Appendix C

The gamma function

For any complex number s not equal to a non-positive integer we define the gamma function by its Weierstrass product,

$$\Gamma(s) = \frac{e^{-C_0 s}}{s} \prod_{n=1}^{\infty} \frac{e^{s/n}}{1 + s/n}. \tag{C.1}$$

Here C_0 is Euler's constant, and we recall from Corollary 1.14 or Exercise B.15 that this constatnt is determined by the relation

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + C_0 + O(1/N). \tag{C.2}$$

From (C.1) it is evident that $1/\Gamma(s)$ is an entire function with simple zeros at the non-positive integers, which is to say that $\Gamma(s)$ is a non-vanishing meromorphic function with simple poles at the non-positive integers as depicted in Figure C.1. On considering the N^{th} partial product in (C.1) and appealing to (C.2), we obtain *Gauss's formula*,

$$\Gamma(s) = \lim_{N \to \infty} \frac{N^s N!}{s(s+1)\cdots(s+N)}. \tag{C.3}$$

By taking $s = 1$ we see that $\Gamma(1) = 1$. Moreover, from (C.3) it is also immediate that

$$s\Gamma(s) = \Gamma(s+1). \tag{C.4}$$

Hence by induction we find that

$$\Gamma(n+1) = n! \tag{C.5}$$

for non-negative integers n. As will become apparent, the gamma function not only interpolates the values of the factorial, but does so quite smoothly.

The function $\Gamma(s)\Gamma(1-s)$ has a simple pole at every integer. Since the same can be said for $1/\sin \pi s$, it is reasonable to investigate the relation between these

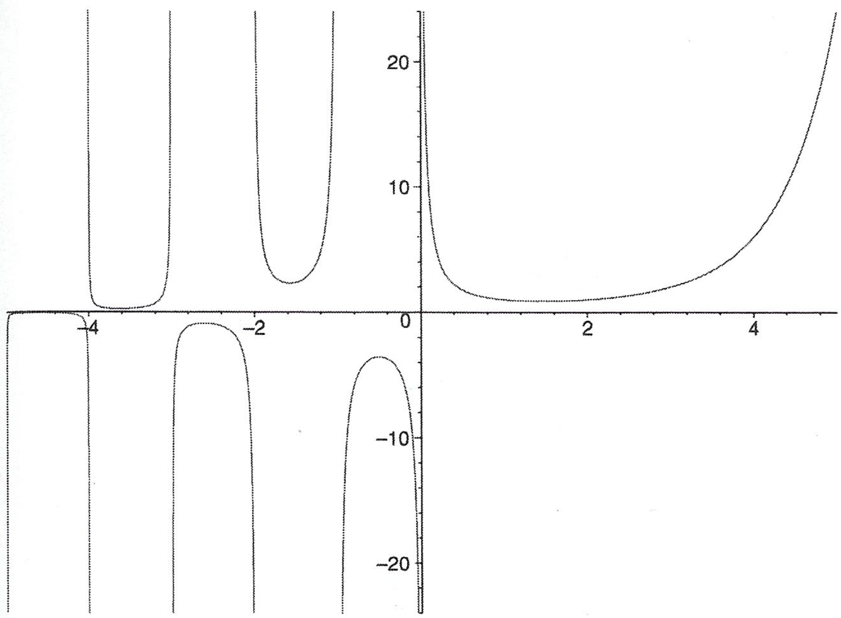

Figure C.1 Graph of $\Gamma(s)$ for $-5 < s \le 5$.

two functions. To this end we let $p_N(s)$ denote the expression on the right in (C.3), and note that

$$p_N(s)p_N(1-s) = \frac{N}{s(N+1-s)}\prod_{n=1}^{N}(1-(s/n)^2)^{-1}.$$

On the other hand, we recall that the Weierstrass product for the sine function may be written

$$\sin s = s\prod_{n=1}^{\infty}\Big(1-\frac{s^2}{(\pi n)^2}\Big).$$

On comparing these formulæ we conclude that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{C.6}$$

We take $s = 1/2$ to see that $\Gamma(1/2)^2 = \pi$. But from (C.1) it is clear that $\Gamma(1/2) > 0$, so we have

$$\Gamma(1/2) = \sqrt{\pi}. \tag{C.7}$$

From (C.1) we see that $\Gamma(s)$ never takes the value 0, and that it has simple poles at the non-positive integers. Let k be a non-negative integer. Since

$\sin \pi s \sim (-1)^k \pi (s+k)$ as $s \to -k$, and since $\Gamma(k+1) = k!$, it follows from (C.6) that

$$\Gamma(s) \sim \frac{(-1)^k}{k!(s+k)} \tag{C.8}$$

as $s \to -k$.

