u>any</u> element $hk \in HK$. Then $(hk)^{-1} \in HK$ since HK is a group. But $(hk)^{-1} = k^{-1}h^{-1} \in KH$. Now the collection of inverses of the elements of HK gives <u>all</u> elements of HK. Hence $HK \subseteq KH$. Similarly (by symmetry) $KH \subseteq HK$. Thus $HK = KH$, which proves the theorem.

Corollary: (Example 5): If $H \triangleleft G$ and K any subgroup of G, then HK is a subgroup of G.

Proof: Since $H \triangleleft G$, we have $aH = Ha$, $\forall a \in G$. Letting 'a' run through the elements of K and taking the union, we get $HK = KH$. Hence by the theorem, HK is a subgroup of G.//

We conclude this lecture with an important property of normal subgroups:

Theorem 2: Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$. Then <u>every</u> element of H <u>commutes</u> with <u>every</u> element of K.

Proof: We need to show that given any $h \in H$ and $k \in K$, we have $hk = kh$. This is equivalent to

$$hk(kh)^{-1} = hkh^{-1}k^{-1} = e \qquad (15)$$

In (15), observe that since H is normal, we have $kh^{-1}k^{-1} \in kHk^{-1} = H \Rightarrow kh^{-1}k^{-1} \in H$. Next since K is normal, in (15) we have $hkh^{-1} \in hKh^{-1} = K \Rightarrow hkh^{-1} \in K$. Thus $h(kh^{-1}k^{-1}) \in H$ and $(hkh^{-1})k^{-1} \in K$ and the two are the same. Thus ~~the~~ $hkh^{-1}k^{-1} \in H \cap K = \{e\}$. This proves (15) and hence Theorem 2.