Similarly we observe that $\Gamma(s)\Gamma(s+1/2)$ has a simple pole at $0, -1/2, -1, -3/2, -2, \ldots$, and that the same is true of $\Gamma(2s)$. We now establish a relation between these two functions by observing that

$$\frac{p_N(s)p_N(s+1/2)}{p_{2N}(2s)} = 2^{1-2s} \frac{N+1/2}{N+s+1/2} p_N(1/2).$$

On letting $N \to \infty$ and using (C.7) we obtain *Legendre's duplication formula*,

$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s). \tag{C.9}$$

On taking logarithmic derivatives in (C.1) we find that the *digamma function* $\frac{\Gamma'}{\Gamma}(s)$ can be written

$$\frac{\Gamma'}{\Gamma}(s) = -\frac{1}{s} - C_0 - \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right). \tag{C.10}$$

Setting $s = 1$, we see in particular that

$$\frac{\Gamma'}{\Gamma}(1) = -C_0. \tag{C.11}$$

Since $\Gamma(1) = 1$, this is equivalent to

$$\Gamma'(1) = -C_0. \tag{C.12}$$

We write $z = re(\theta)$ in the power series expansion $\log(1-z)^{-1} = \sum_{n=1}^{\infty} z^n/n$, let $r \to 1^-$, and apply Abel's theorem to see that

$$\sum_{n=1}^{\infty} \frac{e(n\theta)}{n} = -\log(1 - e(\theta)) \tag{C.13}$$

provided that $\theta \notin \mathbb{Z}$. By applying this formula for various rational values of θ we can express the series in (C.10) in closed form, for any rational value of s. For example, by taking $\theta = 1/2$ we find that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

which with (C.10) gives

$$\frac{\Gamma'}{\Gamma}(1/2) = -C_0 - 2\log 2. \tag{C.14}$$

Also, since

$$\frac{-1-i}{4}e(n/4) - \frac{1}{2}e(n/2) + \frac{-1+i}{4}e(3n/4) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

by taking $\theta = 1/4, 1/2, 3/4$ in (C.13) we deduce via (C.10) that

$$\frac{\Gamma'}{\Gamma}(1/4) = -C_0 - 3\log 2 - \pi/2. \tag{C.15}$$

Similarly,

$$\frac{\Gamma'}{\Gamma}(3/4) = -C_0 - 3\log 2 + \pi/2. \tag{C.16}$$

We now consider the asymptotic behaviour of the gamma function.

Theorem C.1 *Let $\delta > 0$ be given, and let $\mathcal{R} = \mathcal{R}(\delta)$ be the set of those complex numbers s for which $|s| \ge \delta$ and $|\arg s| < \pi - \delta$. Then*

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|) \tag{C.17}$$

and

$$\Gamma(s) = \sqrt{2\pi}\, s^{s-1/2} e^{-s}(1 + O(1/|s|)) \tag{C.18}$$

uniformly for $s \in \mathcal{R}$.

The second estimate here is *Stirling's formula* for the gamma function, which generalizes his estimate (B.26) for $n!$. From this we see that

$$|\Gamma(s)| \asymp \tau^{\sigma-1/2} e^{-\pi\tau/2} \tag{C.19}$$

as $|t| \to \infty$ with σ uniformly bounded.

Proof From (C.2) and (C.10) we see that if $N > |s|$, then

$$\frac{\Gamma'}{\Gamma}(s) = \log N - \sum_{n=0}^{N} \frac{1}{n+s} + O(|s|/N).$$

By the Euler–MacLaurin summation formula (Theorem B.5) with $f(x) = 1/(x+s)$, $a = 0^-$, $b = N$, $K = 2$ we find that

$$\sum_{n=0}^{N} \frac{1}{n+s} = \log(N+s) - \log s + \frac{1}{2s} + \frac{1}{2(s+N)} + O(|s|^{-2}).$$

On combining these estimates and letting N tend to infinity we find that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O(|s|^{-2}). \tag{C.20}$$

This estimate is more precise than (C.17), and still greater accuracy can be obtained by choosing a larger value of K.

To derive (C.18) we begin by taking logarithms in (C.3) and applying the Euler–MacLaurin summation formula, or we integrate (C.20) from s to $s+\infty$ along a ray parallel to the real axis. In either case we find that

$$\log \Gamma(s) = s \log s - s - \frac{1}{2}\log s + c + O(1/|s|),$$

and it remains to determine the value of the constant c. This may be done in a number of ways. For example, we could appeal to (C.5) and (B.26). Alternatively, we can take logarithms in (C.9) and apply the above to see that $c = (\log 2\pi)/2$. Then (C.18) follows by exponentiating. □

The gamma function can be expressed as a definite integral in various ways. We now establish two important integral representations for the gamma function.

Theorem C.2 (Euler's integral) *If* $\Re s > 0$, *then*

$$\int_0^\infty e^{-x} x^{s-1}\,dx = \Gamma(s). \tag{C.21}$$

Proof By integrating by parts repeatedly it is easy to verify that

$$\frac{N!}{s(s+1)\cdots(s+N)} = \int_0^1 (1-y)^N y^{s-1}\,dy.$$

We make the change of variable $x = Ny$ and recall Gauss's formula (C.3) to find that

$$\Gamma(s) = \lim_{N\to\infty}\int_0^\infty f_N(x)\,dx$$

where

$$f_N(x) = \begin{cases} (1-x/N)^N x^{s-1} & \text{for } 0 \le x \le N, \\ 0 & \text{for } x > N. \end{cases}$$

To complete the proof we employ the dominated convergence theorem. Put $f(x) = e^{-x}x^{\sigma-1}$. Then $\int_0^\infty f(x)\,dx < \infty$ when $\sigma > 0$, and $|f_N(x)| \le f(x)$ uniformly in N and x. Since

$$\lim_{N\to\infty} f_N(x) = e^{-x}x^{s-1}$$

for each fixed x, the formula (C.21) now follows. □

Let $\mathcal{C}(\rho)$ denote the circular arc $\{z = \rho e(\theta) : 0 \le \theta \le 1/4\}$. It is easy to verify that

$$\int_{\mathcal{C}(\rho)} |e^{-z} z^{s-1}|\,|dz| \to 0$$

as $\rho \to \infty$. Thus by Cauchy's theorem the formula (C.21) still holds if x is replaced by a complex variable z that goes to infinity along a ray from the origin, $z = \rho e(\theta)$, $0 \le \rho < \infty$, provided that $-1/4 \le \theta \le 1/4$.

For $r > 0$ we let $\mathcal{H} = \mathcal{H}(r)$ denote the Hankel contour, which consists of a path that passes from $-ir - \infty$ to $-ir$ along the ray $x - ir$, $-\infty < x \le 0$, and then from $-ir$ to ir along the semicircle $re(\theta)$, $-1/4 \le \theta \le 1/4$, and then from ir to $ir - \infty$ along the ray $x + ir$, $-\infty < x \le 0$.

Theorem C.3 (Hankel) *For any complex number s,*

$$\frac{1}{2\pi i}\int_{\mathcal{H}} e^z z^{-s}\,dz = \frac{1}{\Gamma(s)}. \tag{C.22}$$

Here z^{-s} is assumed to have its principal value.

As in the preceding theorem, the contour of integration may be altered substantially without changing the value of the integral. For example, the ray from ir to $-\infty + ir$ may be replaced by a ray in the direction $e(\theta)$, provided that $1/4 < \theta < 1/2$.

Proof It is clear that the left-hand side is an entire function of s. Thus it suffices to prove the identity when $\sigma < 1$. For such s we let $r \to 0^+$, and note that the integral along the semicircle tends to 0. The remaining integrals tend to

$$e^{i\pi s}\int_0^\infty e^{-x}x^{-s}\,dx - e^{-i\pi s}\int_0^\infty e^{-x}x^{-s}\,dx = 2i(\sin \pi s)\Gamma(1-s)$$

by (C.21). To complete the proof it suffices to appeal to (C.6). □

Euler's formula asserts that the gamma function is the Mellin transform of the function e^{-x}. We now establish the inverse.

Theorem C.4 (Mellin) *If $\Re z > 0$ and $c > 0$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s}\,ds = e^{-z}.$$

Proof From Stirling's formula we see that

$$\int_{-K+iK}^{c+iK} |\Gamma(s)z^{-s}|\,|ds| \longrightarrow 0$$

as $K \to \infty$, and similarly for the integral from $-K - iK$ to $c - iK$. Moreover, if we first apply (C.6) and then Stirling's formula, we find that

$$\int_{-K-iK}^{-K+iK} |\Gamma(s)z^{-s}|\,|ds| \longrightarrow 0$$

as $K \to \infty$ through values of the form $K = n + 1/2, n \in \mathbb{Z}$. (We are assuming here that the path of integration is a line segment joining the two endpoints.) Thus by the calculus of residues

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)z^{-s}\,ds = \sum_{k=0}^{\infty} Res\Big(\Gamma(s)z^{-s}\Big|_{s=-k}.$$

From (C.8) we see that the above is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k = e^{-z}.$$

□

The digamma function can be examined in a similar way. In view of (C.17), this function is not absolutely integrable on the line $\sigma = c$, and thus we cannot define its Fourier transform in the classical manner. We now formulate a useful substitute.

Theorem C.5 *Let $a > 0$ and $b > 0$ be fixed. If $x < 0$ and $T \ge 1$, then*

$$\int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt)e(-xt)\,dt = -\frac{\Gamma'}{\Gamma}(a+ibT)\frac{e(-xT)}{2\pi i x} + \frac{\Gamma'}{\Gamma}(a-ibT)\frac{e(xT)}{2\pi i x} - 2\pi b^{-1}e^{2\pi ax/b}(1-e^{2\pi x/b})^{-1} + O(x^{-2}T^{-1}),$$

while if $x > 0$ and $T \ge 1$, then

$$\int_{-T}^{T} \frac{\Gamma'}{\Gamma}(a+ibt)e(-xt)\,dt = -\frac{\Gamma'}{\Gamma}(a+ibT)\frac{e(-xT)}{2\pi i x} + \frac{\Gamma'}{\Gamma}(a-ibT)\frac{e(xT)}{2\pi i x} + O(x^{-2}T^{-1}).$$

Proof We write the integral as

$$\frac{1}{i}\int_{-iT}^{iT} \frac{\Gamma'}{\Gamma}(a+bs)e^{-2\pi xs}\,ds.$$

Suppose that $x < 0$. Let $\mathcal{C}$ be the contour passing by line segment from $-\infty - iT$ to $-iT$ to iT to $-\infty + iT$. By the calculus of residues and (C.10) we find that

$$\int_{\mathcal{C}} \frac{\Gamma'}{\Gamma}(a+bs)e^{-2\pi xs}\,ds = -\frac{2\pi i}{b}\sum_{n=0}^{\infty} e^{2\pi x(n+a)/b} = -\frac{2\pi i}{b}e^{2\pi ax/b}\big(1-e^{2\pi x/b}\big)^{-1}.$$

We parametrize the integral $\int_{-\infty-iT}^{-iT}$, and integrate by parts, to see that it is

$$\int_{-\infty}^{0} \frac{\Gamma'}{\Gamma}(a + b\sigma - ibT)e(xT)e^{-2\pi x\sigma}\, d\sigma$$

$$= -\frac{\Gamma'}{\Gamma}(a - ibT)\frac{e(xT)}{2\pi x} + \frac{be(xT)}{2\pi x}\int_{-\infty}^{0}\left(\frac{\Gamma'}{\Gamma}\right)'(a + b\sigma - ibT)e^{-2\pi x\sigma}\, d\sigma.$$

But

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \sum_{n=0}^{\infty}(n+s)^{-2} \ll 1/|t|$$

for $|t| \ge 1$, and hence the last integral above is $\ll x^{-2}T^{-1}$. Similarly,

$$\int_{iT}^{-\infty+iT} \frac{\Gamma'}{\Gamma}(a + bs)e^{-2\pi xs}\, ds = \frac{\Gamma'}{\Gamma}(a + ibT)\frac{e(-xT)}{2\pi x} + O(x^{-2}T^{-1}).$$

We obtain the stated result on combining these estimates. The case $x > 0$ is treated similarly, but with a contour from $+\infty - iT$ to $-iT$ to iT to $+\infty + iT$. □

Exercises

1. Show:

   (a) $$|\Gamma(it)|^2 = \frac{\pi}{t \sinh \pi t};$$

   (b) $$|\Gamma(1/2 + it)|^2 = \frac{\pi}{\cosh \pi t};$$

   (c) $$\Im \frac{\Gamma'}{\Gamma}(s) > 0 \text{ if } t > 0;$$

   (d) $$\frac{\partial}{\partial t} \log |\Gamma(s)| < 0 \text{ when } t > 0;$$

   (e) For any given σ, $|\Gamma(s)|$ is a strictly decreasing function of t on the interval $0 < t < \infty$.

2. (Gauss 1812) Prove *Gauss's multiplication formula*:

$$\prod_{a=0}^{q-1} \Gamma(s + a/q) = (2\pi)^{(q-1)/2} q^{1/2 - qs}\Gamma(qs).$$

3. Show:

   (a) $$\frac{\Gamma'}{\Gamma}(1-s) - \frac{\Gamma'}{\Gamma}(s) = \pi \cot \pi s;$$

   (b) $$\frac{\Gamma'}{\Gamma}(s+1) = \frac{1}{s} + \frac{\Gamma'}{\Gamma}(s);$$

   (c) If n is an integer, $n > 1$, then

$$\frac{\Gamma'}{\Gamma}(n) = -C_0 + \sum_{k=1}^{n-1}\frac{1}{k}.$$